Fundamentals of Transport Processes II
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Lecture - 05
Vector Calculus

So, welcome to this lecture 5 of our course on Fundamentals of Transport Processes, we were going through integral and differential calculus with vectors in order to set the stage for deriving conservation equations for the fluid velocity field, which itself is a vector. As I said we are doing some preparatory courses in order to familiarize ourselves with how one can do calculus integral and differential, considering vectors as objects in themselves rather than trying to do them for the individual components. So, our starting point vector calculus in this one dimension, I had defined for you what is meant by the derivative of a function and what is meant by the integral.

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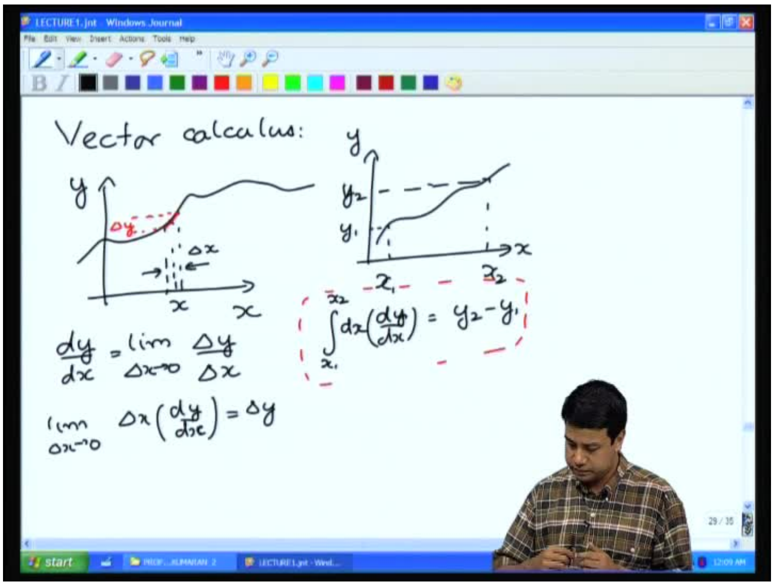

So, if I have a single valued function y of x, single valued means that at each value of x there is a unique value of y, if I want to define the derivative at a point. I take a small integral delta x around this point x and find out what is the variation in y; when f travel with a small integral delta x, that is delta y. In the limit as delta x goes to 0 delta y will also go to 0, but the ratio need not and the limit as delta equals to 0, the ratio is what is called the derivative of the function at that particular location. So, limit as delta x goes to

0 of delta y b delta x is equal to d y by d x alternatively delta x times the derivative, the interval times the derivative gives, you the difference in the different function y. An integral form of this is that integral between 2 end points of d x times the derivative is the difference in y between those 2 end points.

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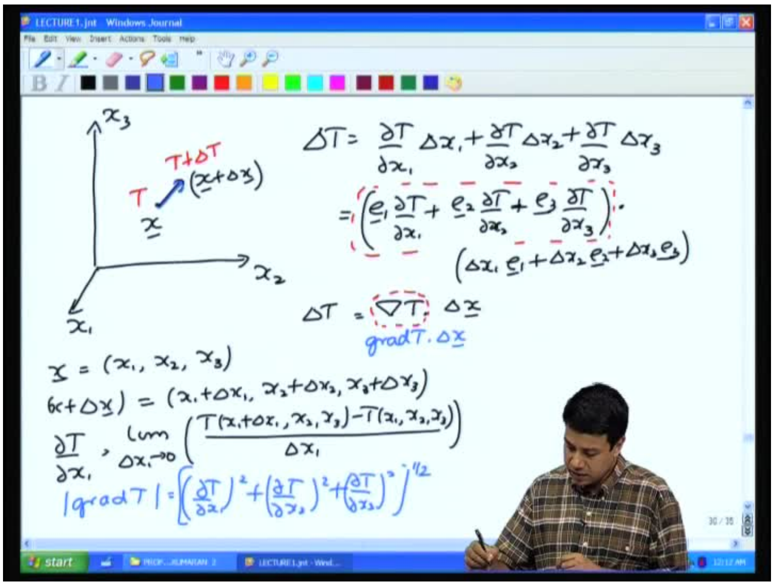

So, we use this analogy to defined of first derivative function, that is the gradient grad T, it is defined such that delta t that is if I sitting at one particular location x and I move a small distance delta x to a new location. The change in temperature say between these 2, the final minus the initial point is equal to this gradient of T dotted with the displacement gradient of T dotted with the vector displacement that was moved. So, the gradient of T tells, you how the temperature is varying is changing locally as you move some distance around, this point. So, even though the gradient I had defined in terms of the partial derivatives grad T was equal to partial T by partial x 1 e 1 plus partial T by x 2 e 2 plus partial T by partial x 3 times e 3.

So, even though grad T was defined in terms of these partial derivatives, the gradient vector itself has a direction in the magnitude that is independent of the coordinate system, when I am using to analyze, the problem. It is a well defined, because if I go from one location to the next location, the difference in temperature has to be grad T times delta x, regardless of what coordinate system, I use to represent grad T and delta x, the result, that I get has to be exactly the same independent of coordinate system. So, grad T is a vector, which is independent of the underlying coordinate system, it has a magnitude, which is independent of underlying coordinate system. That means the magnitude of grad T is equal to partial T by partial x 1, the whole square plus square the half power. This is independent of coordinate systems used and has a direction also, that is independent of coordinate systems used, I had given you a physical interpretation of grad T.

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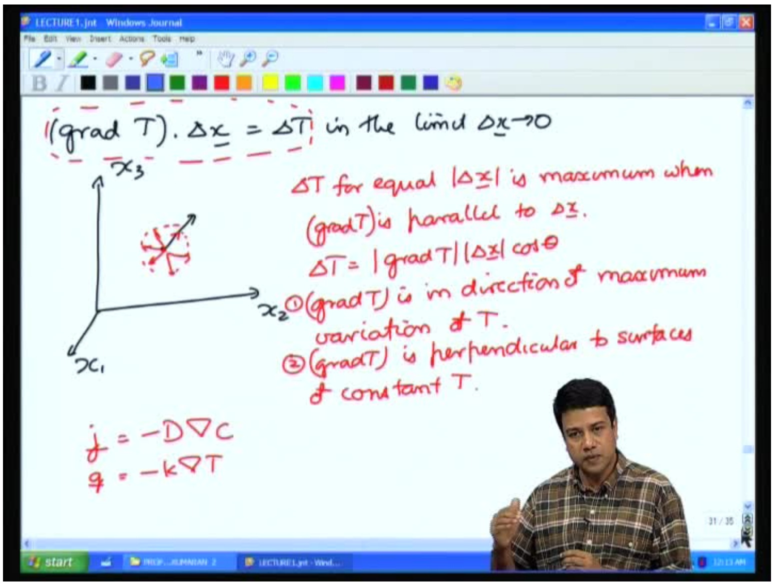

If I sit at 1 particular location and I move a small distance, the distance is the same the direction is different. So, I move a small distance of equal length in different directions and I look at the change in temperature, change in temperature is grad T dotted with delta x. So, that change in temperature is going to be a largest were, grad T and delta x are in the same direction. What that means, is that grad T is in the direction of maximum variation of T and if I move in the direction that is perpendicular to grad T, grad T itself is a vector at a given location.

If I move in a direction that is perpendicular to grad T then grad T dot delta x is equal to 0, because cos theta the angle between them is 90 therefore, cos theta is equal to 0. So, grad T dotted with delta x is equal to 0, we should move in a direction perpendicular to grad T, that means if the gradient of a scalar field is directed in the direction of maximum variation of that scalar field. It is perpendicular to surfaces of constant value of that field. So, in order to identify surfaces of constant temperature, I just need to locate the gradient of that temperature and then look perpendicular to that perpendicular to the gradient at every point in the surface of constant temperature.

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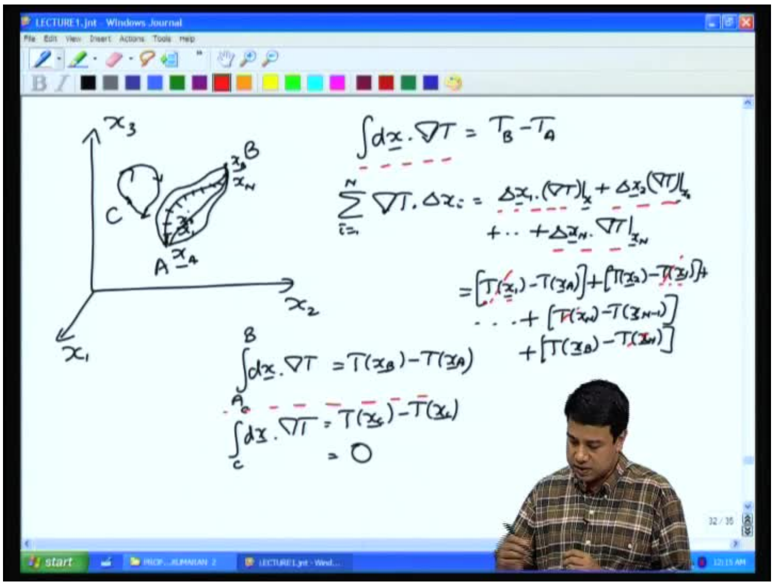

We are also derived the integral relation corresponding to grad T and that is that, if I take grad T integrated with a differential displacement and then integrated from 1 initial to a final position, that is equal to the difference in temperature between that initial and final location. This we had seen by dividing this entire path between these initial and final positions into small differential intervals and then summing up grad T dot delta x, on each of these intervals. And obviously, the temperature at at at each point along that path will cancel out, because it has a positive sign in on one side and negative side on the other side, you just left with the temperature difference between the initial and final locations. So, that gives us the integral relation for grad T.

The integral between 2 locations a and b of the differential displacement d x dotted with grad T was equal to that difference in temperature between the initial and final locations. This has consequences of corollaries, if I take different paths between that same 2 end points integral of d x dot grad T has to be the same on all of those paths. If I take different paths between the same 2 end points, this integral has to be exactly the same on all of those paths. Secondly, if I start from one location come back to the exact same location, this integral has to be 0, d x dot grad T, the integral has to be 0, if you start from a given location and come back to the exact same location.

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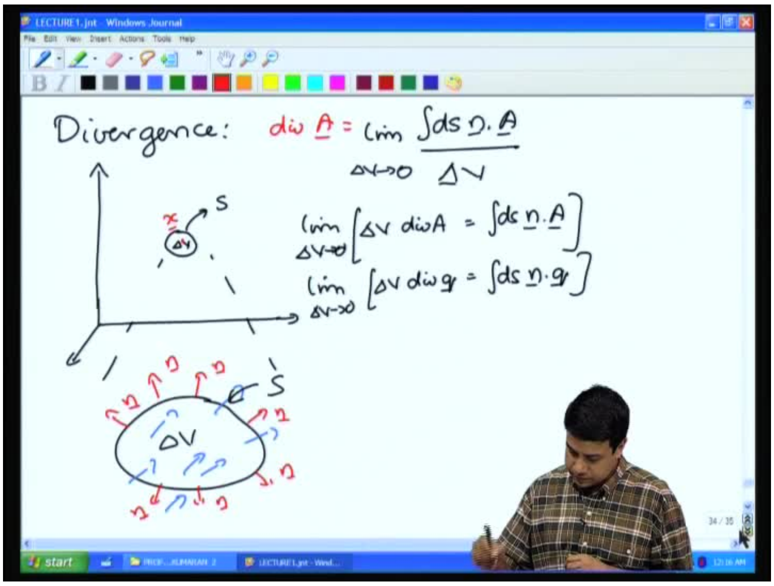

Our next derivative was the divergence divergence acts on A vector in general and we had defined it as if I want the divergence at A given location x vector, I go to that point x and I construct A small differential volume delta V around this point. I will expand it out this volume delta V for you delta V has A surface S each point along the surface, you can define the unit normal n, that is shown by the red in this figure. At each point you define the outward unit normal, there you get outward from this volume.

You take the dot product on the surface of the unit normal times, the vector A itself as I said vector A is A single valued function at each point on that volume. So, you take the unit normal dotted with that vector A, itself integrate over the entire surface and divide by the volume in that case, you get divergence. Since, I have taken n dot A the result that, I get is A scalar, I I have taken n dot A integrate over the surface divided by the volume. So, the divergence will also have dimensions of this vector divided by distance

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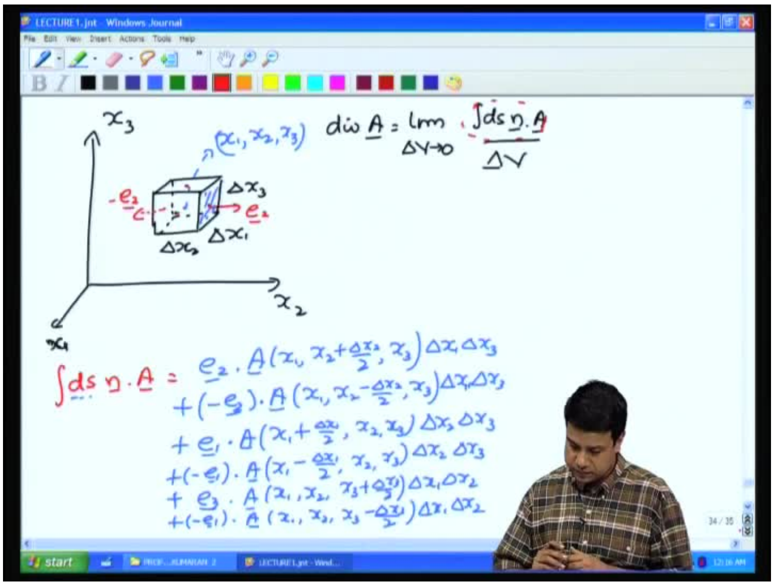

And I give you a physical interpretation of divergence in terms of in terms of.

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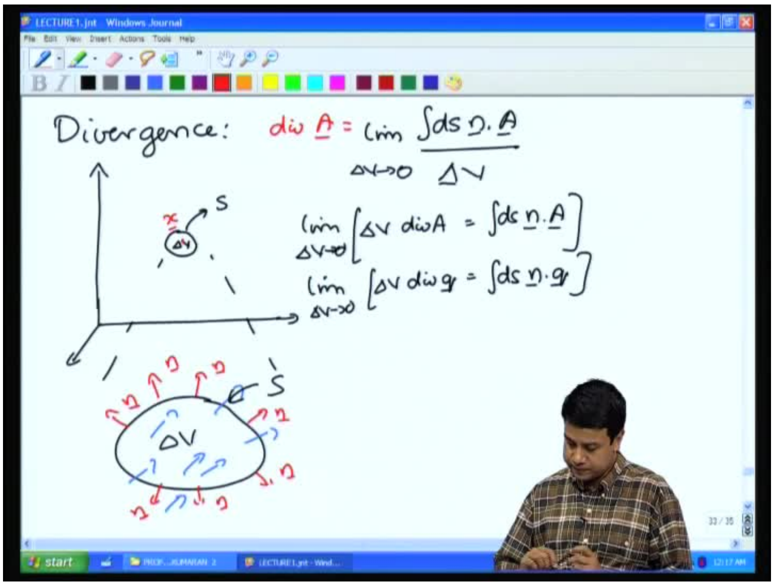

The heat flux or the mass flux integral d s n dot q is the total amount of heat coming out of the surface per unit time. So, for this differential volume, the total amount of heat coming out of the surface per unit time is equal to the volume times the divergence of q. So, where in this case the divergence has dimensions of q divided by distance.

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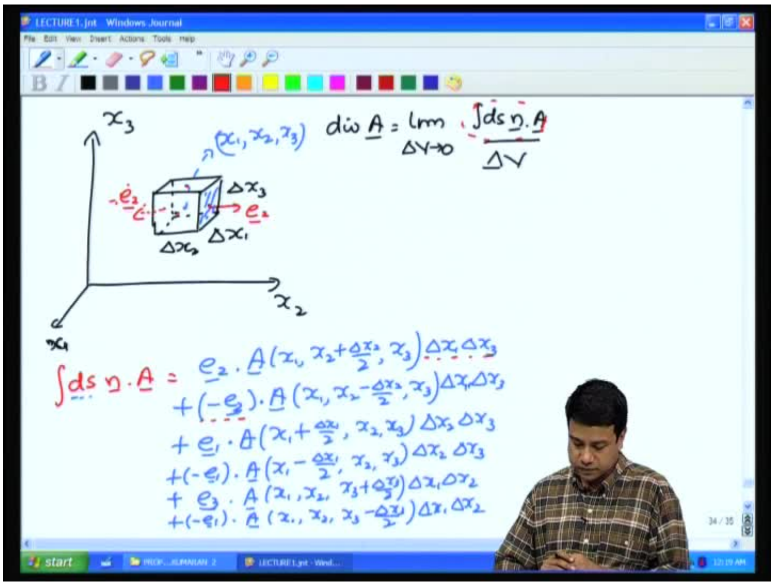

We had calculated the divergence in A cartesian coordinate system limit as delta V goes to 0 integral d s n dot A divided by delta V. So, in this cartesian coordinate system, we construct a differential volume with surfaces along perpendicular to lines of constant coordinator. So, you have 2 surfaces perpendicular to the x 1 direction, the front and the back 2 surfaces perpendicular to the x 2 direction right and left 2 surfaces perpendicular to the x 3 direction up and top and bottom and you have to take, this d s n dot A on each of these surfaces.

So, on the surface on the right n is the outward unit normal, it is in the plus e 2 direction and since my my my volume is center at x 1 x 2 x 3 surface on the right is at x 1 x 2 plus delta x 1 by 2 and x 3. So, taking e 2 dotted with A vector at the location x 1 x 2 plus delta x 2 by 2 x 3 multiplied by the surface area. In this case the surface area is delta x 1 times delta x 3. So, that is on the right side, on the left side the unit vector is in the minus e 2 direction, because it is outward to the surface, the unit vector in the minus e 2 direction. Because, it is outward to the surface dotted with A at x 1 x 2 minus delta x 2 by 2 and x 3 multiplied by the surface area once again, we do this for all surfaces right left top bottom and front and back.

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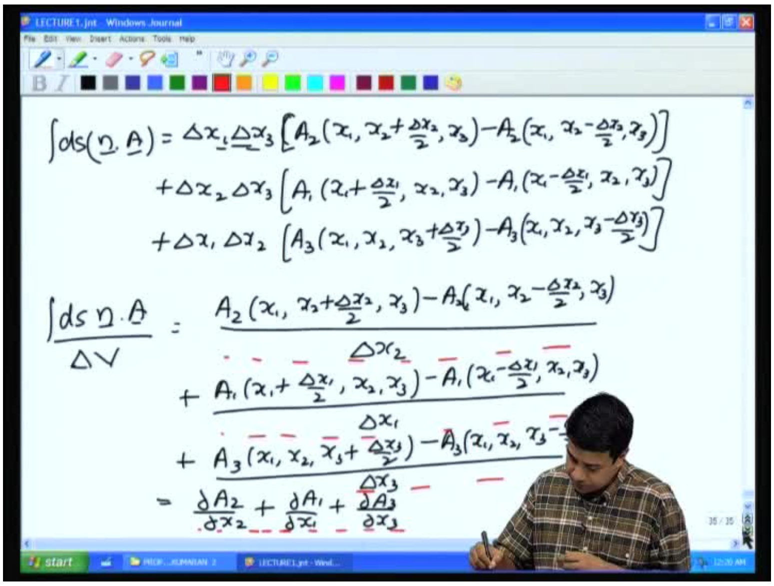

And so I get a total of 6 terms here, 3 of which are positive, 3 of which are negative and then I have to divide by the area and when I divide by the area, I get A 2 at x 1 x 2 plus delta x 2 by 2 x 3 minus A 2 at x 1 x 2 minus delta x 2 by 2 x 3 divided by delta x 2. Similarly, in the x 1 direction and in the x 3 direction and finally, the final expression for the divergence turned out to be acquire A simple 1, partial A 1 by partial x 1 plus partial A 2 by partial x 2 plus partial A 3 by partial x 3, In the cartesian coordinate system.

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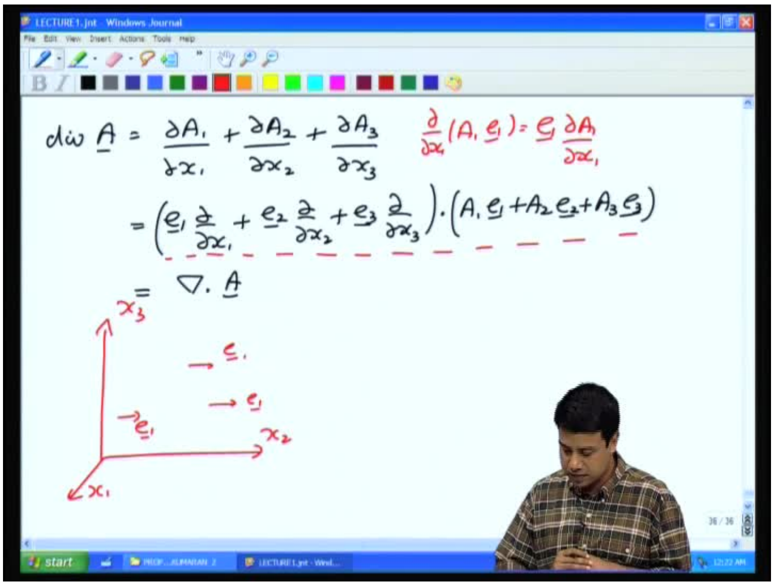

So, let us write that out divergence of A vector is equal to I can also write this as e 1 partial by partial x 1 plus e 2 partial by partial x 2 plus e 3 dotted with A 1 e 1 plus A 2 dotted with A 1 e 1 plus A 2 e 2 plus A 3 e 3. So, this basically is equal to gradient dotted with this A vector. So, that is why the divergence is often written as del dot A, note that this, I had done quite simples, because when I take the derivatives partial by partial x 1 of A 1 e 1, I have assumed here, that partial by partial x 1 of A 1 e 1 is equal to e 1 times partial A 1 by partial x 1, that was the assumption, when I expand it out the derivative. This is 2 for a cartesian coordinate system, when I have a cartesian coordinate system, this unit vector e 1 does not change as I go from one location to the other.

It is exactly the same at every location, the magnitude is the same the direction is the same, therefore the derivative of e 1 with respect to x 1 is identically, equal to 0, see it is quite a simple point, but as it turns out when we go to other coordinate systems. This is no longer true, because the unit vector is actually, do depend upon position and that is something that, we will be careful about while, we extend, this 2 other coordinate systems. So, there is a divergence the definition of divergence is formula delta dot A and what is the integral relation for the divergence.

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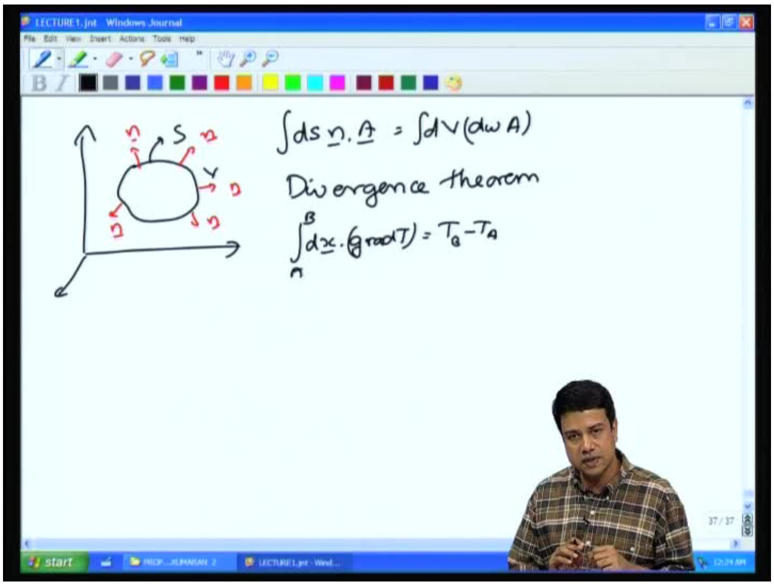

So, this is the derivative the equivalent integral relation, because that for any volume V with A surface S for any volume V with A surface S integral over the surface of n dot A, we get the integral of the volume of divergence of A, where n is the outward near component of the surface. Seems very similar to the formula that, I had for the definition of the divergence, except that this is not in the limit as delta V equals to 0. This is at for any volume, which is A closed volume, which is bounded by A closed surface for any volume integral over the surface of n dot A is equal to integral over the volume of divergence of A.

So, this is the integral theorem, this is called the divergence theorem note that, it relates the integral over the surface to an integral over of volume that is the integral over of volume integral is related to A surface integral. If, you recall the integral relation for the gradient was defined as integral d x dot grad T is equal to T b minus T A. So, this basically related the integral along the line, to the difference in the end points. So, this therefore, the gradient the integral theorem related the integral along A line to the difference at the end points where, as further the the the divergence it relates A surface integral to a volume integral. So, let us try to explain, how this comes about.

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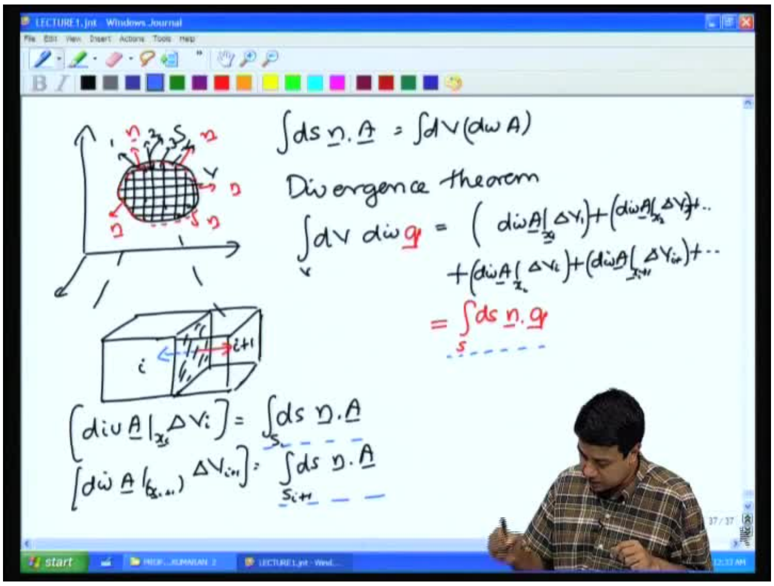

So, let us first take integral over the volume of divergence of A, you integral over the volume, this volume V of the divergence of A. So, how do I calculate this integral over this volume, what I can do is to divide, this volume into a small large number of small differential volumes, I will divide this into a small number of differential volumes, I call this this as volume 1 2 3 etcetera. So, this is volume 1, this is volume 2, this is volume 3, volume 4, I can write this as summation over all of these differential volumes of divergence of A at the location x 1 times delta V 1 plus divergence of A at x 2 times delta V 2 plus etcetera plus divergence of A at x i, times delta V i plus divergence of A at x i plus 1 times delta V i plus 1 plus etcetera.

So, I divide this into small differential volumes and you take the divergence of A at that location times, the volume itself and I sum it over all of those volumes. So, let us take 2 2 typical volumes, I will just expand it out, this is volume i, the next 1 is the next differential volume, this is volume i plus 1. So, if I take divergence of A at x i times delta V i, for this, I can now apply the divergence theorem, because this is now a differential volume for this, I can now apply the divergence theorem, because this is now A differential volume.

So for this differential volume this divergence of A times delta V i over this differential volume can be written as integral over the surface of n dot A, over the surface S i, of n dot A for the adjacent volume divergence of A at x i plus 1 1 times delta V i plus 1 is equal to the integral over the adjusting surfaces i plus 1 d s n dot A where, n is the outward unit normal t is the vector at that particular location on that surface. So, let us look at these 2 adjacent volumes. So, for the volume i on this surface, they have one common surface and the common surface is this 1, this is the common surface between the 2. For the volume i, the outward unit normal points in this direction, it is outward to the surface and that the integral of n dot A outward unit normal dotted with A on this surface for the volume i.

If I go to the next adjacent volume i plus 1, if I go to the next adjacent volume i plus 1 at this common surface the outward unit normal to the volume i plus 1 actually points in the opposite direction. Value of A is the same for both volumes i and i plus 1, because this is the same surface where, the surface for the volume i, the outward unit normal is the right direction for the volume i plus 1, the outward unit normal is in the left direction.

So, on this common surface, this integral d s n dot A, for these 2, it is equivalent magnitude, because A is the same unit normals are in opposite direction that means, that the value of this surface integral contribution to the divergence of A dot del V cancels out on, this common surface. Because, it is between 2 volumes, the value of the function itself is the same on that volume unit normals are in opposite direction therefore, it cancels out.

All surfaces integrals and surfaces, which are in between 2 differential volumes will also cancel out for that same reason, unit normals are opposite, the value of the functions is same. So, the only surface integrals that, I will be left over are those that are not between 2 differential volumes that is the surface, there is the outer surface to this entire volume the outer surface to the entire volume, this is the only surface that will be left out. So, this becomes equal to integral over the outer surface of d s n dot A. So, integral over the outer surface of d s times n dot A.

So, this is the divergence theorem, for divergence the integral relation for divergence just as I had the differential relation for divergence earlier the integral relation for divergence is this 1. It relates the volume integral to A surface integral. Physically, if this for example, was the heat flux or the mass flux, if at I had here was the heat flux or the mass flux let us say heat flux q.

This integral d s of n dot q integral d s of n dot q is the total heat coming out of this volume per unit time, because n dot q is the component of the flux along the outward unit normal and n dot q is the amount of heat coming out per unit surface area per unit time. That means, that integral identical d s n dot q is the total heat coming out per unit area, per unit time, that is equal to the volume integral of the divergence of q throughout this entire volume.

So, what this is saying is that, this volume integral divergence of q integrate over the entire volume. This integral does not depend upon the values of this vector within the volume, you can find out the value of this integral just b knowing, the values of this vector along the surface alone. So, that is that is the that is the physical implication of this particular statement that the value of this divergence of q integrated over A volume does not depend upon, the value of q within that volume it depends only upon value of q on the surface.

So, that is the physical implication of this particular statement. So, that is for the divergence the definition of divergence in A cartesian coordinate system x equal to gradient del dot A, it is equal to del dot A in all coordinate systems everything is that divergence operated self changes and you will see that a little later. And the integral relation for the divergence that is integral over the volume of d V times divergence of A vector is equal to integral surface of n dot the vector.

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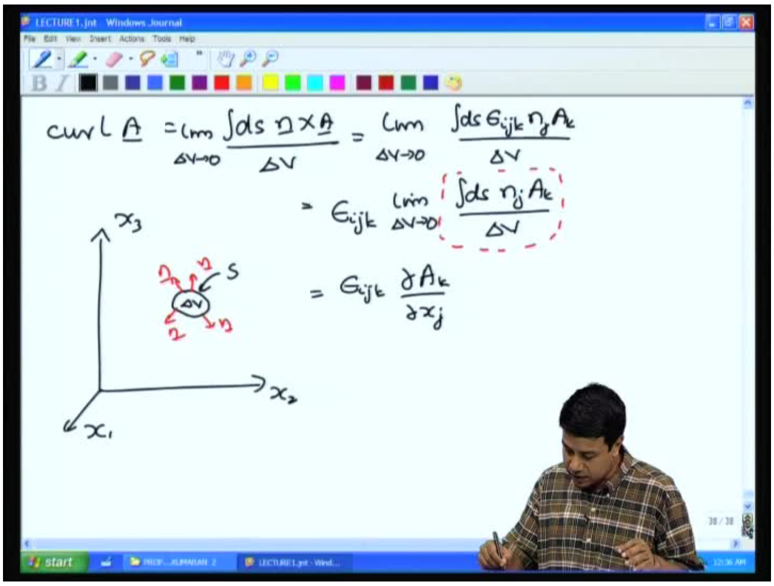

So, the third thing, the third differential quantity that, we shall be dealing with is curl, once again, we take a small volume V, we will call it as delta V with A surface S and with unit normal. The curl of A is defined as integral over the surface of n cross A divided by delta V limit delta V goes to 0, it is defined as integral over the surface n cross A divided by delta V in the limit as delta V goes to 0. Obviously, I have taken the cross product of 2 vectors. So, curl and something A vector, because I have taken the cross product of 2 vectors curl and something A vector and I have on topper surface integral of n cross A divided by the volume. So, once again, it has dimensions of A divided by length. Now, once again, if I wanted to calculate the value of this curl in the catresian coordinate system, I could do the same integral over the surface as that, I done for calculating the value of the divergence.

However it is not really necessary to do that because I have already given, you the way of expressing the cross product as 2 dot products, I already given you in expressing the cross product between 2 vectors as A product of 2 dot products. So if I go to write this in cartesain coordinate system, you get limit as delta V goes to 0 integral d s epsilon i j k n j A k by delta V. This anti-symmetric tensor of course is a constant, it is independent of position, so my anti-symmetric tensor does not depend upon position.

Therefore, I can write this as I can take it out of there, take the anti-symmetric tensor out and I will get epsilon i j k limit as delta V goes to 0 integral d s n j A k by delta V. And of course, this thing I had already evaluated for you in the context of divergence, this I already evaluated for you in the context of divergence and you know that this is equal to epsilon i j k partial A k by partial x j. So, this is equal to epsilon i j k times n j A k, if I integrate over the surface, I will do the same calculation that, I have done earlier.

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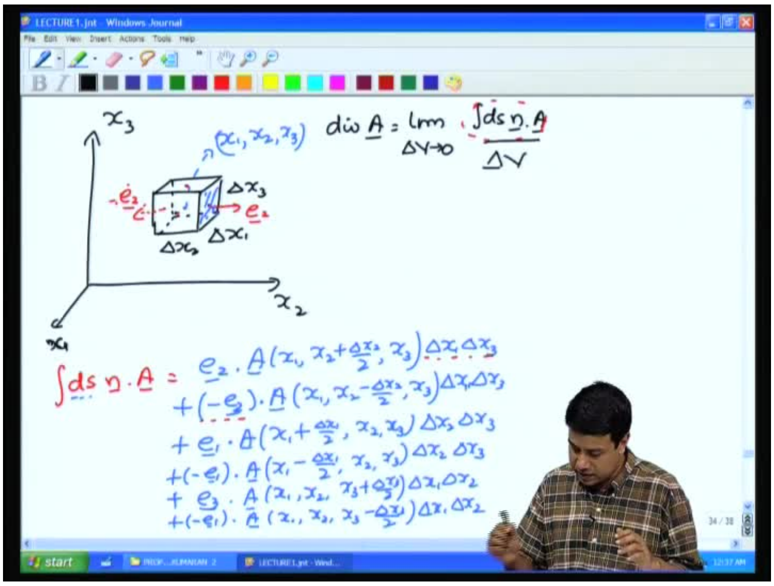

For this cartesian coordinate system n will have one index, A will have the other index and I will take the difference on these 6 surfaces. The unit normal the outward unit normal points in opposite directions on the 2 opposite phases and because that I will get the difference in d A between those 2 opposite phases.

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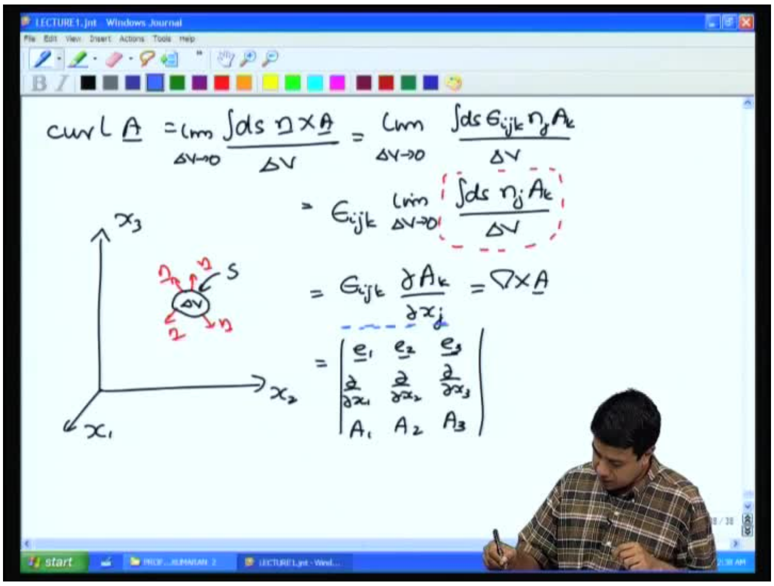

And that will translate to A derivative and so this is just equal to epsilon i j k times partial A k by partial x j. I t can also be written as del cross A, now use to the cross product as in the matrix form e 1 e 2 e 3 d by d x 1 d by d x 2 d by d x 3 A 1 A 2 A 3 that is del cross A and you will get exactly the same result, if you use this format. So, that is the derivative form of the curl, curl of a vector gives, you another vector is equal to n cross A integral over the surface divided by delta V and for a cartesian coordinate system.

If my surface, if I if i take a cubic volume, this can just be written as epsilon i j k partial A k by partial x j for del cross A. Note that in this particular relationship, there is one unrepeated index, that is i the other 2 are repeated that means, that there is only one unit vector and therefore, the result that, I get is A vector. So, this is nothing but the gradient of a dot dotted 2 times with this anti-symmetric tensor.

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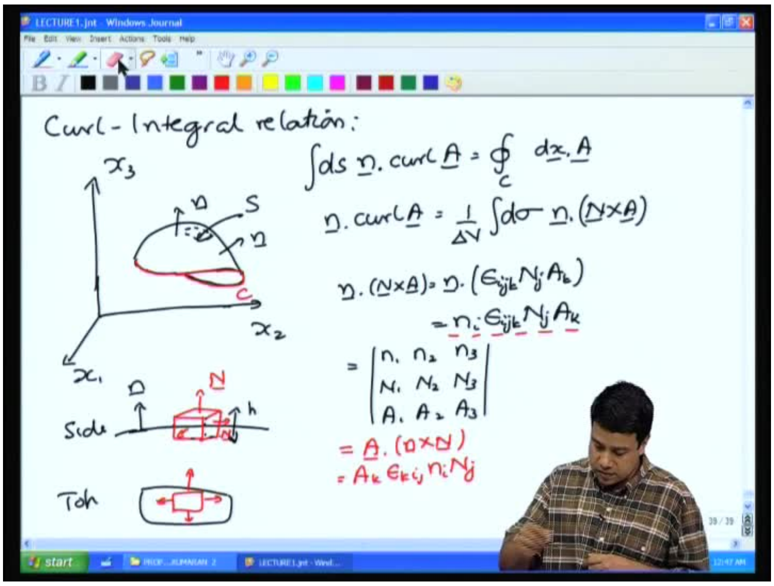

What is the integral relation, if you recall for the gradient the integral relation, related the integral along the line to the difference between the end points. For the divergence, the integral relation related the integral over a surface to the integral over a volume for the curl the integral relation will relate the integral over a surface to the integral over a line. So, that is the integral relation of the curl. So, let us let us look at that so if I have some open surface have some open surface with some perimeter. So, this this is the perimeter C of the surface and this thing this thing is the surface S. So, this is an open surfaces S with a perimeter C with a unit normal n at various points along the surface and what the integral theorem states that integral over the surface of the unit normal dotted with the curl of A vector is equal to integral over.

This closed loop C of d x dotted with A integral of n dot curl A over let, I take C here integral of n dot curl A over this entire surface over, this entire surface is equal to integral over the closed loop of d x dotted with A. So, that is the integral theorem for curl, now how do we prove this. So, if you recall the definition of the curl related in a surface integral and a volume, the curl as defined was a surface integral divided by the volume.

So, we cannot prove it straight away, you cannot straight away relate a line and a surface integral, what you need to do is to construct a small volume around the surface. So, let us take a little patched surface here, let us take a little patched surface. So, if I look side on from this surface, I will see this as the surface and I can also look from the top. So, this little patched surface, if I look at the top, it will look something like this that is a little patched surface, if I look from the top and from the side. So, this is side and this is top view.

So, I construct a small little cubic volume around this surface, because the volume actually, started as a surface it is both above and below the surface, it has a certain height h, I will remove the surface little bit. So, this is the surface and this volume is a height h and there is a unit normal to this surface n. So, if I take the volume on to the bottom here, it will look something like this, there is a volume looking at from the top and from side view, it will be both above and below the surface.

So, I have a unit normal n to the surface itself, in my definition of curl, I have to take n cross A, in my definition of curl, I have to take n cross A for the differential volume. So, I have to define a unit normal to the differential volume also, in this case I have to define a differential volume for in different an outward unit normal for this differential volume itself, that outward unit normal is directed in different directions at different surfaces. So, I will call that outward unit normal as capital N.

So, it is it is right and left front and back and above and below it is directed in different directions, this outward unit normal for the volume is capital N. So, for this little differential volume, if I take note that small n is the unit normal to the surface it is directed only in one direction to the surface. Capital N is the outward unit normal to the cubic volume that, I have constructed around the surface capital N is the outward unit normal to a cubic volume that is constructed around the surface.

So, if I would look from the top, I will have unit normals in all of these directions as well as one coming out of the boat, the other one going into the boat. So, this is these are unit normals to the top and bottom surfaces. So, if I take this n dot curl an dot curl a is equal to one over for this differential volume 1 by delta V integral over all the surfaces let me call this differential volume surface as sigma in order to not confuse at with the surface S that, I already have.

Sigma represents the surfaces of this cubic differential volume that I have constructed sigma of n dot curl A of n dotted with curl A was capital N cross A, n dot curl A was capital N cross A. This n dot n cross A, we are probably familiar with it is what is called the triple product. So, n dot n cross A is equal to I am taking the dot product of this times the curl epsilon i j k n j A k and when, I take the dot product, the indices are repeated. So, this just becomes equal to n i epsilon i j k n j A k, it is a good place to now introduce the triple product.

So, this you are familiar with this can be written as n 1n 2 n 3 N 1 N 2 N 3 a 1 a 2 a 3 that is the matrix form in indicial notation, we will just write it in this particular fashion. However, you know that n dot N cross A right, you can interchange the order, because you can, if you interchange 2 rows, you will get back the same matrix. So, I can also write this as A dot n cross, I can also write this as A dot n cross N, because this is a triple product and the same thing will get out of here.

So, this will be equal to A k epsilon k i j n i n j when I go from epsilon i j k to epsilon k i j, I have interchanges indices 2 times, first I interchange i and j then I interchanged j and k. So, because of that, I get back the same result. So, the triple product, if you go in a cyclical fashion it remains unchanged good. So, this is the triple product and let us say that, I can write this is as a dot n cross N. So, let us use all of this.

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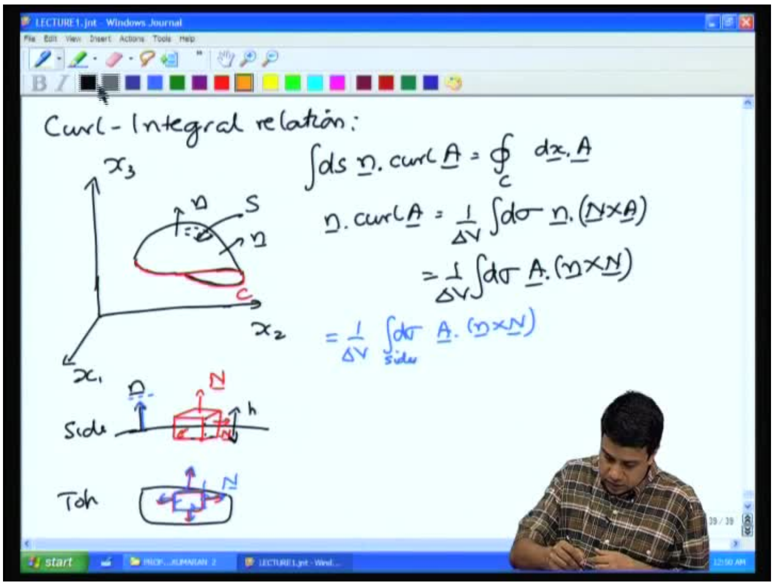

So, this I can write it as 1 by delta V integral d sigma of A dot n cross N. So, what is this A dot n cross N, n is this vector, it always points upward, n is this vector, it always points upward. Capital N is the vector that is perpendicular to the surface on the top and bottom it is upward and downward on the front and back it is pointing out and into the in the right and left it is to the right and the left.

Whenever it points the same direction as small n, n cross n is equal to 0, because if the 2 are direction is the same direction the cross product of 2 parallel vectors has to be equal to 0. So, the integral over the top and bottom surfaces is identically equal to 0 what 1 is left with is the integral over the sides. So, this is also equal to 1 by delta V integral d sigma over the side alone of A dot n cross N.

And what is the integral over the sides over the sides, the outward unit normal n is pointing in this direction whereas, the outward unit normal capital N is actually pointing in these directions. So, if I take n cross N, I get A vector, which is perpendicular both to small n as well as capital N that means, that integral has to go along this path that means, the integral has to go along the perimeter of the intersection between this volume and the surface, it has to go along the perimeter of the intersection between this volume and the surface. So, that vector along the perimeter, I will call it, as you can use a different symbol here. So, this is my intersection of this differential volume on the surface, this was this was capital N.

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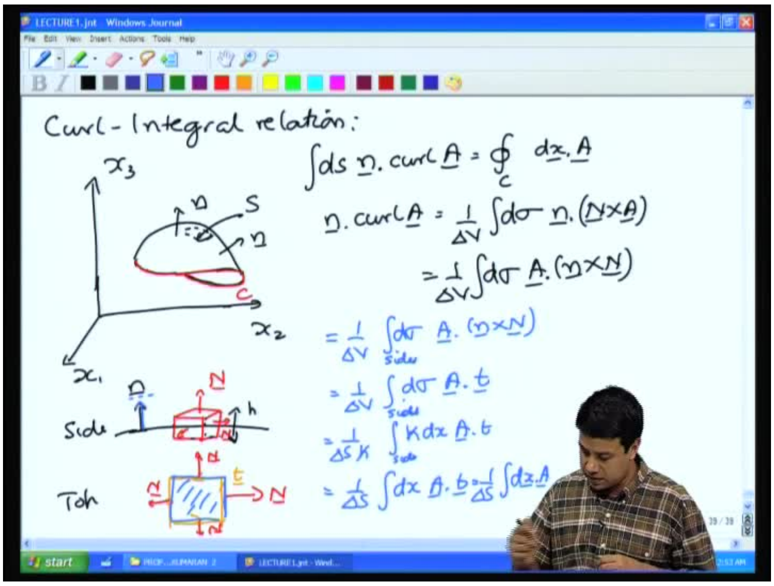

Small n is coming out of the board at me small n is coming out of the board at me capital N is in this direction. So, if I take n cross n and I use A right handed coordinate system, I get a vector that is pointing in this direction over here on this direction, you will A vector is pointing in this direction, because n is this way over here, it will be coming in this direction. Because, capital N is this way and at the bottom n is taken downwards.

So, I will get a vector n cross n, which is in this direction, so this I will call the tangent vector to this contour intersection between the volume and the surface. So, this thing becomes 1 over delta V integral over the sides of d sigma of a dotted with this tangent vector. A tangent vector along the 4 sides directed in the direction of the cross product between capital N and small n. In this particular case, since we are using the right handed coordinate system, we use the right hand rule.

So, I have integral only over the sides, the top and bottom integrals are 0, I have integral only of this sides, the surface area of the sides is equal to this contour length this contour length times the height h, is equal to the contour length times, the height h. The volume is equal to this patch of surface, this area of this patch of surface times, the height h the volume of this box is equal to this the volume of this patched surface times height h. The surface area of this surface is equal to this path length times the height.

So, I get 1 over delta S times h integral over the sides of h d x where, d x is small displacement along this path and h is the height perpendicular times A dot T and the 2 h’s will cancel out and I will get 1 over delta S integral. Now, since the h’s are cancelled out the integral is only over d x. So, it is integral over the length d x d x A dot t and this I can also write it as 1 over delta S integral d x vector dotted with A. I can also write this is 1 over delta S times d x vector dotted with A where d x, vector is equal to the tangent vector times the displacement.

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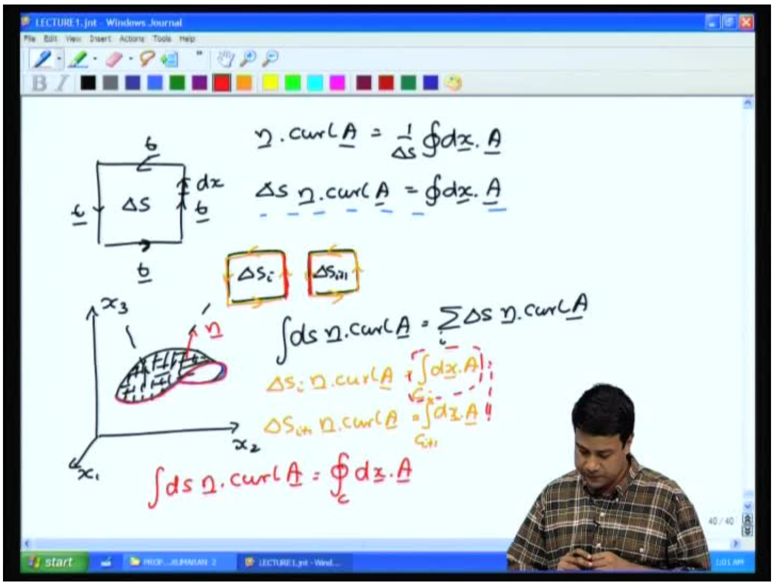

So, let us put that all together. So, I have this path, now I ground the surface, this is the surface delta S, this is the tangent vector at each point and this differential displacement is d x. So, this vector displacement is d x times T, which is just d x vector therefore, from m previous lecture from m previous page, I have n dot curl A is equal to integral of d x vector dotted with A 1 over delta S integral d x dot vector dot A. For this small little patch delta S that is to only for this small little patch delta S. Note that this is an integral over the entire path. So, this is a cyclical integral, which goes all the way from the beginning to the end. So, it is the integral over the entire path.

So, this is for each little patch on that surface of if you recall, we had this entire surface here and we have taken only a little patch on that surface. For that little patch, we showed that integral of d delta n dot curl A is equal to 1 over delta S times integral d x dot A. Alternatively, delta S n dot curl A is equal to integral d x dotted with A, but my original integral theorem for was for the entire surface, my original theorem was for the entire surface.

So, how do we get rid from this microscopic relation, once again, we use the same little bit of logic that, we used for calculating the divergence. So, I have this surface here with this perimeter surface S normal n and this perimeter C is the perimeter of the surface. Hence perimeter C is the perimeter of the surface, this relation that I have is for each little patch on that surface. So, what I can do is take this entire surface and cut it into small little patches cut into small little patches cut it into small little patches for each little patch, this thing works.

So, I take 2 adjacent patches here, let us say I take 2 adjacent patches i and i plus 1. So, I have 2 adjacent patches, delta S i and delta S i plus 1. I know that the total integral integral d s n dot curl A is equal to summation over all the patches i of delta S n dot curl A over all these patches. For this patch S i right, this integral of delta S n dot curl A over this patch S i is equal to is equal to integral d x dot A over the contour C i where, C i is this particular contour integral, over this contour delta S i plus 1 of n dot curl A is equal to integral over the patch C i plus 1 of d x dot A.

So, for this patch, you go in this direction, there is 1 line that is common between these 2 a piece of the contour, that is common between these 2 and that is this. The right side phase for S i, the left side phase for S i plus 1, they are identical I have just separated this out to show that they are 2 separate patches, but they have a common boundary between them. If, I take this integral delta x d x dot A, A is the same on that line it is a single valued function for the left side, I am going in an anti-clock wise direction around this path. So, I am going upward my x vector is upward for the left side, x vector is downward for the right side, because once again, I am in the according to the right hand rule, in the anti-clock wise direction.

So, the values of this contour integral of this contour integral on these on this 1 and this 1 are equal in magnitude and opposite in sign and therefore, they will exactly cancel out. That is true for all pieces of the contour that are in between any new any 2 adjacent surface patches all pieces of the contour that are in between 2 adjacent patches. The direction of the tangent vector will be opposite, if we use the correct right hand coordinate rule where, as the value of the vector itself is the same.

So, these 2 will cancel out on all adjacent patches and the only thing that, you will be left with is the integral over pieces of the contour that are not between 2 adjacent patches. What will be left with is pieces of the contour that are not between 2 adjacent patches and that is the only piece of the contour that is not between 2 adjacent patches, because this perimeter of this surface, the only piece of contour that is not between 2 adjacent patches is the perimeter of this surface.

Therefore, this proves that integral over the surface of n dot curl A is equal to integral over the perimeter of d x dot A exact same bit of logic that, we had used in the divergence in that case, we had looked at the surface integral between adjacent surfaces. Unit normals were in opposite directions and therefore, we cancelled out for all parts of the surface, which are between adjacent volumes, same thing here for all parts of the contour, which are between 2 adjacent surface patches.

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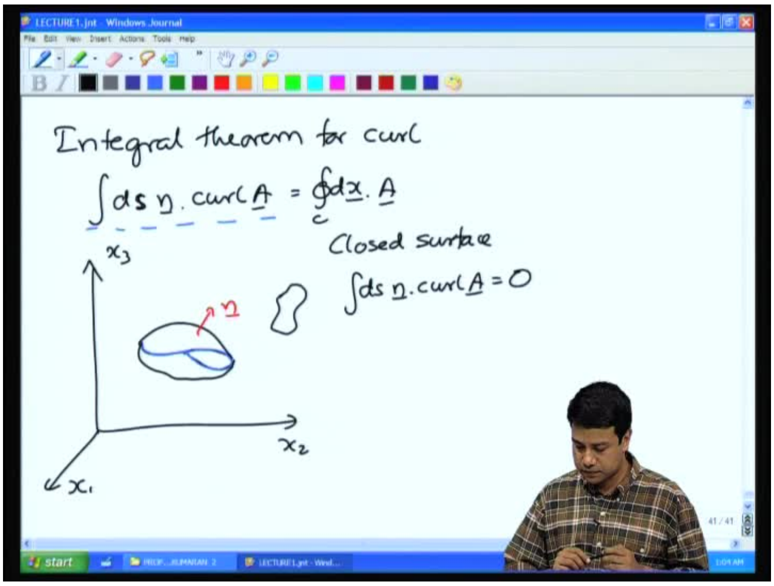

This is the integral theorem for curl integral d s n dot curl A is equal to integral d x vector of over the contour. So, this will a surface integral to a line integral and like the other theorems, it also has consequences. The first consequence is that this integral over the surface of n dot curl A, I have reduced it to an integral over the line of d x dot A that means, that this surface integral does not depend upon the value of this vector on the surface. It only depends upon the value of this vector on the perimeter itself, it does not depend upon the values of the vector at all points in the surface, it only depends upon the value of the vector on the perimeter of the surface, that is one important consequence. Second important consequence, I could have many surfaces, which have the perimeter just as in the case of the the the integral for the gradient, I had told you that, I could have many different paths between 2 points. In this particular case I could have many different surfaces, which have exactly the same perimeter.

One could be the surface, I could have some other surface, which is not the same as this one, which has exactly the same perimeter and the implication of this integral theorem is that n dot curl A over all of these surfaces has to be exactly the same. So, for an open surface with the perimeter any surface, which has the perimeter has to have exactly the same value of n dot curl A provided n is the outward unit normal and the curl is taken with the right hand rule. Second consequence, if I had a closed surface a closed surface closed surface has no perimeter. The perimeter of closed surface is basically a point, that means for a closed surface integral over the surface of n dot curl A has to be equal to 0 for any closed surface integral over the surface of d s n dot curl A has to be exactly equal to 0.

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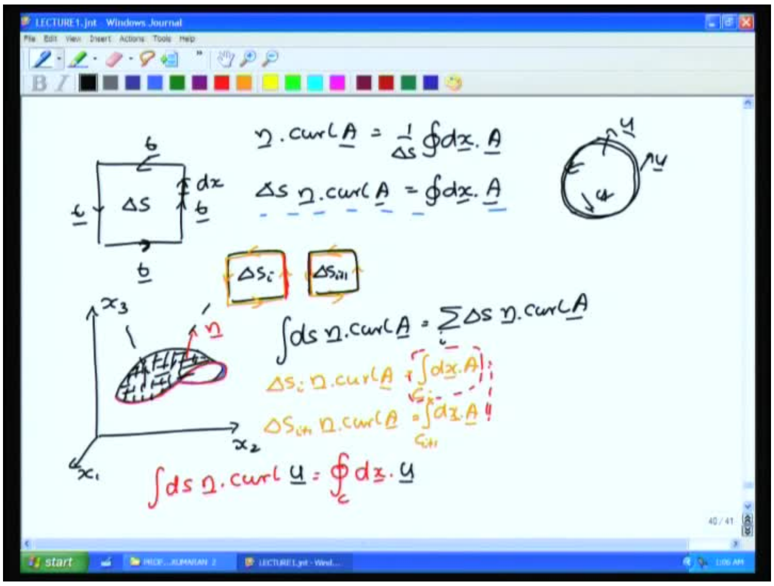

Physically what is n dot curl A, mean it is preciously would have done earlier. This integral over the d s of n dot curl A is basically, equal to the value of this function dotted with a tangent along the perimeter. So, in this case, if I, if a was for example, of velocity field, this integral d s n dot curl of the velocity is equal to integral d x dot the velocity displacement dotted with the velocity integrate over the entire perimeter. So, if you have a perimeter that looks like this, I have the velocity vector at various points pointing in various directions, I take a small patched surface integrate A tangent along the surface times A velocity, I go all the way around. It will tell me whether there is a circulation along this path or not, there is the velocity circulating around this path or not.

So, that is the physical implication of the curl. So we have a 3 fundamental elements of differential vector calculus each of these has an identity independent of a underlying coordinate system, it has an equivalent integral relationship and it has its own physical interpretation. So, when we do vector calculus, we will work in terms of these, we will not work in terms of derivatives in an in a specified coordinate system. So, in the next lecture, I will briefly discuss in this case, we derived it for a cartesian coordinate system, which was relatively simple. What happens, if you do not have a cartesian coordinate system, we often work in terms of spherical or cylindrical as we saw in fundamentals of transport processes one.

In that case how do you derive vectors or the gradient divergence and curl in those coordinate systems. So, I will briefly see that in the next lecture, because in those coordinate systems as I said the unit vectors themselves are functions of position. And we will look at how to define gradient divergence curl in those coordinate systems, before we proceed into look at fluid mechanics property.

So, I will briefly deal with these quantities in a non-cartesian coordinate system but still orthogonal that is the coordinates are not directed or the unit vectors do depend upon position. But, at each point in space, all 3 unit vectors are perpendicular to each other. So, its its an orthogonal coordinate system, we will see that briefly in the next lecture before proceeding to fluid mechanics. So, kindly revise what all was done in this class, because this forms the basis of what we will be using later on.