**Fundamentals of Transport Processes II**
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**Lecture - 39**
**Turbulence - Part 2**

So, welcome to lecture 39 of our course on fundamentals of transport processes. As you recall in the last lecture we were discussing turbulent flows. Many characteristics of turbulent flows; one is there primarily at high Reynolds number, because you require the non-linear terms in the equation to have multiple solutions for the conservation equations. As I said even at high Reynolds numbers the laminar flow is still a valid solution of the equations of motion. The only problem is that the laminar flow is unstable, where low Reynolds number the laminar flow is stable, which means that if I have a flow and if I put in a small disturbance that disturbance decays it damps out and you get back the laminar flow as time progress.

Whereas the Reynolds number increases that comes a transition Reynolds number, which depends upon the flow configuration for which if you put in a disturbance that disturbance does not decay, but amplifies. In other words any spontaneous disturbance and in nature there are always disturbances in nature, if you have a pipe flowing there is always, some sound wave some vibrations something or the other which will act as a disturbance for the velocity profile. That disturbance will spontaneously take the flow from the laminar state it will spontaneously grow up, because the flow is unstable and it will take the disturbance that the flow from the laminar state to a turbulent state. A highly irregular chaotic state only at high Reynolds numbers with large velocity fluctuations.

So, even though the flow is statistically steady that is in the limit as time progresses if you take an average over time. If your period of average in this larger than the frequency of the fluctuations, those averages will be independent of time. However, at a given location there will be a fluctuation in the velocity profile in all components of the velocity as a function of time and this irregularity is characteristic feature of the turbulent flow. In other words the fluctuations are not regular in the in the sense of being periodic oscillations with one particular frequency. But rather they are chaotic they are they are highly irregular oscillations which do not have a well-defined frequency characterizing them.

They are three dimensional, so even though I may have a two dimensional flow. For example; the flow in a channel is a two dimensional flow in which the mean velocity is only in the flow direction along the walls whereas, there where that the velocity gradient is only perpendicular to the walls. Even though the mean velocity may be two dimensional the fluctuating velocity is in general completely three dimensional. At each location you have velocity is in all three directions and these are fluctuating velocities. In the flow direction there will be a mean velocity, in other two directions the average of the fluctuating velocities will be 0. But the fluctuating velocities themselves will be non-zero never the less. Because, of these velocity fluctuations the amount of energy that is dissipated is much larger than what you expect for the laminar flow with nice straight extreme lines.

Because, you have these additional fluctuations, which are cause additional strain rate locally and additional dissipation, due to which it is a highly dissipative flow. Also the diffusion of mass momentum and energy takes places not a primarily not due to molecular diffusion. But rather due to the motion of these eddies, which are correlated parcels of fluid of various length scale, which are transported across the fluid, resulting in enhanced transport of mass momentum and energy. For this reason they are highly diffusive and also highly dissipative, because the momentum transport rate is much larger than what you expect for a laminar flow. And for that reason the flow is the friction factor for example; in a pipe or a channel turbulent flow is much larger, sometimes orders of magnitude larger than that in a laminar flow.

So, last class we were trying to look at how we would analyze this highly fluctuating velocity profile. As I said one way to analyze it is to separate the velocity into a mean velocity plus a velocity fluctuation and then try and obtain equations for the mean velocity alone. If you could obtain an equation for the mean velocity alone then one could solve that equation and find out the velocity profile throughout the flow. As we saw in the last lecture it is not possible to obtain an equation for the mean velocity alone.

Because, when you write an equation for the… you take the momentum conservation equation and average it over time you do get terms, which depend up on the velocity fluctuations. That is because as I said there is momentum transport taking place due to velocity fluctuations and that innovatively has to come in to the momentum conservation equation. Then we went and looked at how does the energy how is the energy in the flow produce that is at the large scales what produces the flow energy and how is that energy dissipative. And that let us to the discussion of the smallest scales in the flow to micro scales. So, let us briefly review that and then progress with how we would go about modeling these turbulent flows.

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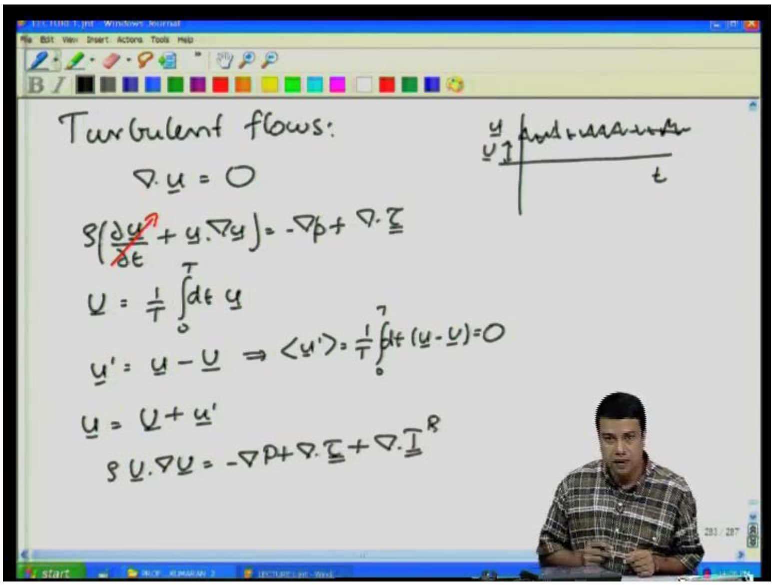


As I discussed in the last lecture turbulence is a continue phenomenon, it is not does not when the when the flow becomes turbulent it does not mean that the continues approximation is breaking down. The equations of motion are still the continued navier stokes equations, the solutions are such that the velocity is highly irregular. So, the starting equations are as usual the navier stokes mass momentum equations, the navier stokes and mass momentum equations. And since, the velocity is fluctuating function of time the velocity has some fluctuating function of time, an irregular fluctuating function. We can separate it out in to a time average mean velocity, this is the time average mean velocity, which I call with as capital U, the time average mean velocity which I call as capital U plus the fluctuating velocity.

So, the mean velocity is defined as 1 over T integral 0 to T d t of U vector, it is average to over a time period which, is long compare to the largest time scale of the velocity fluctuations. Usually the largest time scale of the velocity fluctuations will correspond to the mean strain rate at that location. And for our present argument discussion we are only looking at steady flows that is there is no time dependence of the velocity profile, so we just look at only steady flows. The fluctuating part is the local instantaneous fluid velocity minus the mean velocity capital U. Since, it is the instantaneous velocity minus the mean velocity, it implies that the average of u prime, which is 1 over T integral 0 to T d t of u minus U, this is equal to 0, the average of the fluctuating velocity has to be equal to 0, because I have already subtracted out the mean velocity.

So, this fluctuating velocity has 0 average and one would hope that I can derive an equation for the mean velocity alone by taking an average of the entire mass conservation equation, after substituting for the velocity in the pressure as the sum of the mean plus the fluctuating part. So, I substitute u is equal to capital U plus u prime vector in to the mass conservation equation and then I write down entire the equation. So, in the last lecture I had written this explicitly as del dot tau vector, where tau was the shear stress. You can do either one of the two whether you can take the either the stress or the velocity gradient explicitly.

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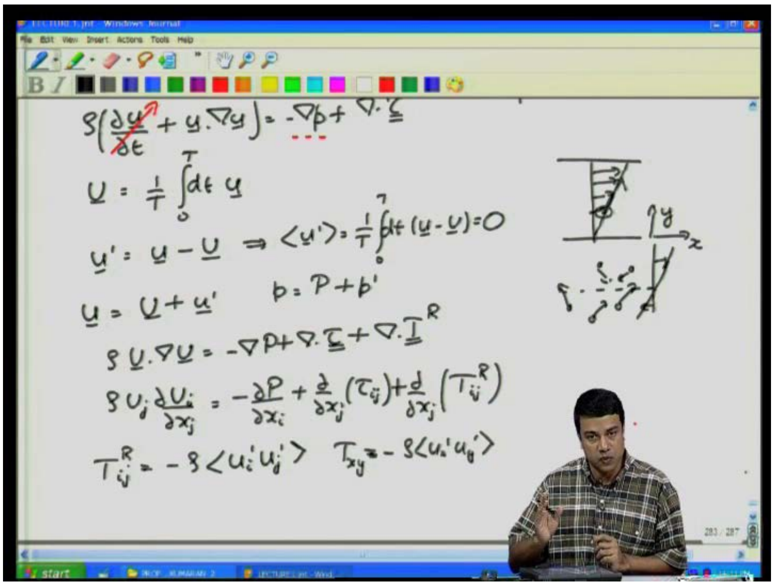

So, when we wrote this when we substituted this and took the average we found we got an equation of the form rho times U dot grad U is equal to minus grad P plus delta dot tau plus the divergence of a second term. Or if I write it in indicial notation you will get rho U j partial U i by partial x j is equal to minus partial P by partial x i plus partial by partial x j of tau i j plus partial by partial x j of T i j. This second term is what is called the Reynolds stress this is because of the velocity fluctuations. The Reynolds stress as you recall can be written as minus rho times the average of u i prime u j prime.

As I said the average of the velocity itself just for completeness I should also say that the pressure is written as the mean pressure plus the fluctuating part of the pressure, just for completeness the pressures written as a sum of a mean plus the fluctuating part. Now, I take the average of the entire momentum conservation equation I have the gradient of the pressure here, I am writing the time average I just get the gradient of the mean pressure over. So, this is the additional contribution to the stress due to the turbulent velocity fluctuations. Note that it is an inertia term it is minus rho times u i prime u j prime. So, it is effectively an inertia term, it is a convective transport of momentum due to the velocity fluctuations.

Obviously, for example; if I take the shear stress component of this I will get T x y is equal to minus rho times u x prime u i prime. If u x and u y prime the fluctuating velocities are completely uncorrelated, this would be identically equal to 0. So, if the if they were independent variables the Reynolds stress will be 0. The reason you have a Reynolds stress in a turbulent flow is because, these velocities are actually correlated, rather they are anti-correlated clearly, only if u x prime u i prime is negative will the Reynolds stress be positive for this particular flow. So, what it implies is that a parcel of fluid going in the plus u x direction has a lower velocity u i then the average, parcel of fluid going in the minus u x direction has a higher u i velocity than average.

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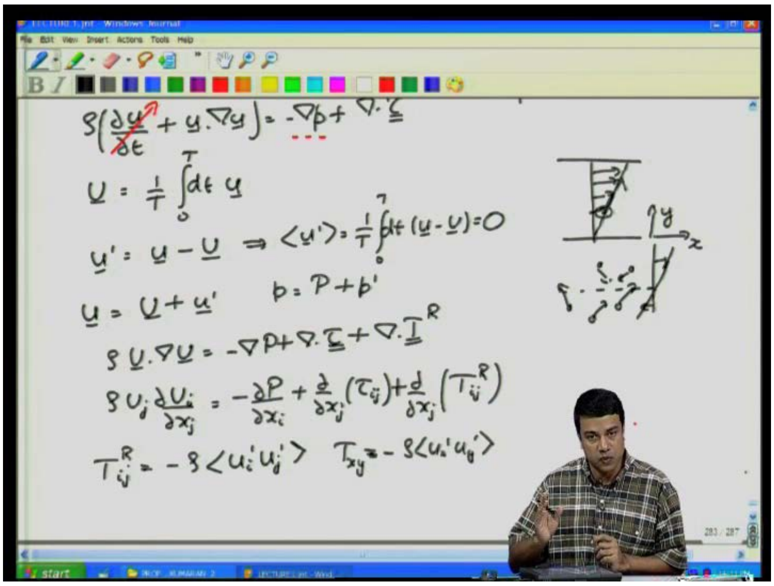

So, the idea is as follows if I have a shear flow, let us say I have a linear shear flow for simplicity. If it is a turbulent flow, let us take a coordinate system here x and y coordinate. If it is a turbulent flow momentum is going to get transported from one location to the other, only if a partial of fluid going in the plus x direction I am sorry, a partial of fluid going in the plus y direction has a negative u x component. So, that if a partial of fluid over here has a velocity that is lower than average, when it goes up it is going to take its momentum deficit along with it and its going to get speeded up by the flow above. Where is the parcel of flouring coming down with a minus u i velocity will have a higher velocity u x.

So, therefore as it comes down it will get slow down and therefore, it will transport momentum to the fluid that is below. If you recall when we did our discussion of diffusion molecular diffusion the argument was exactly the same. We took particular location across which there was a certain velocity profile and we said that molecules that are coming upward are coming on an average one mean free path below the surface. Molecules that are going down wards are going are coming from on average above the surface, those molecules are the molecules coming downwards bring additional momentum with them, whose to transfer it to the fluid below. Whereas, molecules going upwards deficit of momentum with to transfer it to the fluid above only then will you have momentum transport across the surface resulting inertia stress. So, this results inertia stress if u x prime and u i prime are anti-correlated for a positive strain rate.

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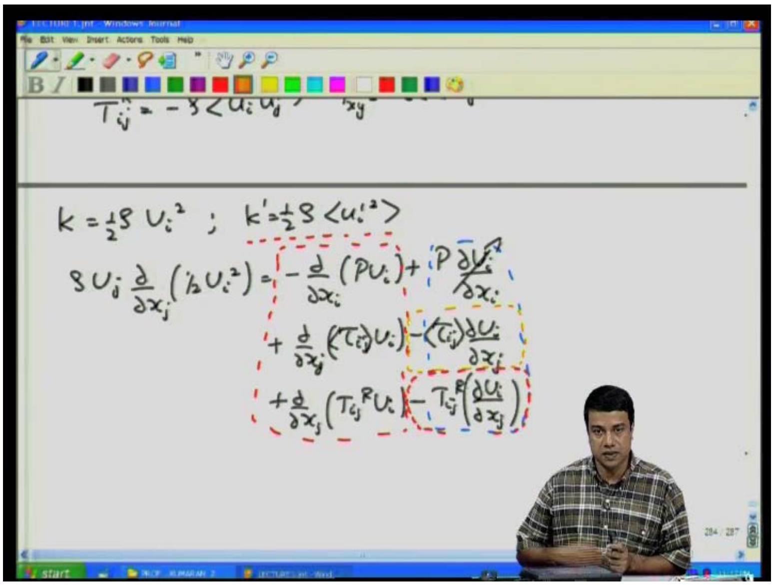


So, what is it the generates these fluctuations these fluctuating velocities is highly irregular fluctuating velocities. Obviously, these fluctuating velocities have associated with them fluctuating strain rates, these fluctuation fluctuating strain rates these variations in velocities are going to result in viscous dissipation. So, if I had a fluctuating velocity viscous dissipation would tend to slow it down, unless for some reason that is the source of energy which is replenishing the energy of these fluctuating of these velocity fluctuations.

So, we next looked at the mechanism by which, velocity fluctuations are generated by writing down equations for the mean kinetic energy and the fluctuating kinetic energy. The mean kinetic energy per unit volume is of course, equal to rho times I am sorry, the mean kinetic energy is equal to half rho U i square and the kinetic energy of fluctuations is equal to half rho average of u i prime square, the average of u i prime square. So, I try to write an equation for the mean kinetic energy and what we found was and so that is what the equation that we got in the previous lecture for the mean kinetic energy of the mean flow.

These terms here the term on the left of course, is the convicted derivative of the turbulent kinetic energy. That is the rate at which the kinetic energy is transported this is a steady state flow, so the time derivative is equal to 0. So, when the time derivate is equal to 0 the left hand side is just the convicted derivative, because there is no time variation of half u square. And the right hand side, the three terms on the left are basically the divergence of something integrates and favor to right this for a volume these three terms will be the divergence of something integrated over the fluid volume, which is basically the transport the transport the flux across the surface of the volume. So, this does not represent a net change in the energy of the fluid itself, but rather just the transport from one volume to the adjacent volume.

The other three terms are actually what represents the net rate of change of energy should be minus sign here that is important, they represent the net change in the energy. The first one the pressure times the divergence of the velocity is the work done, in this case if the flows incompressible that is identically equal to 0, if the flow is incompressible that is the divergence of the velocity is equal to 0. The second term is the viscous dissipation, the viscous stress times the means strain rate the dissipation of energy due to the mean flow. So, this is the mean strain rate this should be an average stress this is the stress this is not the fluctuating stress, but rather the average stress. So, this is an averaged equation, so this represents the mean strain rate.

The last term is the dissipation due to the Reynolds stress. So, this is not a viscous dissipation it is a dissipation due to the fluctuating energy of the molecules. The viscous dissipation of course, converts energy, mechanical energy into heat energy into thermal energy and increases the temperature. The Reynolds stress does not converted into thermal energy because it does not increase the fluctuating velocity of the molecules, it increases the fluid fluctuating velocity. So, it does not converted into Reynolds stress into I am sorry, does not converted in to heat.

So, where does that energy go? The energy thus dissipated due to the Reynolds stress, where does that go? That basically increases the kinetic energy of the fluctuations. For this fluid there is a kinetic energy of the fluid, which is equal to half rho into U square where U is the total fluid velocity. That kinetic energy we have separated into a kinetic energy for the mean flow capital U square plus a kinetic energy for the fluctuations u i prime square. You can easily see that if I take the square of capital U i plus u i prime, the cross term 2 U i times u i prime averaged will be 0, because the average of u i prime is equal to 0.

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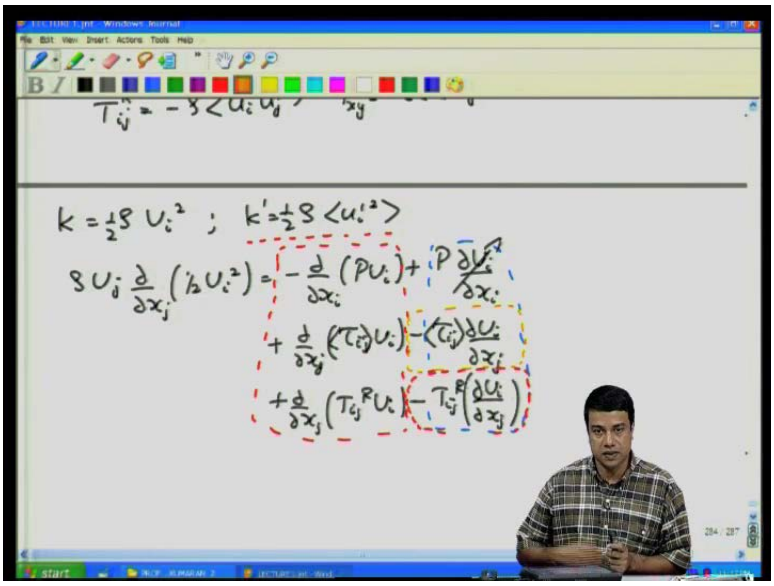

So, the kinetic energy separates out neatly in to two parts; one due to the mean velocity and the other due to the velocity fluctuations. This dissipation term due to Reynolds stress is converting the energy from the mean flow kinetic energy to the kinetic energy of the velocity fluctuations, in other words transferring energy from the mean flow to the fluctuations. As I said because these the dissipation in the stresses due to the Reynolds stress are much larger than that due to the molecular the viscous stresses.

Therefore, one would expect that this term the Reynolds stress dissipation term is actually much larger than the molecular, the viscous stress dissipation term. Because, as I said on a turbulent flow T i j Reynolds is much larger than the average of the molecular stress and the viscous stress. So, therefore this is the dominant mechanism which is taking energy out of the mean flow. So, where does that energy go? Now to see that one has to write a conservation equation for the kinetic energy of the fluctuations.

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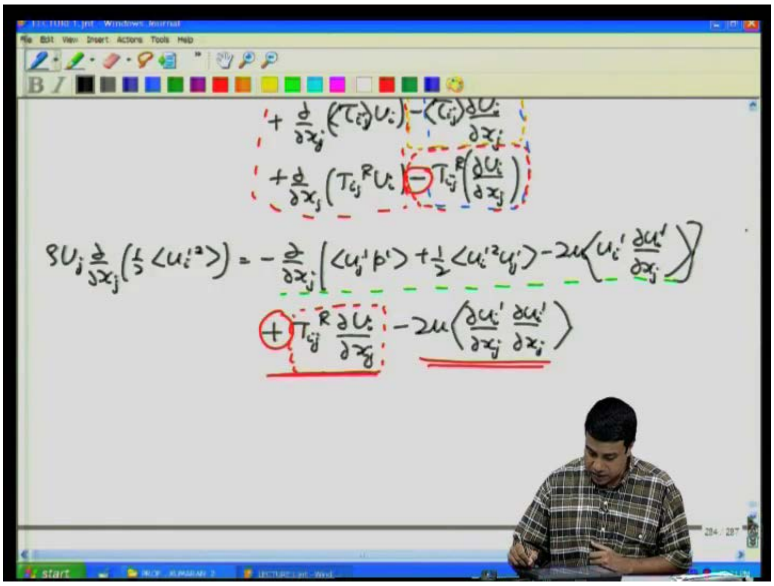

I do not go through the details of the calculation in the last lecture, but just wrote down the final equation, it becomes rho U j partial by partial X j of half u i prime square, the fluctuating velocity is equal to... So, that was the energy for the turbulent kinetic energy fluctuations. The first terms that you see here are of course, just the divergence of something and if you write that down for a surface for a particular volume this converts into a surface integral and just represents the exchange of energy between adjacent volumes. This term here appears with a negative sign, if the sign that is opposite to that in the mean kinetic energy equation. So, that acts as a transfer of energy from the mean flow to the fluctuations.

So, this basically acts as the source term in the equation for the fluctuating kinetic energy that is balanced by the dissipation due to the velocity fluctuations. So, this the source term the Reynolds stress transfer of energy from the mean flow to the fluctuations is balanced by the dissipation the viscous dissipation due to the fluctuations and this should be not a kinematic viscosity, but rather the dynamical viscosity itself. So, this acts as transfer of energy to the fluctuations, so energy is going from the mean flow to the fluctuations, the molecular viscous dissipation of energy in the mean flow equation is small. So, energy is going from the mean flow to the fluctuations and in the equation of fluctuations kinetic energy this energy that is coming down from the mean flow is getting dissipated due to molecular ah due to viscous dissipation.

So, that is the mechanism of dissipation in a turbulent flow, because the viscous dissipation of the fluctuations is much larger, than the viscous dissipation due to the mean flow. You get a friction coefficient or drag coefficient, which is much larger than what would you expect for a laminar flow in a turbulent velocity profile. So, next we looked at the question of where does the energy go, you have fluctuations across of course, a various length scales starting from the largest one's which are comparable to the system size. The channel width or the border layer thickness for examples, then you have smaller and smaller eddies of various dimensions down to the smallest scales.

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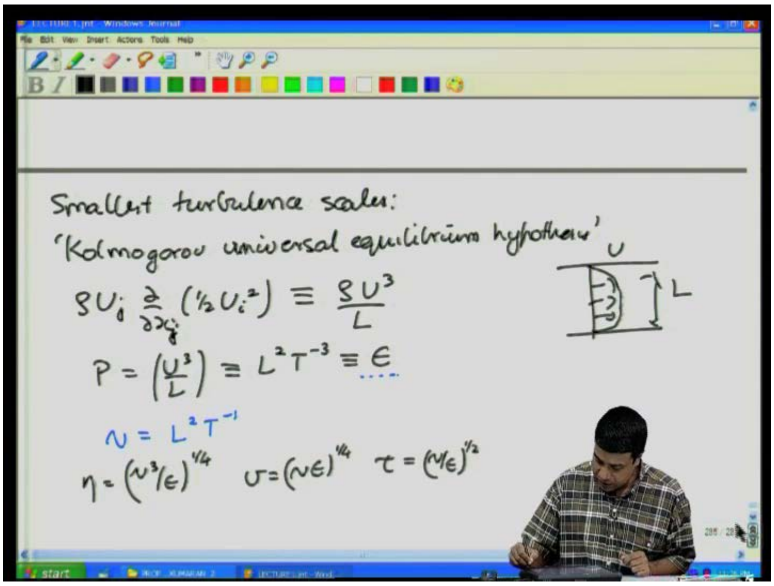


So, what is the smallest scale of turbulence? This smallest scale of turbulence we had inferred on the basis of the Kolmogorov equilibrium, universal equilibrium hypothesis as this called. And the ideas as follows; for the smallest scales in a turbulent flow it is assumed that the length and the time scales are much smaller than the largest flow scales. So, we said the largest flow scales that kinetic energy is getting pumped in from the mean flow to the fluctuations. So, lets us for definiteness let us associate for example, some length scale L and some velocity scale U to the larger scales, inside the larger scales that energy is being pumped in to the fluctuations.

What is the rate at which energy is being pumped into the fluctuations? As you know on the left hand side in the equation for the mean kinetic energy we had partial U i by partial x j. So, this will scale as rho U square into I am sorry, the kinetic energy equation we had this this was partial by partial x j of half U i square, on the left hand side of the kinetic energy equation. So, this is the rate at which energy is coming in from the mean flow, this one can see is approximately equal to rho times U cubed by L.

Usually rather than writing the energy per unit volume half rho U square is the energy per unit volume, it is more convenient to write it down in terms of the unit mass. Because, that reduces one mass dimension that we have to deal with once we write in terms of the kinetic energy per unit mass, you no longer have to deal with the mass dimension. So, one can write the production of energy, per unit mass rather than per unit volume by dividing it by the density. So, this is the rate at which this production of energy U cube by L, this has dimensions of lengths square T power minus 3. The rate of production of energy, this is the energy that goes down to the smallest eddies and gets dissipated.

So, clearly this production rate of production of energy in a fluid steady state has got to be equal to the rate of dissipation of energy in the flow, epsilon is called the rate of dissipation of energy. For the smallest scales themselves we assume that the length scale and the velocity scale are sufficiently small that they do not depend directly up on the macroscopic velocity scale U or the macroscopic length scale L. Because, if you have a region that is very small compared to the macroscopic scales it is got it respond only to the local environment not to the large spatial extent of the flow or the spatial or the or the or the macroscopic velocity.

So therefore, it depends only it does not depend upon the macroscopic length scale L or the velocity scale U. But of course, it depends on those only through this rate of dissipation of energy epsilon, because this energy is ultimately have to going to have to be dissipated by the small scale flows. So, the length and the velocity scale do depend upon epsilon, but not independently on U or L. So, the smallest scales can depend up on epsilon, they could also depend up on the kinematic viscosity, they could also depend up on the kinematic viscosity mu with dimensions of length square T inverse. So since, the length the length and velocity scales of the smallest scale eddies depend only up on epsilon and mu just from dimensional analysis you can infer what the dependences are it is quite easy.

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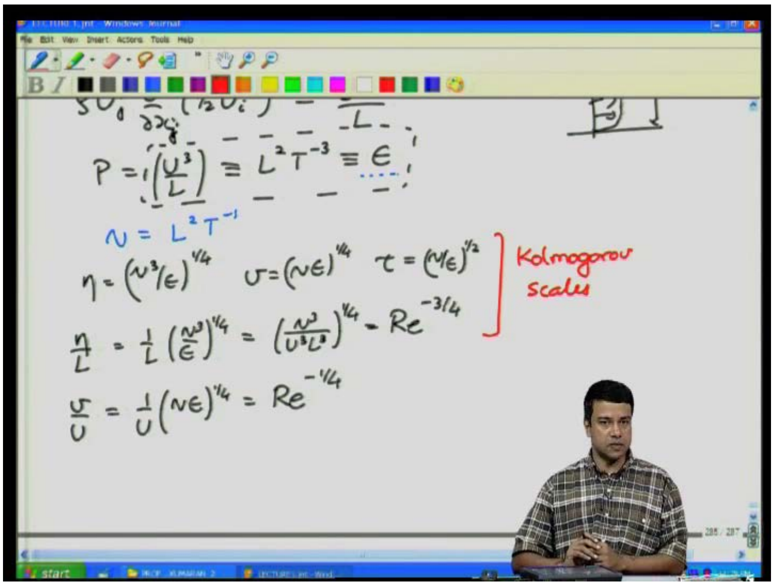


Just from dimensional analysis if I have to get a length scale from epsilon and mu, the only way that I can get a length scale is to write eta is equal to u cube by epsilon or one-forth. The velocity scale v for the smallest scales is new epsilon for one-forth and the time scale tau for the smallest scales is u by epsilon power half. You can easily verify that these dependences basically produce a length a velocity and a time scale. Now, how do these compare to the macroscopic scales? Eta by L is equal to 1 by L into new cube by epsilon power one-forth. For epsilon I can substitute U cube by L, because I know that the production has got to be equal to the dissipation from here.

So, for epsilon I can substitute U cube by L and I will get u cubed by U cubed, L cubed power one-forth, which is equal to the Reynolds number per minus 3 by 4. So, clearly the macroscopic lengths scale is much smaller than the macroscopic scale it goes as R e power minus three-fourth, this is in the limit of high Reynolds number. So, clearly in the limit of high Reynolds number this goes as R e power minus 3 by 4, which is which means that the smallest length scale is much larger than the microscopic scale.

Similarly, v by capital U that is the ratio of the of the smallest velocity scale to the large scale of flow, can be written as 1 over U times u epsilon power one by fourth, which goes as R e power minus one-fourth. If you substitute epsilon is equal to U cube by L, if substitute epsilon is equal to U cube by L you will get R e power minus one-fourth. So, clearly the velocity scale this the smallest eddies is still much smaller than the velocity scale of the mean flow. So, these length and velocity scales or what are called the Kolmogorov scales, the smallest length and velocity scales in a turbulent flow.

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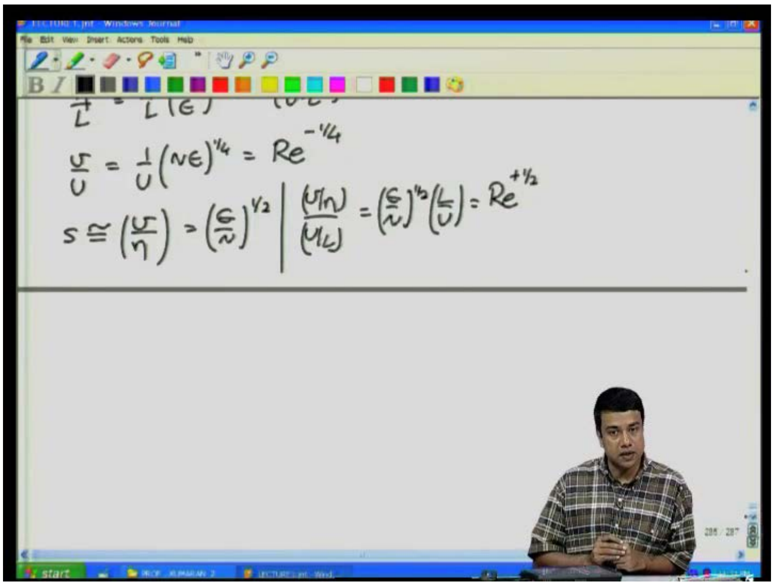


Even though the both the length and the velocity scales in the smallest eddies are much smaller than those in the large scale flow in the large scale flow. What about the strain rate? The scale for the strain rate for the small scale flow is approximately the smallest velocity divided by the Kolmogorov scale, this is an this a scale for the strain rate. That is the gradient of the velocity field the gradient of the velocity field at the smallest scales. And if you work this out this will turn out to be equal to epsilon by nu power half. If I take the ratio of the strain rate for the smallest scales divided by the strain rate for the large scale flow, the strain rate for the smallest scales is v by eta, the strain rate for the large scale flow is equal to U by L.

So, if I take the ratio of those two, v by eta by U by L. This is equal to epsilon by nu power half into L by U, which if you substitute U cubed by L for epsilon this turns out to be equal to Reynolds number plus half. So, even though the velocity scale is much smaller, the length scale is much smaller the strain rate in the small scale, smallest eddies is actually much larger than the strain rate of the large scale flow. The strain rate in the smallest eddies is much larger than the strain rate in the large scale flow.

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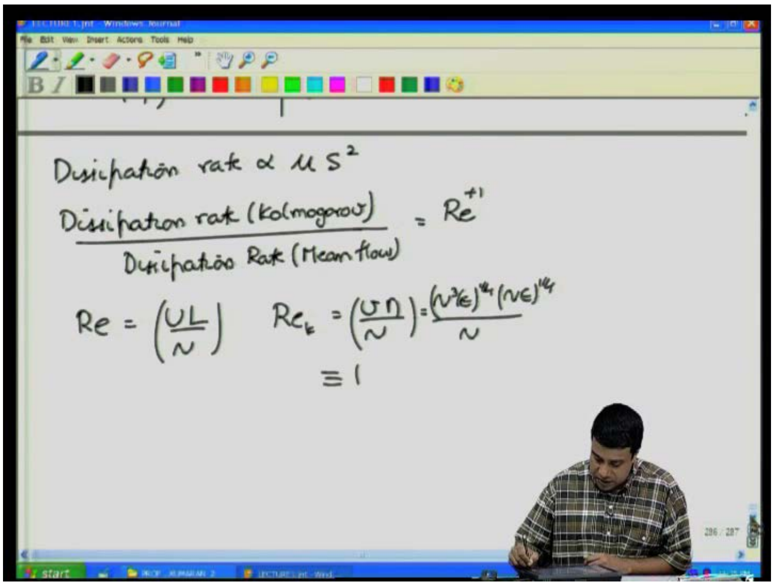


So, this is the strain rate how about the energy dissipation rate. The energy dissipation rate is proportional to the kinematic is proportional to viscosity times the strain rates square. As I said the rate to dissipation of energy is equal to shear stress times strain rate, which is equal to the viscosity times the strain rate square. So, the strain rate ratio of strain rate in the small scale flow to the large scale flow is R e power plus half that means the dissipation rate, in the Kolmogorov scales divided by the dissipation rate, in the mean flow is equal to R e power plus 1 is equal to Reynolds number power plus 1, because it is goes as a strain rate square.

This is telling us something very important when we wrote the equation for the mean kinetic energy or the mean momentum, we said that the inertial terms were order R e larger that the viscous terms. So, when we wrote the mean kinetic energy equation, the inertial terms and the production terms were R e larger than the viscous terms in the limit of high Reynolds number. And because, of that the dissipation due to the mean strain rate and the viscous dissipation due to the mean strain rate, is smaller than the inertial terms in the kinetic energy equation, inertial terms are R e larger than the viscous dissipation rate.

What this is telling us is that the dissipation rate in the smallest scales is R e larger than the dissipation rate in the large scales. That means that the dissipation rate in the smallest scales is comparable to the inertial terms in the large scales. That is what that is what determines the balance of energy, the flow itself creates smaller and smaller length scales until at the smaller length scale the dissipation rate balances the production rate in the largest macro scales. Because, the dissipate rate balances the production rate, the flow is able to dissipate all the energy that is pumped in into the macroscopic scales into the largest scales.

That is because, it creates sufficiently small scales both the length and the velocity scales of the small scales are much smaller than the length and velocity scales of the large scale flow. The strain rate is much larger, it sufficiently large to balance the production of energy in the mean flow, the dissipation due to the smallest scales is sufficient to balance the production of energy in the mean flow.

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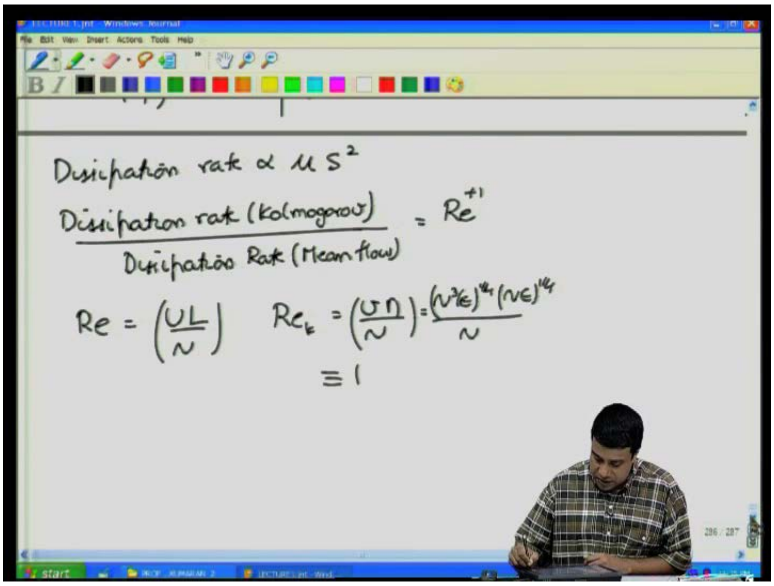


And the other point is that I got Reynolds number I define the Reynolds number for the large scale as U L by the kinematic viscosity. I can similarly, define a Reynolds number for the Kolmogorov scales as v eta by nu. I can also define a Reynolds number for the Kolmogorov scales as v eta by nu. And as we know eta is given by u cubed by epsilon power one-fourth and v is given by nu epsilon power one-fourth by nu, this as you can see is equal to 1.

So, it is creating sufficiently small scales, the Reynolds number based up on the large scale flow is large. However, the flow itself is creating smaller and smaller fluctuations, fluctuations are smaller and smaller scales. In such a way that for the smallest scales inertial and viscous terms are comparable the Reynolds number is above one, inertial and viscous terms are comparable and because of that you have a balance between the production and the dissipation of energy at these smallest scales. The energy is produced in the largest scales is dissipated at the smallest scales, because the strain rate there is much larger than that in the large scales.

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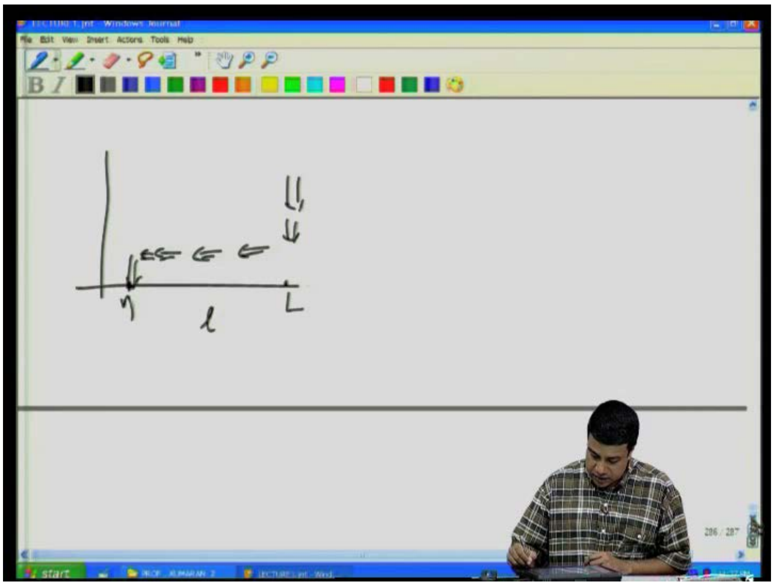


So, you have what is called a cascade of energy in a turbulent flow. If I plot for example, as a function of the length scale I have the large scale, the length scale of the flow and the large scale L corresponding to the macroscopic flow and I have the small scale eta. Energy is coming in, being transferred from the mean flow to the fluctuations at this large scale L, as was shown by mean kinetic energy equation and that gives transfer to smaller and smaller scales. Until it is dissipater at this smallest Kolmogorov scale at the scale where the Reynolds number becomes order one and therefore, there is a balance between the inertial and the viscous terms in the momentum conservation equation.

There are other things that you can say about this kind of an energy transfer, though one thing is of course, that that energy is getting transferred from the large scale flow down to the small scale flow. We got the small scale just on the basis that at this smallest scale there is that the scales themselves the velocity length and time scales do not depend up on the large scales, but only upon the rate of dissipation of energy and the kinematic viscosity.

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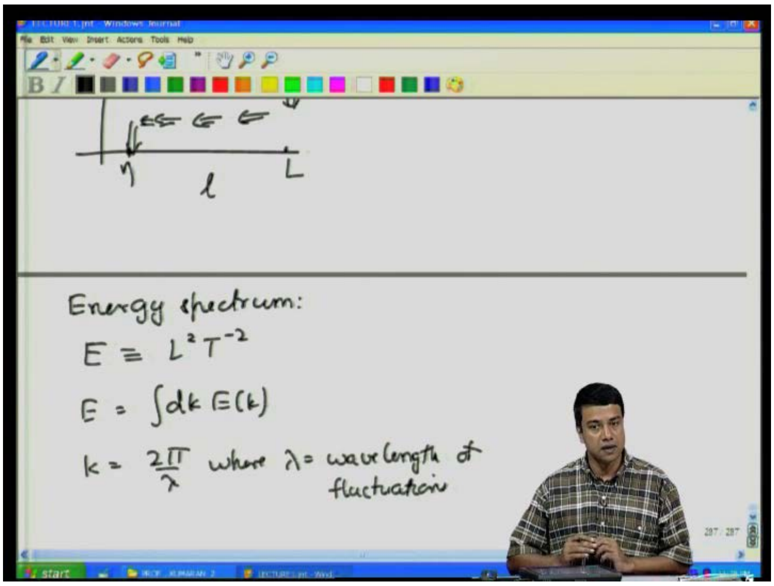


One can infer how this the amount of energy in these flows varies, is called the energy spectrum. The energy itself the energy itself the kinetic energy itself I will call it as E here, this has units of L square T power minus 2, that is because, I have scaled it by the density. So, this energy per unit mass, which has dimensions of L square T power minus 2. You know that in fourier transforms you can always write this, the energy can be written as integral d k, E of k, where k goes from some small value to a large value.

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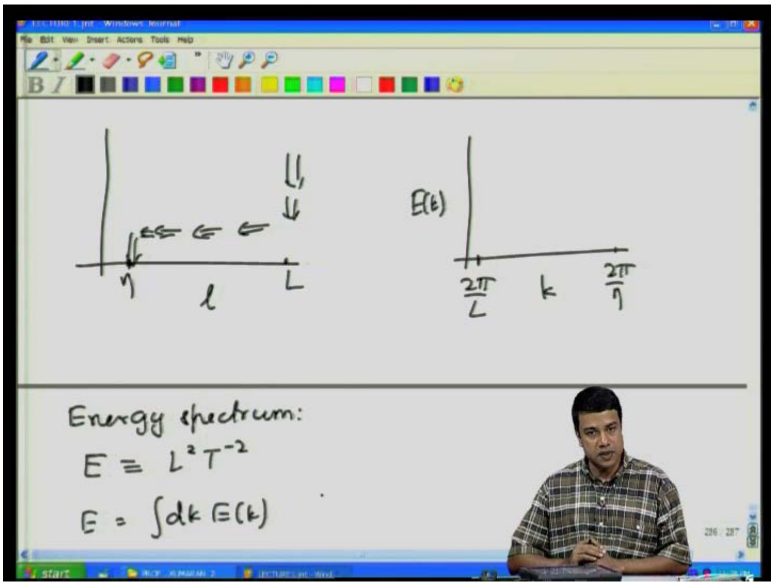

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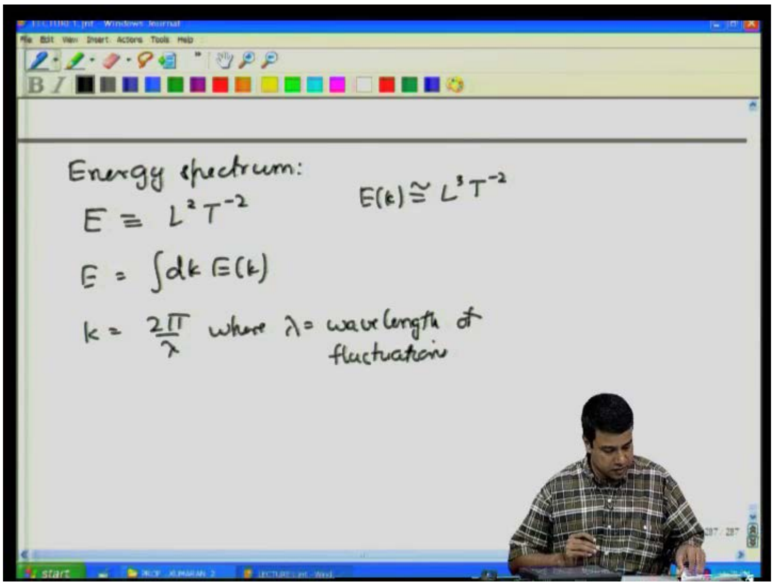


This k is equal to 2 pi by lamda, where lamda is equal to wave length of fluctuations, you take lamda as some characteristic eddie size. So, then e of k will give you the amount of energy in eddies of that characteristic eddie size. So, the standard and Fourier analysis if for those who are not familiar with this kindly go back and refer to your notes on Fourier transforms. So, basically instead of writing down the size of the eddie itself, you write down the wavelength, which is 2 pi by the size. So, this is convenient way of expressing the spectrum.

So, since k is equal to 2 pi by lamda that means, that if I write this in terms of k rather than in terms of the rather than in terms of the length scale, 2 pi by L will come down here, because it is one over L is much smaller than one over eta. So, you can easily see that since, k has dimensions of length inverse that means that E of k has to go as length cubed T power minus 2. Because, energy has dimensions of length square T power minus 2, k has dimensions of 1 over length is 2 pi by lamda.

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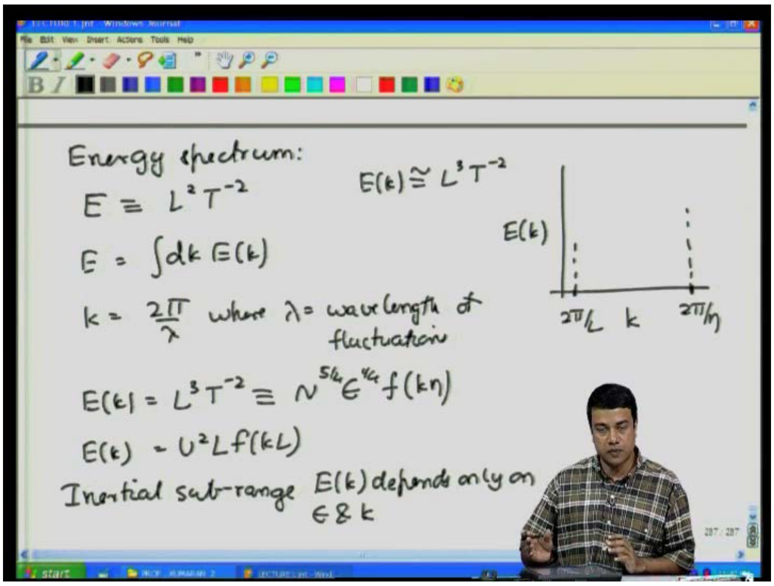


So, you can you can get some universal features of this E of k the energy spectrum itself. For example; if you are at the smallest Kolmogorov scales just from dimensional analysis, the smallest energy scale has to go as is equal to length cube T power minus 2, this will go as nu power 5 by 4 epsilon power one-forth times some function some function of k times eta. So, that is how you would expect the scaling to be at the smallest Kolmogorov scales, because the energy spectrum cannot depend upon U and L, it has to depend only up on nu and eta. So, that is at the smallest scales here, which corresponds to the largest value of k. So, at this end of the spectrum, which corresponds to the largest value of k, it has to scale it has to be some function of U cubed epsilon power minus nu power five-fourth epsilon power one-fourth times some function of eta.

But the largest scales of course, it has to scale only as u and L. So, this will basically turnout to be equal to U square L times, some function of k times L. So, that have the largest scales the scale in accordance with U and L. So, for this scale I have two limits here, let me just plot it again here as function of k, 2 pi by eta, 2 pi by L, E of k. At these two scales I know how it has to scale, it has to scale in some way as function of k eta or k L depending up on the limit that you are taking. However, in the intermediate regime when you are in between these two, it cannot depend up on the kinematic viscosity either. Because, for the intermediate regime the kinematic viscosity is much smaller than the I mean the viscous effects in the intermediate regime are negligible compared to inertial effects, viscous and inertial comparable only at the Kolmogorov scale.

If you are in between the macroscopic scale in the Kolmogorov scale there is an intermediate region were the small scale kinematic viscosity does not matter, as well as the large scale U and L do not matter. In this intermediate regime the only thing that matters is the energy that is following through the energy that is following through the entire system epsilon itself. So, in this inertial sub range as it is called E of k depends only on epsilon and k. It cannot depend up on nu, because the length scales are much larger than the Kolmogorov scales, cannot depend up on U and L, because the length scales are much smaller than the microscopic scales.

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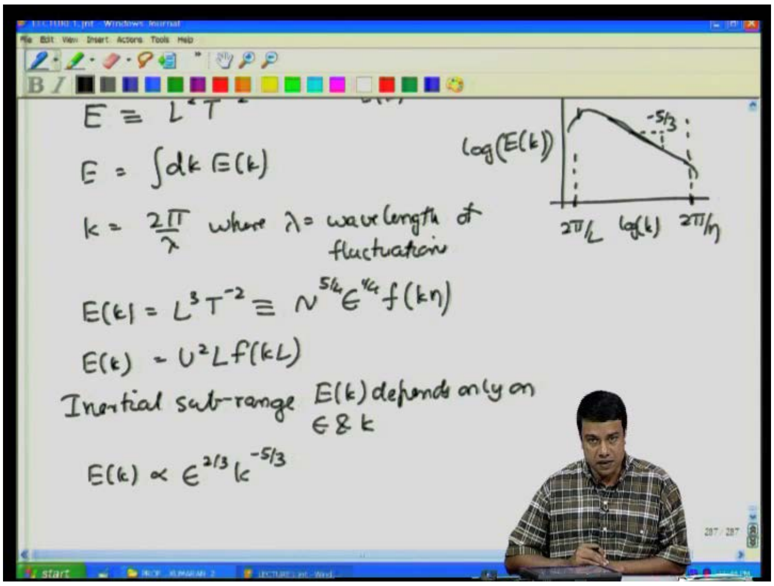


So, it can depend only upon epsilon and k and just from dimensional analysis you can easily see that this intermediate regime. The energy just from dimensional analysis it has to scale as E of k is equal to this rather proportional to epsilon power two-thirds k power minus 5 by 3, that is the only way we will get the correct dimensions epsilon power two-thirds k power minus 5 by 3. So, this predicts a scaling law for the energy in eddies of various intermediate sizes between the Kolmogorov scale and the microscopic scale.

So, what this is saying is said this energy spectrum has to go have a minus five-third part. This I should state here plotting log, it is just log of the energy, it has a minus five-third part as in the intermediate spectrum. So, what this is saying is that the energy in the macroscopic scale, the kinetic energy in the macroscopic scales is much larger than the kinetic energy in the Kolmogorov scales, most of the energy is in the large scale flow.

The energy in the small scale flow decreases as the eddie size decreases with a scaling law, which goes as k power minus five-third or length scale to the power plus five-third it decreases the length scale to the plus five-thirds part. In a manner similar to the energy spectrum one can also get the dissipation spectrum.

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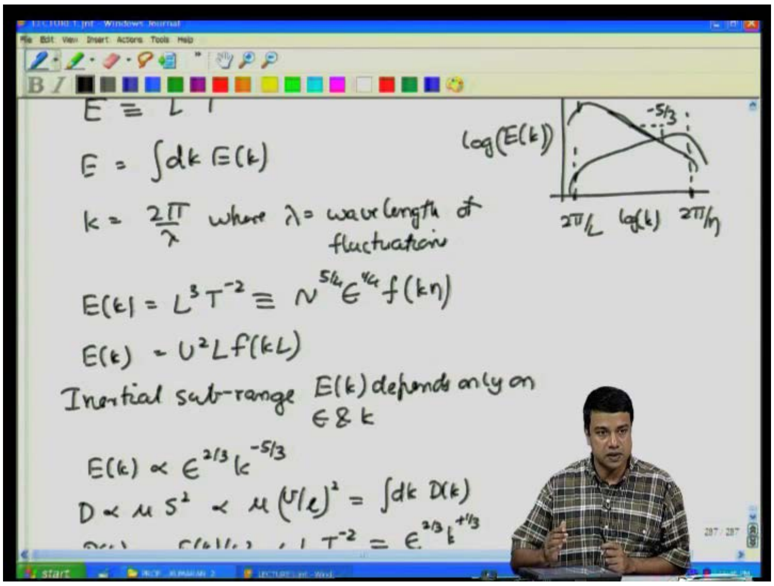


As I said the dissipation of energy is proportional to mu into the strain rate square, is proportional to mu times the velocity divided by the length scale whole square. And in my Fourier notation I can write this as integral d k times D of k. So basically, the dissipation rate goes as 1 over the energy divided by length square. So similarly, dissipation spectrum will also go as will have dimensions of the energy spectrum will go as the energy spectrum divided by L square, which is proportional to L times T power minus 2, just from dimensional analysis. And in the intermediate regime this can depend only up on epsilon and k and the only way it can depend up on that is as epsilon power two-thirds k power plus one-third.

Because, it has dimensions of one over length square, so that it is proportional to k square times the energy spectrum. So, in my figure that I had here the dissipation spectrum actually increases as k power plus one-third in the meter dredge. So, what this states is that the energy in the mean flow is much larger than the energy in the smallest scales whereas, the rate of dissipation of energy is largest for the smallest scales. k is equal to 2 pi by eta corresponds to the smallest scales of the flow and the rate of dissipation of energy is largest for the smallest scales. So, this gives us a comprehensive idea of the flow energy the lengths scales and the dissipation scales in the flow.

How do you model these turbulent flows? The most commonly used model is what is called the k epsilon model, which is based up on what I had just done, the kinetic energy of the flow and the dissipation rate of the flow. It recognizes the fact that you have to have both kinetic energy and dissipation, both of these have to be modeled these could be in homogeneous through the flow, you have to write additional equations for the kinematic energy in the dissipation rates. Here the shear stress is written as the turbulent viscosity times the strain rate. So, you write the shear stress tau i j is equal to of course, there is a molecular viscosity plus there is a turbulent viscosity times grad U indicial notation is by partial x i. Where the turbulent viscosity is modeled as rho constant times the kinetic energy square divided by the rate of dissipation of energy.

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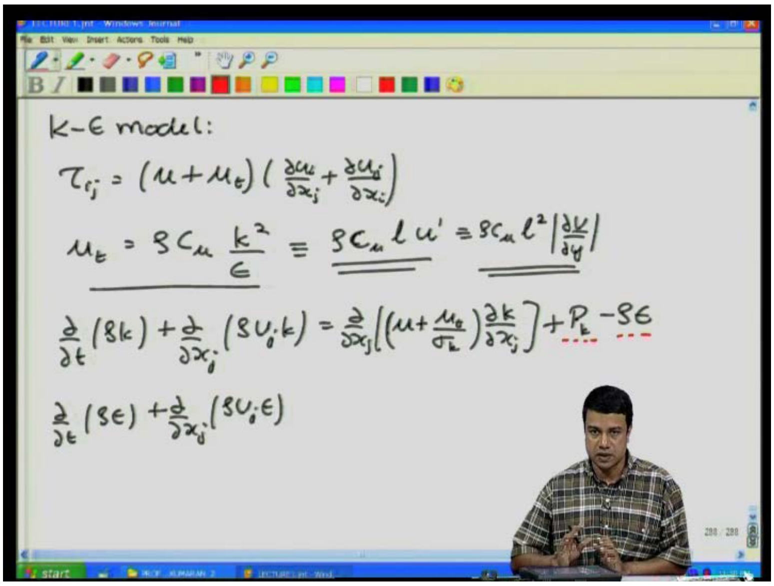


The kinetic energy goes as length square per time square and the dissipation of energy goes length square by time cubed and you can see that this combination. Basically, it gives you the velocity times the length scale basically, it gives you velocity times the length scale. The other models that are often used for this are what are called the mixing length models. The mixing length model is similar it has some constant in it, it has a mixing length and it has a velocity of fluctuations. Rho times a diffusivity, diffusivity has a length scale unit and a velocity scale unit and u prime is a velocity of fluctuations l is the length scale. And this velocity of fluctuations is often written as rho c mu times l square times the local gradient, the length scale times the local gradient gives you a velocity.

So, these are what are called the mixing length models. Whereas, if you write in terms of the kinetic energy and the rate of dissipation of energy it turns out be the k epsilon model expression for the kinematic viscosity. So, this is the conservative relation of course, to find the turbulent viscosity I need to know what are k and epsilon and typically equations are written for each of these. The left hand side for both of these are identical they has a rate of change of kinetic energy and in the rate of change of dissipation of energy. Both are substantial derivatives in a reference frame moving with the mean fluid velocity. In the right hand side for the kinetic energy you have a diffusive term basically the diffusion of kinetic energy due to in-homogeneities in the kinetic energy plus the production term minus the rate of dissipation of energy.

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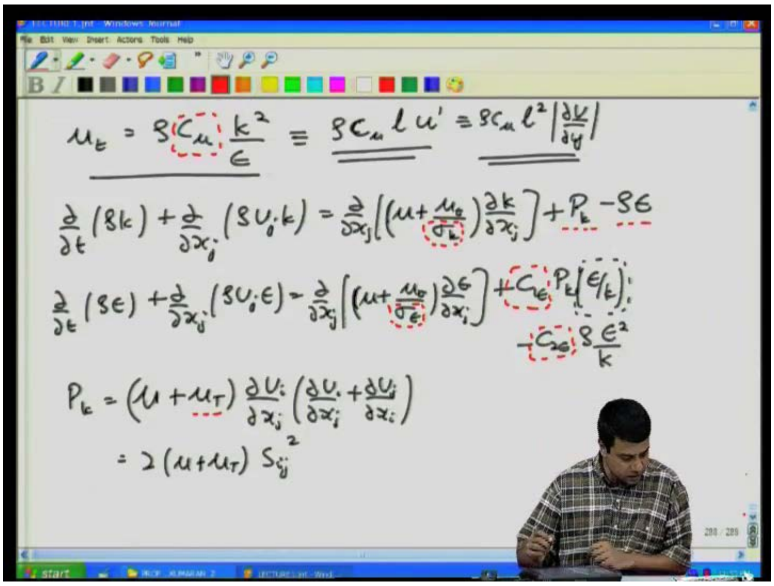


So, this term is the production due to the mean flow, which I will come back to later, this is the rate of dissipation of energy. Of course, if you would have steady state the homogeneous flow production would exactly balance dissipation. And in the equation for the fluctuate for the dissipation of energy I have a similar term here, plus this is a term, which is the production term, but it has to have an inverse time unit. Because, the rate of dissipation energy is was has it does not have the same dimension as the kinetic energy itself. Epsilon has dimension of 1 over T times k, that 1 over T is the turn over time, which is taking to accomply this epsilon by k over here. The turn over time, which is taken to accomply this epsilon by k, which has dimensions of 1 over T.

So, that is the transfer of energy term from the mean kinetic energy to the fluctuations. And then I have a final dissipation term which goes as minus C 2 epsilon times rho epsilon square by k. Once again epsilon by k is one over time, it gives basically a turn over time and epsilon divided by that time gives me the rate of change of epsilon. So, side the both equations are the familiar convective derivative on the right hand side in both cases have a diffusion term. The kinetic energy have production in flow the dissipation due to the kinetic energy and in the dissipation term I have the mean flow energy transfer and then I have a molecular energy dissipation term. Each of these are constants, which are basically fitted, each of these are constants which are basically fitted all of these constants are fitted constants as well as the constant C nu here.

The production is basically the mean kinetic energy production, which is basically is equal to the turbulent mu plus mu T times partial U i by partial x j into this is equal to 2 mu plus mu T times the mean strain rate square. Note that this mu T here takes into account the factor this is a Reynolds stress. So, there is the augmentation of the stress to the Reynolds stress times the strain rate square, basically gives me the production of energy and that energy is dissipated. So, that the balance being the production and dissipation is incorporated in these equations for the mean kinetic energy in the mean dissipation rate. And as I said all of these are constants which had to be fitted. The common ones that I used based up on a large class of turbulent flows for these fitted constants are basically, C mu in the equation for the viscosity is usually assumed as close to 0.09.

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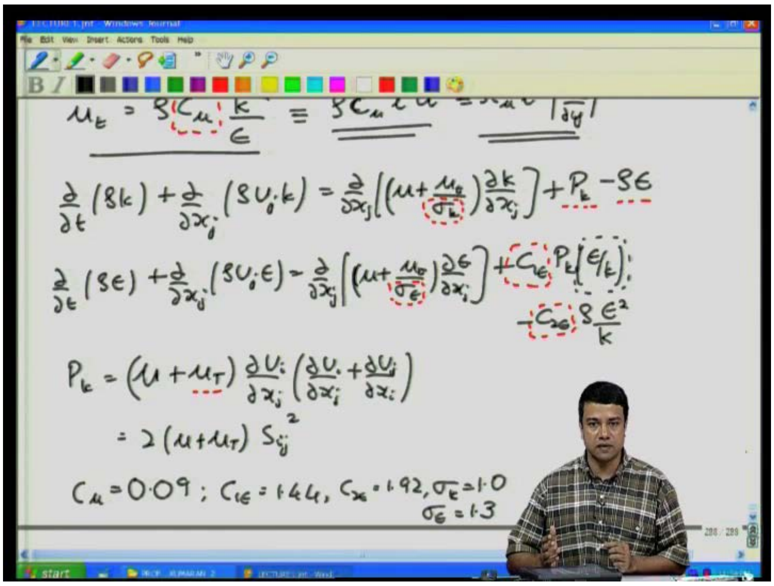

This is the same whether it is a mixing length model or it is the k epsilon model. c 1 epsilon is equal to 1.44, C 2 epsilon is equal to 1.92 sigma k. So, actually very close to 1 and sigma epsilon is taken as 1.3. So, these set of equations for k and epsilon basically, give you what you need for the constitutive relation for the shear stress, mu T for the mean velocity equation mean momentum equation. These two coupled with the mean momentum equation basically form the k epsilon model for turbulent flows.

Of course, there are various variations of this the k omega model and so on you could have more constants depending upon, whether you have buoyancy effects and other effects and you could also take into account the vorticity equation and so on. But this is the one that is most commonly used and basically it consists of an inertial a convective derivative of each of these quantities is equal to diffusive part plus a production, minus a dissipation. Put these in to calculate k and epsilon use k and epsilon to get the kinetic the turbulent viscosity and put that into the mean flow.

So, I have taken this occasion to explain how the length scales of turbulence flow comes what are the smallest and the largest length scales. And how the energy varies in between these two scales and how elements of this energy transfer mechanism are taken into account in turbulence models for turbulent flows, which contain fitted constants of course. But they are broadly based up on the physical picture of the energy cascade for turbulent flows. In the next lecture we will look at a concrete example of how we do modeling of turbulent flows, wall bounded turbulent flows, that is flows in a channel, which will be most useful for our case. So, we will continue with that discussion of turbulent flow in the next lecture. We will see you then.