Fundamentals of Transport Processes II
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Lecture - 36
Vorticity Dynamics - Part I

This is lecture number 36 of our course on fundamentals of transport processes. Welcome to this lecture. We started off this course by deriving the mass momentum conservation equations, the Navier stokes equations for flows of Newtonian fluids and the momentum conservation equation contains a balance between inertial and viscous forces. And I said the momentum conservation equation is in general a non-linear differential equation. So, there is no general procedure, which can be used to solve this equations for all situations. One needs to make approximations depending upon the situation and their consideration in order to solve this equation.

In the previous lectures, we had looked at different approximations that could be made based upon physical considerations, based upon the ratio of inertial and viscous forces. The momentum conservation equation basically balances the inertial and the viscous forces in the fluid. If the inertia is small, then there is a dominance of viscosity therefore, diffusive effects are dominant compared to convective effects. And we looked at ways to solve the equation, when the flow is diffusion or viscous dominated. In that case the equations mostly reduce to Laplace equations, which could be solved by a variety of techniques.

Next, we look at inertial dominated flows when we looked at inertial dominated flows first of all we made the approximation that the flow was inviscid. That means, that the viscosity is equal to 0, the second was irrotational that is there is no vorticity anywhere within the flow, an important approximation irrotational. And if you assume to the flow was inviscid and irrotational then one derives the potential flow equations for the Navier strokes equations reduce to the potential flow equations because, the flow is irrotational the velocity can be expressed as the gradient scalar velocity potential because the curl of the velocity is equal to 0.

And the vector momentum conservation equation just reduce to the scalar Bernoulli equation for the pressure, and these simplified equations were solved subjected to boundary conditions. Of course, since we have neglected the viscous terms in the conservation equations, we can no longer satisfy all of the boundary conditions.

When we neglect the viscous terms we are neglecting the shear stresses exerted by bounding surfaces on the fluid because the shear stress depends upon viscosity. So, one can have only pressure forces acting within the fluid for this reason one can satisfy only the normal velocity and the normal stress, boundary conditions at the surfaces. It is not possible to satisfy the tangential velocity and the tangential stress boundary conditions at the surfaces in potential flow.

So, we looked at how to solve the potential flow equations in various situations both 3 and 2 dimensional and in all of these cases the simplification made was that the flow was the there was no tangential stress at the surface. And therefore, we could satisfy only the normal velocity and the normal stress conditions.

However, in a real situation the velocity does in fact, have to come to 0 at bounding surfaces because in the physical situation the no slip condition has to be satisfied at the surfaces. And that let us to the analysis of boundary layers, thin layers near surfaces where viscous effects continue to be of equal magnitude as inertial effects, even in the limit as the Reynolds number goes to infinity. So, there is conviction that is taking place in the stream wise direction the simplest boundary layer that we looked at was the boundary layer for the flow past the flat plate.

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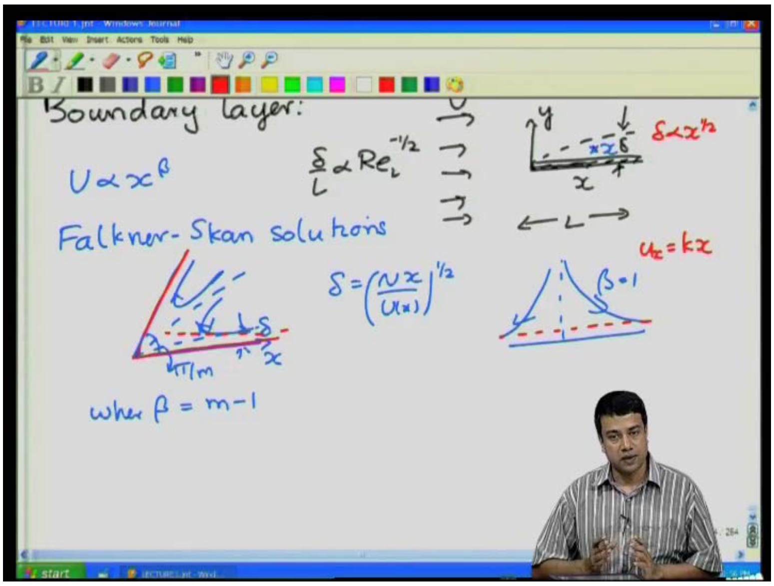

The simplest one that we had looked at was the boundary layer for the flow past a flat plate, you had a velocity constant velocity U coming in and there was a boundary layer here this is the stream wise direction the cross stream direction. If you just use the outer potential flow that would predict that the velocity is equal to capital U everywhere. However, that does not satisfy the no slip condition at the surface of the plate itself, and for that reason you have to incorporate the viscous effects within a boundary layer of thickness delta of thickness delta which is small compare to the length scale of the flow. This thickness delta is dictated by the flow itself because the only way that the velocity will come to 0 at the surface is if there is shear stress of the surface.

The Reynolds number based upon the microscopic L length scale L is large therefore, you cannot have a balance between inertia and viscosity over length scale is comparable to capital L. However, if there is a velocity gradient in the flow very close to the surface over a very small thickness delta, delta being small compared to L. So, the velocity starts from 0 at the surface and reaches the free stream value of or reaches close to the free stream value of capital U within a layer of thickness delta and delta is small compared to L. Then, the velocity gradients are much larger than you expected based upon the simple scaling of the velocity and the length scale L because delta is much smaller than L the velocity gradient is much larger than U by L.

And in that case one could have a balance between inertial and viscous forces provided delta is small enough. How small does delta have to be for the balance between inertial and viscous forces to be or for viscous forces to be comparable to inertial forces, even as Reynolds number goes to infinity. We found that delta by L has to be proportional to the Reynolds number power minus half.

So, this is a length scale that is set up by the flow itself while the requirement that the only way that the velocity is going to come to 0, is if there is a balance between inertial and viscous forces within a thin region. And then in order to obtain similarity solutions we use the additional piece of information that at a given location x, within the flow at a given location x within the flow the value of the boundary layer thickness at that location will not depend upon the total length L because, the downstream edge of the plate is downstream of this location x.

So, at a given location x the only length scale that is relevant is x itself, which is the distance from the upstream edge because convection is transporting momentum downstream. Therefore, the any boundary conditions that are applied at a downstream location to this point x cannot effect the flow at the given location x. And using that additional piece of information, we had managed to find similarity solutions for this particular case and there were other cases for which we managed to find similarity solutions. For any velocity profile for which U is proportional to x power beta U the mean velocity varies at some power of x in this particular case beta is equal to 0. In case of the mean potential flow velocity varies as x power beta does admit a similarity solution and those were known as the Falkner Skan solutions.

This is a family of solutions for the flow for the potential flow in a corner of angle pie by m, where beta is equal to m minus 1. So, for all of these flows it is possible to get similarity solutions and we found out the scaling of the boundary layer thickness, the shear stress and so on depending upon this parameter beta. So, only for this class of solutions that we got boundary layer this class of potential flows, that we managed to get boundary layer solutions, but this gave us additional insight into the dynamics of the flow.

So, if we plot for example, as a function of this downstream distance x here function of this downstream distance x alright we have a boundary layer here and this boundary layer thickness delta depends upon x we had derived the scaling between delta and x it depends upon whether the flow is accelerating decelerating or it is a constant. Delta was basically equal to U x by U of x power half that was the scaling for delta at a given location x and of course, U goes as x power beta. So, there is a dependence of delta on beta as well.

So, what we found was that if x is equal to if beta is equal to 1 if beta is equal to 1 which correspondent to a stagnation point flow we found that the boundary layer thickness was a constant it was independent of x that is because the flow is being accelerated downwards. For this particular flow U x in the potential flow you find that U x is proportional is equal to some constant times x. So, it is accelerating downstream the boundary layer thickness is constant for the flow past a flat plate that we had done earlier we found that delta was proportional to x power half. So, it increases as you go downstream.

If the angle is less than 90 degrees as in this case as I have shown here you here the angle is less than 90 degrees, the boundary layer thickness actually decreases as you go downstream, the boundary layer thickness decreases as greater than 90 degrees the boundary layer thickness increases as you go downstream. In all these cases we managed to get a similarity solution.

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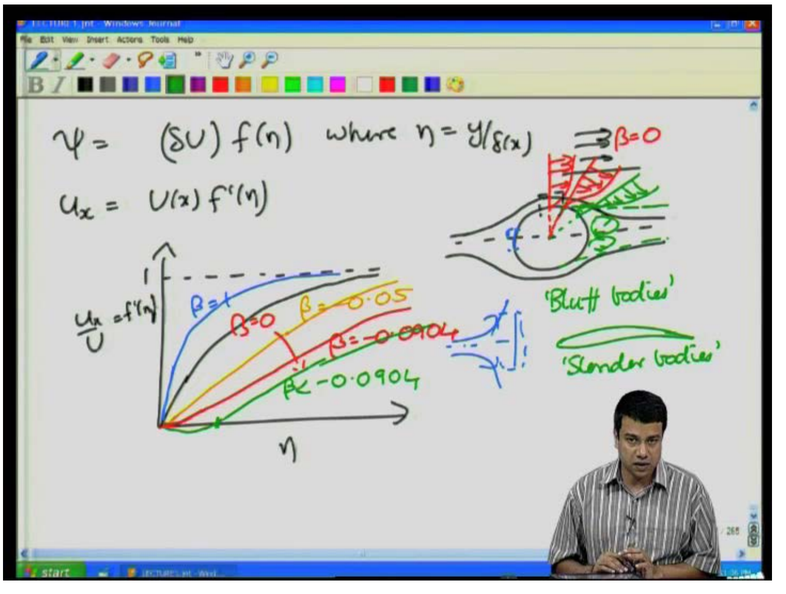

The solution was always of the form we wrote a stream function of the form U x, f of eta where eta is equal to y by delta of x, y is the cross stream direction and x is the stream wise direction. On this basis we found the velocity U x is equal to U of x times f prime of eta it should be here delta U times f of eta was the stream function, where delta is the boundary layer thickness on that basis U x was equal to f prime of eta. And this gave us a very sight inside that was that, the velocity profile U x by U which is equal to f prime of eta you should plot that as a function of the similarity variable eta.

Of course, for the free stream this f prime has to go to 1 in the limit as eta goes to infinity this f prime has to go to 1, the profile of course, depends upon beta. So, for example, for the Blasius boundary layer for the flow past the flat plate, beta is equals to 0 corresponds to this profile that I have here. If I increase beta that is if I go to the case where I have a stagnation point flow, you will get a similar velocity profile except that it will reach the free stream value faster, beta is equals to 1 which corresponds to the flow past a flat plate corresponds to something that looks like this.

So, as beta increases the gradient at the origin of the velocity increases or the shear stress at the origin increases, as beta decreases the gradient decreases. If you decrease beta below point below 0 you find that at some point. So, at beta is equal to 0.5 it looks like this, you reach a point beta is equal to 0.090 minus I am sorry we should write this as minus at beta is equal to minus 0.05 you get something like this beta. And initial point where beta is equal to minus 0.0904, where the slope of this curve at the origin is equal to 0. That means, that the derivative of the velocity with respect to the y coordinate is equal to 0 at that point when the derivative of the velocity with respect to the y coordinate is equal to 0.

That means, the shear stress is equal to 0 because tau x y is equal to mu times dux by d y, and that point the velocity becomes 0 the slope is 0 at the origin should decrease beta further you end up going in this direction. That means, that it should decrease beta further you will end up with the small region, where there is velocity in the opposite direction very close, this corresponds to a re-circulating region. And we had drawn the analogy between this and the flow past a sphere we had taken the flow past cylinder in the last lecture. One can do a similar calculation for the flow past a sphere, but let us just stick to our example of the flow past a cylinder.

So, as this potential flow stream lines go around this the potential flow stream lines are of course, symmetric. If there is no circulation the potential flow stream lines are symmetric. At this frontage what you get actually corresponds to beta is equal to 1 because that corresponds to a stagnation point flow, if you if you expand this region out it corresponds to a stagnation point flow.

So, here there is an accelerating stream line the velocity is increasing as the distance along the surface increases. So, it corresponds to a stagnation point flow. As you go downstream if you reach the top and bottom surfaces you find that at these two points, the flow resembles just the flow over a flat plate, the flow resembles the flow over a flat plate for which beta is equal to 0 neither any acceleration nor any deceleration.

So, the velocity profile at this surface will look something like this, as you go downstream the flow actually decelerates on the upstream side because the fluid has to go around the cylinder, the flow accelerates. On the downstream side it comes back down because, it has to fill in the volume that is behind the cylinder and the flow decelerates. And in this decelerating flow at some point beta becomes equal to minus 0.0904 at some location here at some location here, beta becomes equal to minus 0.0904.

That means, that at this location the shear stress itself at the surface is equal to 0 and if you go a little further downstream, beta becomes less than minus 0.0904 and you end up with a re-circulating region. Very close to the surface the velocity is actually opposite to the free stream velocity you end up the re-circulating region. So, you have a 2nd point here a 2nd point here corresponding to this point here at which the velocity is equal to 0.

So, I have 2 points at which no slip is satisfied that means, that the 2nd point also acts something like surface with a no 0 velocity condition. That there is a tangential velocity comes to 0 at this point the normal velocity of course, is equal to 0 because it is perpendicular to the stream lines and this region behind is what forms the wake, and you have re-circulation within the wake.

So, even though the potential flow predicts that the velocity is symmetric because the boundary layer separates in on the decelerating side, you end up having a wake behind the cylinder in this wakes vorticity effects are not small. The pressure within the wake is much smaller than the pressure layer in the potential flow and for that reason you get a net force, when we calculated the force on the cylinder we found that it was 0 in the absence of circulation. That was because of symmetry the pressure in the Bernoulli equation was proportional to half rho U square and U square on the upstream and the downstream sides in potential flow are the same normals are in the opposite direction. So, there is no net force.

The boundary layer theory is telling us that when the flow decelerates on the downstream side of the cylinder, there is boundary layer separation. And the potential flow solution is no longer valid, there is a wake region at the back where there is re-circulation. In this re-circulating region the pressure is much smaller than it is on the front side of the wake of the cylinder on the upstream side of the cylinder, and for that reason you will have a net track force. In these kinds of objects in which there is boundary layer separation at the back are often called as bluff bodies.

Whereas objects that are designed in such a way that there is no boundary layer separation they are they are designed to be arrow dynamic in shape. So, that there is no boundary layer separation these are often called as slender bodies. So, there is the distinction between bluff bodies and slender bodies. Slender bodies are designed such that under normal circumstances, there is no boundary layer separation on the downstream side. That means, you have to make sure that the deceleration is never too large. So, that this the beta in the Falkner Skan solution never goes below minus 0.0904.

So, this tells us that even though potential flow may predict solutions they are not always valid because potential flow solution will be valid in most of the flow, only if the regions where the viscosity becomes important or restricted to thin regions near surfaces. In bluff bodies like a cylinders or spheres of course, on the upstream side it is restricted to a thin region the thickness of course, goes as R e power minus half. However, on the downstream side because of boundary layer separation, it is no longer restricted to a thin region. Therefore, you do have viscous effects being important within the wake region behind the object after boundary layer separation has taken place.

The flow in the wake is of course, usually turbulent and we will come back to analyze what are turbulent flows, but in the mean while it is useful to look at the dynamics of vorticity in these flows. Because usually turbulence usually involves a lot of vorticity intensification due to stretching and bending of vortex lines. So, we will pause briefly to look at vorticity dynamics in an otherwise potential flow, when you solve the potential flow equations we made the implicit assumption that it is irrotational, there is no vorticity. There could be Vorticity however, and as you can see here vorticity the circulation that we see behind the object is actually generator at solid surfaces, the circulation that we saw in the wake region is actually generated at boundary surfaces. So, we will look at vorticity dynamics, how does vorticity convict and diffuse in flows and how is it generated at surfaces. That is the next topic vorticity dynamics.

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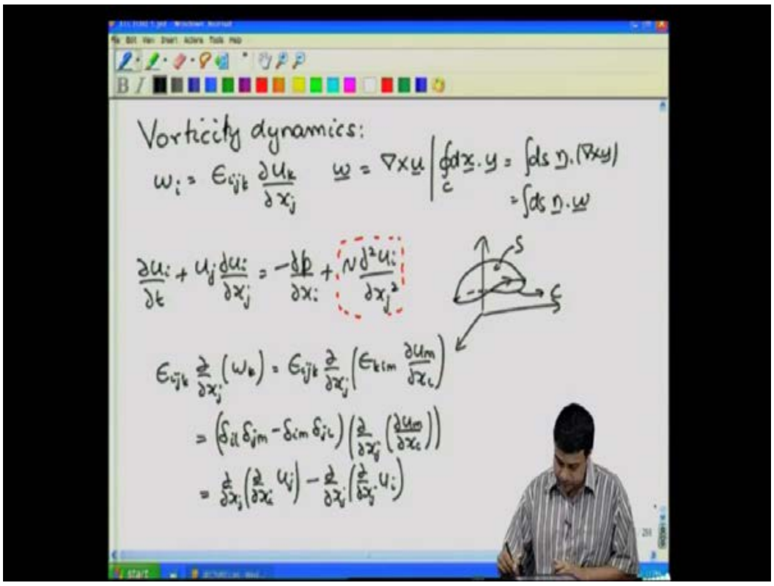

So, first things first we can write an equation for the vorticity omega i is equals to epsilon ijk partial U k by partial xj will be omega is equal to del cross U, the definition of vorticity as you know that this satisfies the stokes theorem or the theorem for the curl. The integral theorem for the curl integral dx dot U is equal to integral over the surface of n dot del cross u this is also integral of the surface of m dot omega.

So, if I have some surface within the flow, so this is the surface s and this is the closed c which is the perimeter of the surface. So, integral over this closed counter c integral over this closed of the displacement element the vector line element dotted with velocity has got to be equal to integral over the entire surface n dot omega.

If you recall our Navier Stokes momentum conservation can equation can be written as partial ui by partial t plus uj partial ui by partial xj is equal to minus partial p. We will take the curl of this to get the vorticity equation shortly, but first we can just note that many of these terms in this equation can actually be written in terms of the vorticity. In particular if you take this diffusion term here, if you take the curl of the vorticity epsilon ijk partial by partial xj of omega k, that is the curl of the vorticity. And omega can once again be expressed in terms of the velocity which is equal to epsilon ijk partial by partial xj of epsilon klm partial um by partial xi.

So, that is the curl of the velocity I used different indices in the 2nd term here just to make sure that we do not have confuse I have already used I and j in the first curl. This epsilon ijk times epsilon klm can of course, be written in terms of delta functions. So, this can be written as delta il delta jm minus delta im delta jl two partial by partial xj of partial um. So, this can be written as partial by partial xj of partial by partial xi uj partial by partial xj partial by partial xj of ui this is just simple algebra.

In the first term I can change the order of differentiation because the gradient vectors are independent vectors. So, this just becomes equal to partial by partial xi of partial uj by partial xj minus this is just partial by the gradient dotted with the gradient. So, this is just partial square ui by partial xj square, this first term is 0 because of the incompressibility condition. This first term is 0 because of the incompressibility condition therefore, the Laplacian of the velocity can be written as the curl of the negative of the curl of the vorticity because, I get the curl of the vorticity is equal to minus if the curl of the vorticity that I started of here is equal to minus partial square ui by partial xj square.

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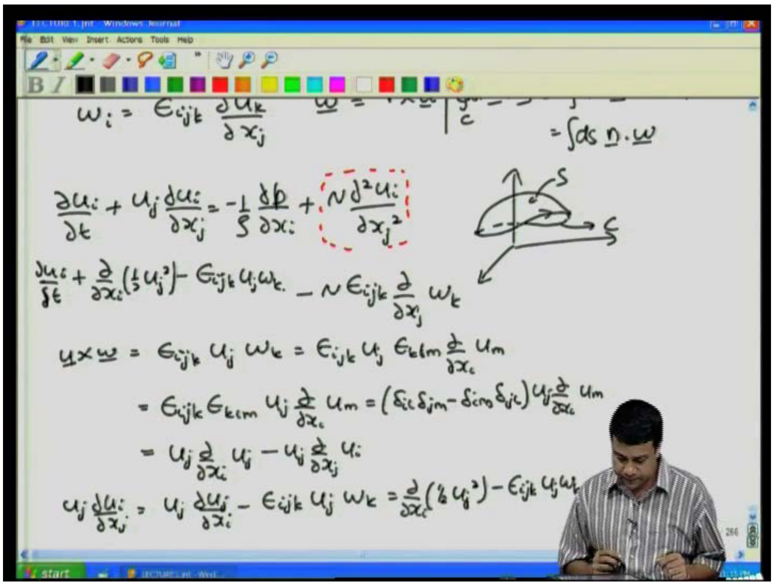

So, this can also be written as minus nu times del cross omega or epsilon ijk partial by partial xj of omega k for the pressure to 1 over density here. So, please note that correction this is equal to minus 1 over rho as partial p by partial xi. I can also write this uj partial ui partial xj in terms of the vorticity, we had done that previously when we actually derived the potential flow equations, we will just do it here quickly if I take u cross omega u cross omega is equal to epsilon ijk uj omega k um am sorry, just writing omega k as epsilon klm partial by partial xi of um. So, this is equal to epsilon ijk epsilon klm uj partial by partial xl of um.

Once again we write epsilon in terms of a familiar deltas this is delta il delta jm times and I can write this as. So, what this implies is that the convective term uj partial ui by partial xj is equal to minus and you recall that uj partial uj by partial xi can also be written as partial by partial xi of half uj square. So, is equal to partial by partial xi of half uj square minus epsilon ijk uj omega k because uj partial ui by partial xj can also be written as partial by partial xj of half uj square. So, I can write the left hand side of this also as partial ui by partial t plus partial by partial xi of half uj square minus epsilon ijk uj omega k, just using vector identities to simplify terms in the conservation equation.

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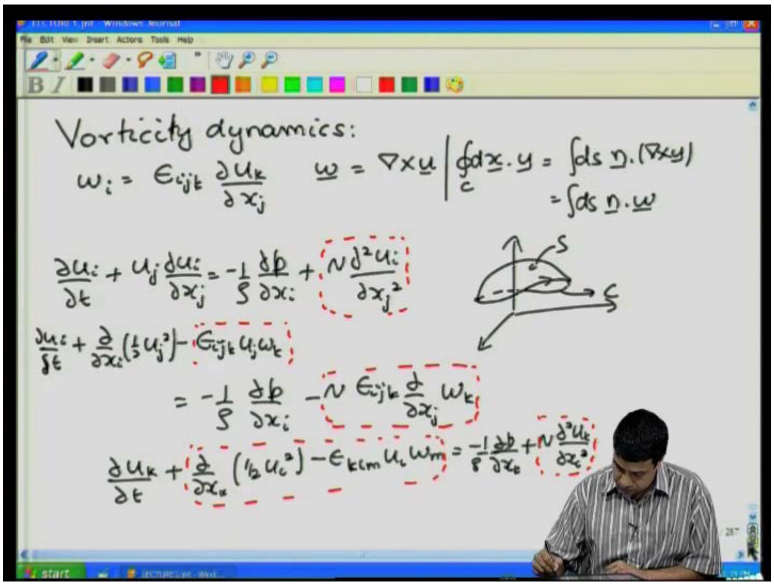

So, this is equal to minus 1 by rho partial p by partial xi plus nu or this a negative sign here minus nu at epsilon ijk partial by partial xj of omega k. So, that is an alternative way of writing the equation and you can see that the Bernoulli's equation actually comes out from here if I take omegas identically equal to 0 everywhere because these two terms end up becoming 0. The vorticity is 0 everywhere and these two terms actually end up becoming 0. And then I end up with the Bernoulli's equation for the relationship between the pressure and velocity potential.

These was the procedure we used for actually deriving Bernoulli equation in that case, we had a body force which you drew out as the great of potential and then included that in the pressure of course, but this was basically the procedure that we had used. Now, what happens for the situation where the vorticity is non 0. So, that is the next question that we will look at, in that case one should be able to get an equation for the vorticity by just taking the curl of the momentum equation for the velocity field. So, I take the curl of this momentum equation in order to get an equation for the vorticity.

Let me start of with my equation in a specific form. So, as the starting point that I will use will be rather than using i and j because when I take the curl I will be using epsilon ijk times partial by partial xj of the momentum equation. In this particular case I will use the free index as k and the repeated index as l and m. So, I will write my equation of this form, partial uk by partial t plus partial by partial xk of half ul square minus epsilon klm ul omega m is equal to minus 1 by rho partial p by partial xk plus nu partial square uk by partial xl square. So, I will start with my equation of this form I have taken this term to be the usual diffusion term, where as for the convection term I will substitute it in terms of the identity that I just derived for you and I take the curl of this entire equation.

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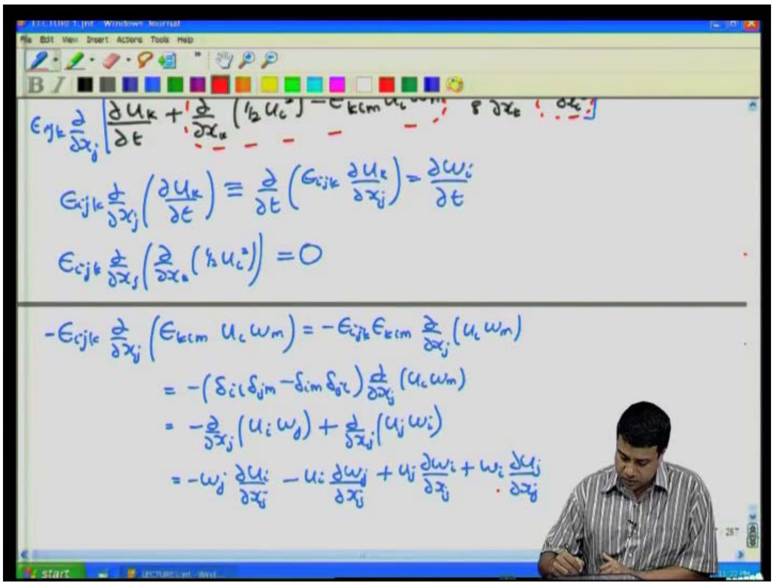

So, I take the curl of this entire equation epsilon ijk partial by partial xj of the entire equation. So, let us work out what it works out two term by term epsilon ijk partial by partial xj of partial uk by partial t can equivalent be written, you can now interchange the order of differentiation because time and position are independent coordinates. This is also equal to partial by partial t of epsilon ijk partial uk by partial xj and what is within the brackets is only the vorticity. So, this is equal to partial omega i by partial t. So, that is the first term.

The second term is epsilon ijk partial by partial xj of partial by partial xk of half ul square, partial by partial xk of half ul square is the gradient of a scalar half u square. The kinetic energy or the velocity square you are taking the curl of that, the curl of the gradient of a scalar has to be equal to 0. The curl of the gradient of a scalar has to be equal to 0.

The next term is epsilon ijk partial by partial xj of there is a negative sign there epsilon klm ul omega m this term turns out to be important. So, I will spend a little bit of time on this one. So, I can just write in terms of the epsilons as minus epsilon ijk epsilon klm partial by partial xj of ul omega m and I use my identity for the deltas once again. So, this will be equal to minus of delta il delta jm minus delta im. So, this becomes equal to minus partial by partial xj of ui omega j plus partial by partial xj of uj omega i.

So, I am just expanding out this term here minus partial by partial xj of u i omega j. And now I can use chain rule for differentiation, we can use the chain rule for differentiation this is equal to minus omega j partial ui by partial xj minus ui partial omega j by partial xj plus uj partial omega i plus omega i. The last term here of course, is equal to 0 because of incompressibility, divergence of velocity is equal to 0 in the Navier Strokes equations.

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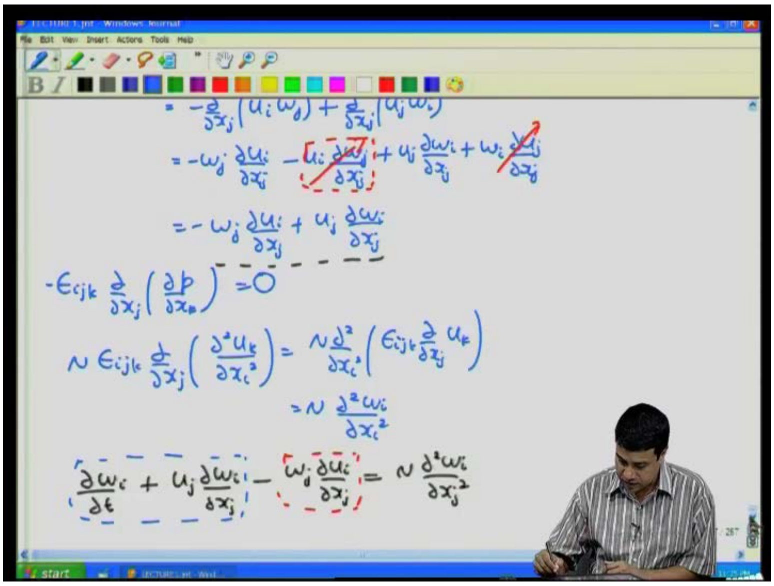

This term here the divergence of the vorticity, the divergence of the vorticity, the divergence of the vorticity is the divergence of the curl of the velocity, you know that the divergence of the curl any vector has to be equal to 0. So, this is also equal to 0. And finally, you are left with minus omega j partial ui by partial xj plus uj. So, that was the second term we had their the next term of course, is epsilon ijk partial by partial xj of partial p by partial xk. Once again the divergence of the gradient of a scalar pressure is a scalar, since we have the gradient of the pressure in the conservation equation if we take the curl of the entire conservation equation you will get 0. So, it is the curl of the gradient of the pressure that has to be equal to 0.

And the last term that we had was epsilon ijk there is the viscosity there partial by partial xj of partial square uk by partial xl square. Once again you can change the order of differentiation to get nu partial square by partial xl square of epsilon ijk partial by partial xj of uk, change the order of differentiation to get it in this form. And of course, you can always do that because the getting partial derivatives and they commute. What is in the brackets is of course, the vorticity itself holds in the brackets is the vorticity. So, this just equal to nu partial square omega i by partial xl square.

So, putting all these together we will get an equation for the vorticity, the conservation equation for the vorticity the first is partial omega i by partial t plus there are two terms here uj partial omega i by partial xj minus omega j partial ui by partial xj is equal to nu partial square omega i by partial xj square. I can take this term to the right hand side take this term to the right hand side the first two terms here are the substantial derivative.

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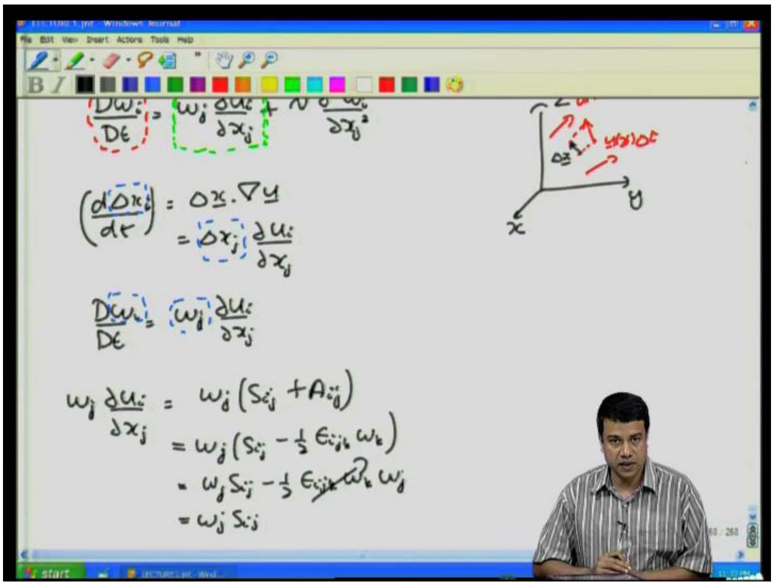

And my final vorticity conservation equation becomes D omega i by Dt is equal to omega j. So, let us look at this equation the term on the left is the rate of change of vorticity in a moving reference frame. So, substantial derivative of the vorticity from the right there are two terms. The second term on the right is the usual viscous diffusion the viscous diffusion of momentum, if you take the curl of the momentum equation you just get the viscous diffusion of vorticity. So, just like this concentration diffusion or heat diffusion or momentum diffusion. In this particular case you do have a viscous diffusion of vorticity as well, and the diffusion coefficient for that is also the kinematic viscosity .

And then you have this other term here, and then you have this other term here this is not a source term because a source term would be independent of the vorticity itself. It is a term that is propotional to vorticity vector times the gradient of the velocity. So, this basically says that there is an increase or decrease in vorticity and that is proportional to the vorticity dotted with the velocity gradient to try to make sense of that one can draw an analogy between the stretching and bending of material lines within a fluid.

So, you have a fluid flow and I have some fluid material element, let me just I have a fluid material element of distance delta x. As time progresses because of the fluid flow this material element is going to stretch and bend. And how does delta x vary? Due to the fluid flow you can see that at some later distance this goes to u at x delta t, this goes to some other location u at x plus delta x times delta t, and you get a new material element.

The stretching of this material element is something we had seen when we looked at deformation, we said that the rate of change the rate of change of this material element with this respect to time can be written as delta x dot the gradient of the velocity.

The gradient of the velocity is the rate deformation tensor is equal to delta xj partial ui by partial xj this is the rate of change of the material element in a reference frame that is moving with the fluid. So, as the material element moves with the fluid one end of the material element moves a distance u times delta t the other end of the material element moves a distance u plus delta u times delta t. Therefore, the change in the distance moved the length of that material element is delta x dotted with partial u by partial x of the deformation tensor.

The vorticity conservation equation if I neglect diffusion is telling me a similar thing, it is telling me that D omega i by Dt is equal to omega j times by partial ui by partial xj. Note the correspondence here between this delta x vector this should be delta xi, if you are writing this in notation and I have omega i here and it goes as delta x dotted with the of the velocity. So, what this is telling you is that vorticity intensification or the change in vorticity takes place in a manner identical to the change in the length of material element. That is vortisity gets stretched and bent in a manner identical to material fluid elements.

So, that is the meaning of this first term here, what is the intensification due to stretching and bending due to the main deformation. So, this is the additional element unlike momentum vorticity is not a conserved quantity, vorticity could be created or destroyed in a flow and the rate of creation or destruction is proposal to the rate at which material elements are expanding in the flow.

This rate is also proportional to just the symmetric part of the rate of deformation tensor. So, just look at this by partial xj can be written as omega j times Sij plus Aij where S and A are the symmetric and anti symmetric parts of the rate of deformation tensor. If you recall the anti symmetric part can be written in terms of the vorticity itself the anti symmetric part can be written as minus half epsilon ijk omega k. There is a relationship between the anti symmetric part and the vorticity vector itself. And this second term omega i Sij is equal to minus this should be omega j, this second part is the cross product of omega cross omega and the cross product of any vector with itself has to be equal to 0.

So, this is equal to 0. So, therefore, I can also write this as omega j times Sij. So, this is only proportional to the symmetric part of the rate of deformation tensor, this vortex intensification mechanism is only proportional to the symmetric part of the rate of deformation tensor.

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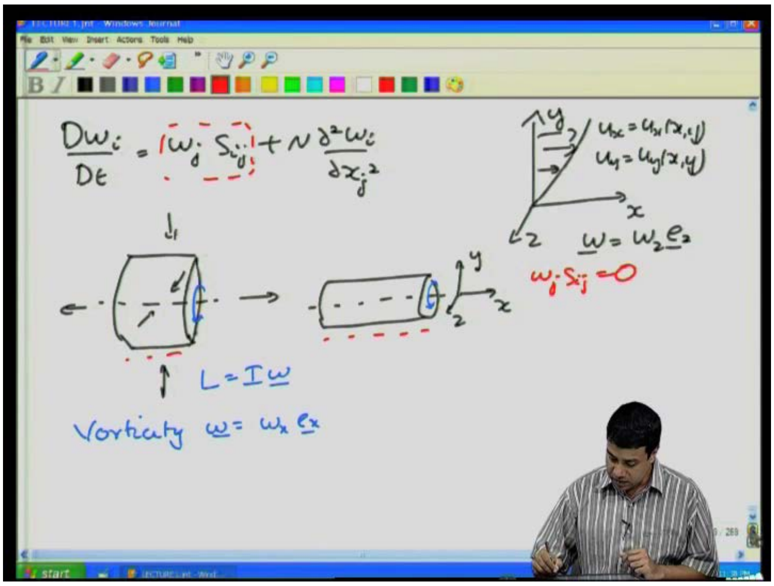

So, my vorticity conservation equation can also be written as D omega i by Dt is equal to omega j Sij plus nu partial square omega i by partial xj square. So, what does this omega j times Sij mean? This is as I said a mechanism for intensification of vorticity due to the stretching and bending of vortex lines. Let us consider a simple two dimensional flow, if you have a simple two dimensional flow with some velocity profile ux which is a is equal to a function of y ux is a function of y. That means, you have a velocity or even it was a general two dimensional flow, it does not have to be unidirection ux is a function of x and y and uy is also a function of x and y that is there is no variation in the z direction, the velocity is purely in the plain of the flow.

So, in that case it is clear that the vorticity vector is the curl of the velocity, it has to be perpendicular to the velocity vector. The velocity vectors are all in the plane of the flow that means, that the vorticity vector can be only perpendicular to the plane of the flow in the plane perpendicular to the that means, that the omega vector is has only 1 component which is in the z direction.

That means, the vorticity vector has only one component which is in the z direction, the velocities are in the x and y directions. The vorticity is only in the z direction that means, that omega j times Sij Sij has only four components s x x s x y s y x and x y y because the velocity is two dimensional the flow is only in the x and y directions, omegas only in the z direction. That means that this terms omega j times Sij is equal to 0 for these two dimensional flows.

So, you cannot have vorticity intensification due to the stretching and bending of vortex lines for a two dimensional flow because the velocity vector is in the plain of flow, the vorticity vectors perpendicular to the plane of flow. The velocity gradient has components only that are in the plain of flow the vorticity is only perpendicular to the plain of flow. That means, that you require a three dimensional flow in order to have intensification of vorticity due to stretching and bending of vortex lines. Physically how does that happen it is a basically a consequence of angular momentum conservation.

So, if I had an element of fluid that look something like this a cylindrical element of fluid by volume conservation if it expands in one direction it has to compress in the other directions. So, let us assume that it is expanding in this direction therefore, it has to compress in the other directions the cylindrical. So, it compresses in all directions around the cylinder it compresses in all directions and it expands in only one direction.

So, this stretching will make this volume element greater length and smaller diameter. Note that this is a three dimensional process this compression in two directions expansion in the other direction, and let us further assume that the vorticity is along the direction of expansion. So, let us just put an x axis here and let us further assume that it is circulating in this direction. So, that the vorticity omega is equal to omega x ex.

Now you can see that between these two configurations the angular momentum of this partial fluid is half I omega square sorry the angular moment of this element of fluid will be equal to l is equal to I times omega, where I is the moment of inertia. When you go from this larger on the left hand side on this larger to the smaller on the right hand side, if the angular momentum is conserved the moment of a inertia has decreased. Because the moment of inertia is proportional to half m times R square, where R is the characteristic distance the length of that cylinder I has decreased therefore, the vorticity has to increase. So, stretching it along the direction of vorticity will basically increase the vorticity compressing it along the direction of vorticity will decrease the vorticity.

So, for Vorticity that is already present in the flow if you stretch it you will increase the vorticity, and if you compress it you will decrease the vorticity. The fact that vorticity intensification or the rate of change of vorticity happens in a manner similar to the length of material lines is also reflected in a theorem called Kelvin’s theorem, which basically states that that the corticity along a moving line element has to remain a constant.

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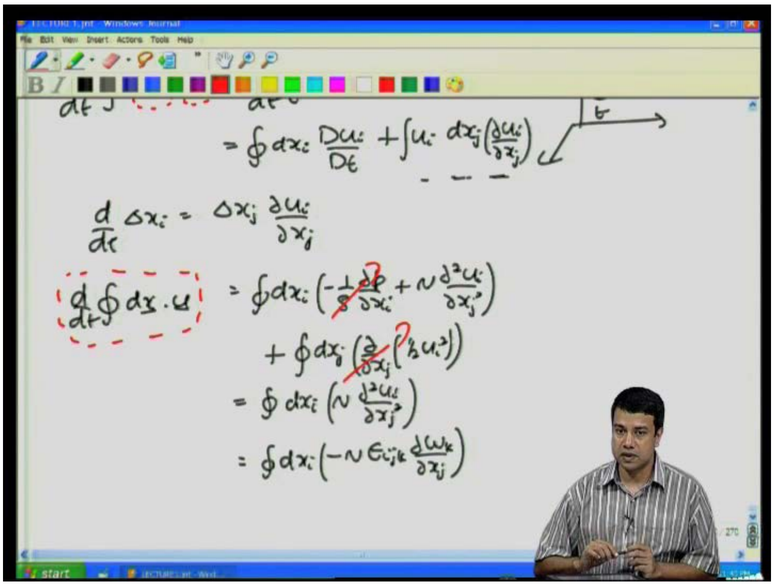

So, this is Kelvin’s circulation theorem you should take the time derivative of the vorticity along some loop within the fluid, this loop is a moving loop. So, it moves along with a material line elements, it goes to from at one time it is here at t plus delta t it goes to some other location this I can write it. I mean there are two reasons, why the circulation changes along this loop one is because the velocity itself is changing, the second is because the line element itself is changing.

So, let us just write the indicial notations for in order to make it clear dxi ui. So, because the velocity is changing there is a rate of change dxi dy by dt. And also because the line element is changing dxi is equal to the rate of change of this line element is dxj times partial ui by partial xj. So, there is a rate of change of the line element delta xi is equal to delta xj times partial ui by partial xj the rate of change the relation that I have just used.

So, this is equal to integral dxi dy by dt is minus plus integral dxj of ui times partial ui by partial xj can be written as integral dxj of partial by partial xj of half ui square.

So, just simplifying this term I have ui times partial ui by partial xj, which I write as partial by partial xj of half ui square. The second term the variant of a function integrated over a closed loop has to be equal to 0 from the gradient theorem. Basically the gradient of some function half ui square integrated over a closed loop dx dot grad 5 integrated over a closed loop has to be equal to 0. In the first term the pressure gradient integrated over a closed loop once again the gradient of a scalar integrated over a closed loop has to be equal to 0.

So, therefore, you find that the rate of change of vorticity dot u ultimately just becomes equal to dxi also it is a kinematic vorticity partial square ui by partial xj square. So, it is equal to kinematic viscosity times the del square of x, if you recall we had also written this as equal to integral dxi of minus nu epsilon ijk partial omega k by partial xj del square u is minus curl of the vorticity.

So, what this is telling us is that there is a change in the circulation along a moving material element only due to viscous diffusion only. So, it basically proves the Kelvin theorem that if you have no vorticity in the fluid then the circulation along a moving material element will not change with time. So, even though the fluid is if the fluid is inviscid you can still have vorticity in the fluid, but that vorticity will be convicted downstream with circulation along any loop remaining unchanged as fluid is being convicted.

So, even an inviscid fluid if you do have vorticity this is telling you that it cannot be generated, it cannot be produced or dissipated in the inviscid fluid which just moves along in such a way that the circulation remains unchanged in the flow. So, this is about vorticity dynamics within the flow, where is Vorticity generated? You have seen an example of that for the flow around a cylinder at the surface because a boundary layer separation you have circulation.

So, vorticity gets generated at solid walls and then it gets convicted through the flow and if there is viscous diffusion it gets diffused as well. What is the mechanism of generation of vorticity at surfaces? We will see that in the next lecture. So, we will continue this vorticity dynamics in the next lecture and then move on to our analysis of turbulent flow, we will continue this in the next lecture. Kindly revise the calculations that we did here, I will go through that briefly in the next lecture. But kindly go through and make sure that you understand all of the algebra that is done here. We will see you next time.