Fundamentals of Transport Processes II
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Lecture - 23
Lubrication flow - Part 2

So, welcome to lecture number 23 of our course on fundamentals of transport processes, where we were in the middle of analyzing low Reynolds number flow, lubrication flow between two surfaces which are very close to each other. As I said these kinds of lubrication flows are often encountered in machinery, where you want to prevent solid to solid contact and therefore often a thin lubricating layer is present between solid surfaces. The thickness of these firms is usually very small.

So, the land scale of the flow is small; it is nearly parallel, because the curvature of the two surfaces is rather gentle, and the fluid used is very viscous, and because of that these three the Reynolds number in these cases is actually small. So, we can analyze these problems using the low Reynolds number, Stokes-flow equations, and we make additional approximations, because the thickness of the fluid firm, the distance between the two surfaces is much smaller than the lateral extent, the extent to which the fluid firm is present.

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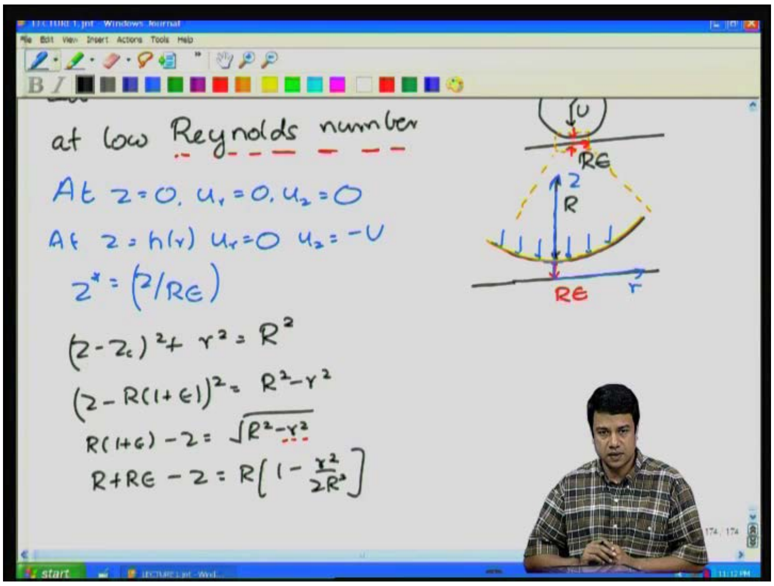

So, we were looking at those classes of problems lubrication flows at low Reynolds number and of course I had to define precisely for you what low Reynolds number is, we had seen it in the last lecture, but we will briefly review it here before going for it. So, the specific problem that we had considered was the flow between a flat surface and the sphere which is coming towards this flat surface with a velocity u. And the assumption here is that the distance between the sphere and the surface which I had called as R times epsilon in the last lecture is much smaller than the radius of the sphere which is capital R.

So, the distance between the sphere and the bottom surface is much smaller than the radius of the sphere. So, we expect that as the sphere is coming down and the distance between the two surfaces is very small; in order for the sphere to come downwards it has to squeeze out the fluid in the gap outwards, because the flow is incompressible. So, in order for volume to be conserved whatever volume is displaced by the sphere as it is coming down has to be squeezed out outward from the film, and because it is squeezed out of a thin gap the velocities are large, the shear stresses are large, and that exerts a large pressure on the sphere which prevents it from coming downwards. So, that was the basic idea.

So, we had expected intuitively that most of the force that is generated on the sphere is going to be generated due to this thin gap between the two surfaces. Therefore, we can focus our attention on this thin gap region between the two surfaces. So, expand this out, and I focus my attention there. Here I have a bottom surface which is flat and I have a top curved surface. The radius, the center of the sphere is of course at a distance which is large compared to the gap thickness. So, this thing is the radius of the sphere R, and this thing is the gap thickness R epsilon. So, we have this axisymmetric configuration where the entire configuration is axisymmetric about an axis perpendicular to the bottom surface which passes through the center of the sphere.

So, because of that I can use a cylindrical coordinate system with this as an axis, because there is no variation as you go around this axis. Therefore, the flow field in this gap, note that I am only focusing on the flow field in this gap, depends only up on the z-coordinate in the cylindrical coordinate system which is vertically upwards, and the R coordinate which is distance from the axis. So, I use a cylindrical coordinate system. This is my z-coordinate; this is my r-coordinate for analyzing the problem. The boundary conditions if you recall at the bottom surface the velocity is equal to 0; that means, that both u r and u z are equal to 0; at the bottom surface z is equal to 0, u r is equal to 0, u z is equal to 0.

At the top surface you have radial velocity is equal to 0; the sphere is coming vertically downward, the sphere is coming vertically downward. Therefore, the radial velocity is equal to 0, and the axial velocity is equal to minus capital U, because it is coming down in the minus z direction. First things first; we scale the coordinates. The z-coordinate in the gap, so at this minimum location; at this minimum location the z-coordinate in the gap varies between 0 and R epsilon. We are going to use the fact that epsilon is a small parameter, and therefore, in the limit as the sphere comes downwards this parameter epsilon goes to 0. Therefore, my gap thickness also is going to 0 in the limit as epsilon goes to zero; however, to solve the problem I should work in a scaled coordinate system in which the boundaries of the flow remain finite in the limit as epsilon goes to 0.

So, I have a parameter epsilon which is going to 0; in the unscaled coordinate the gap thickness also goes to 0. So, every point within the fluid, the z-coordinate of every point within the every fluid, is simultaneously going to 0. However, I should work in a scaled coordinate; that is I should expand my coordinate in such a way that in the limit as epsilon goes to 0 the scaled coordinate remains finite. The scaled coordinate remains finite order one. What do I mean by order one? It does not either go to 0 or go to infinity in the limit as epsilon goes to 0; it remains finite in the limit as epsilon goes to 0.

So, I should define my scaled coordinate such a way that in the limit as epsilon goes to 0 my scaled coordinates, scaled velocity, scaled pressure; they all remained finite, from that I will find out what is the magnitude of the force that is exerted. So, that is the fundamental principle here. I am taking a limit as epsilon goes to zero, but the scaled coordinate has to remain finite. So, therefore it is natural to define z star is equal to z by R epsilon, because at z is equal to R epsilon z star is equal to 1. How do I find out the equation of the bottom surface here? The equation for the bottom surface h of R; we had done it in the last lecture, let me just briefly review that. The equation for the surface is z minus z c the whole square plus R square is equal to capital R square where z c is the height of the center of the sphere.

If you recall the R-coordinate because we have put the axis perpendicular to the surface through the center of the sphere, the R coordinate of the center of the sphere is equal to

0; therefore, there is only a z-coordinate for the center of the sphere. So therefore, this z c which is the z-coordinate of the center of the sphere from the bottom surface you go a distance R epsilon, touch the surface of the sphere; you go further distance R up to the center of the sphere, because the radius of the sphere is r. So, z c is equal to R into 1 plus epsilon. So, I will get z minus R into 1 plus epsilon is equal to square root of R square minus R square; of course, I have to take the square root here. I mean let me go back a little. I have z minus R into 1 plus epsilon whole square is equal to R square minus R square. Now I have to take the square root. The square root on the left hand side has two signs either positive or negative.

For the bottom surface I have to take the negative sign, because at the bottom surface z is less than R into 1 plus epsilon. Therefore, z minus R into 1 plus epsilon has to be negative on the bottom surface. So, we take the negative sign; you just get R into 1 plus epsilon minus z is equal to square root of R square minus R square. And we had found, we had simplified this on the assumption that r is small compared to capital R in the gap. The idea is that in this gap I am going only a small distance from the bottom surface of the gap. Therefore, r is small compared to capital R; with that I can do a Taylor series expansion, and I got R plus R epsilon minus z is equal to 1 minus r square by 2 R square. Doing expansion and keeping only the first term in the series. And as you can see the 1 here will cancel out with this R here, because I have R plus R epsilon minus z and I have R into 1 minus r square by R square here.

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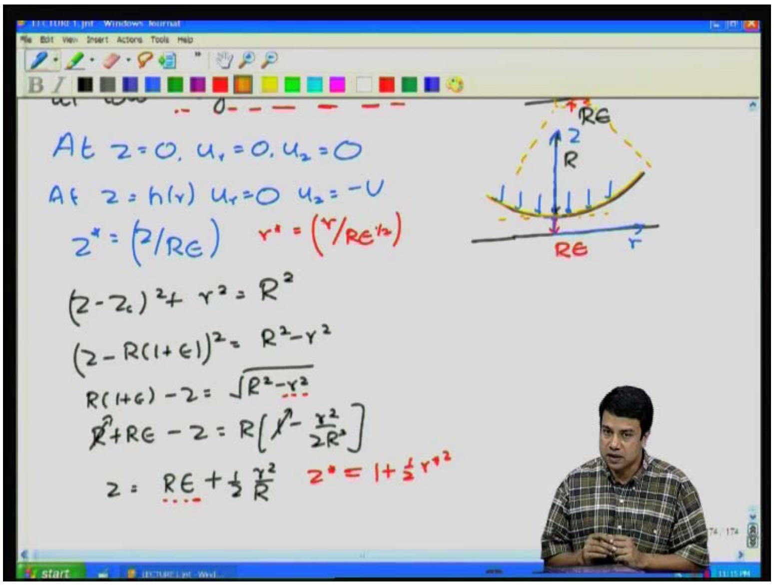

So, the equation for z finally becomes z is equal to R epsilon plus half r square by R, that is we had got in the previous lecture, scaled coordinate z; this provides a natural scaling for z, because I told you that when I define my scaled coordinates that scaled coordinates has to be order one in the limit as epsilon goes to 0. The coordinate z itself is going as epsilon; therefore, if I divide by R times epsilon I get an equation for the scaled coordinates z star is equal to 1 plus half R star square. r star square was defined by dividing this entire equation by R epsilon. Therefore, I find that r star is equal to r by R epsilon power half. In other words the lateral extent of the film goes as R times epsilon power half, the height goes as R times epsilon.

So therefore, the lateral extent is epsilon power minus half larger than the height of the film. So, the lateral extent of the film is much larger than the height of the film in the limit as epsilon goes to 0; that is expected, because if a sphere is coming towards the surface the curvature at the bottom is equal to 0. If I just had a flat disk at the bottom it would be of infinite extent; however, I have a sphere. The curvature the slope is equal to 0 at the bottom, and then the slope increases outward. Therefore, you would expect the lateral extent to actually be larger much larger than the height of the film, and we will use this two advantage in our scaling analysis.

The other important point to note, because the lateral extent is larger in the limit as epsilon goes to 0, we need to squeeze the fluid out of a longer and longer distance in comparison to the height. This dimensional value of R is going to 0 as epsilon power half in the limit as epsilon goes to zero, but the ratio of the lateral extent to the height is actually diverging. So, we had defined the scaled coordinates, and then we had to solve the equations in the scale coordinates.

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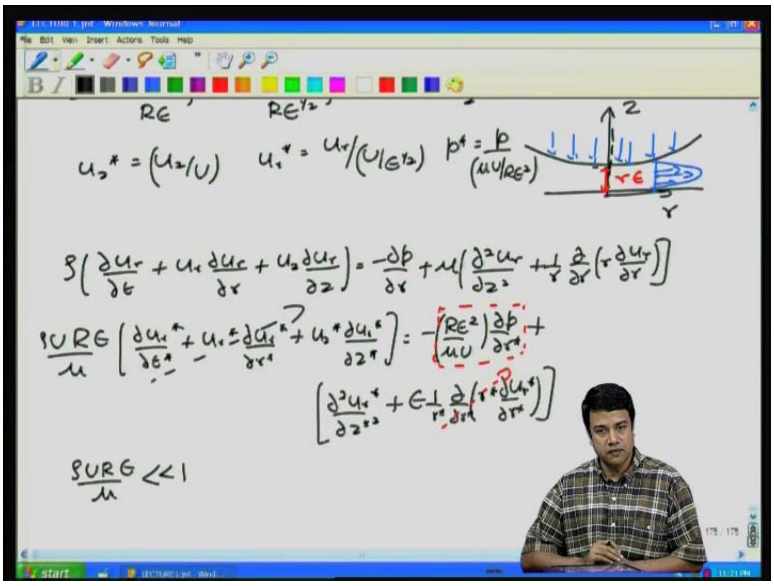

So, z star was equal to z by R epsilon, r star is equal to r by R epsilon power half; they had the height of the film the scaled height is equal to 1 plus half r star square. So therefore, I am defining my boundary conditions at the location 1 plus half r star square. This is r, this is z-coordinate; in this thing this is R epsilon. Next to the scaling are the fluid velocities. The fluid velocities it is natural to scale u z by u itself; it is natural to scale u z by u itself, because u z varies from minus u on the top surface to 0 on the bottom surface; u z varies from minus u on the top surface to 0 on the bottom surface. How do I scale u r? That scaling I obtain from the mass conservation equation is equal to 0.

I express all velocities, I express velocity u z as well as coordinate z and R in terms of the scaled z star r star and u z star, and I will end up with an equation which will give me the scaling for u r. And if you recall in the last class we got that scaling as u r by u by epsilon r half, and with this scaling my conservation equations become equal to 0. So, that is the mass conservation equation. Important point to note; the velocity u R is actually much larger than the velocity u z. If this scale velocity u r star is order 1; that means that the dimensional velocity u R is proportional to u by epsilon power half; even though u is finite epsilon goes to 0, the velocity u R goes to infinity. So, basically I have a large fluid velocity coming out of this thin gap.

The reason is because the radial extent of this gap is large compared to the height. The fluid that is displaced goes as the area that is of the gap which is basically the radius square; all of that volume that is displaced that is u times radius square has to come out of the sides. The side the height of the sides is proportional to the radius times the height, and the ratio of those two gives me that the radial velocity has to be proportional to u by epsilon power half. So, next we go to the momentum conservation equation in the r direction. The momentum conservation equation in the radial direction; once again I express everything in terms of the scaled coordinates except for the pressure. We still have not found out what the pressure is, but once you express everything in terms of the scaled coordinates and divide throughout by the largest viscous term that is expected because we expect the viscous terms to dominate in this case divide by the largest viscous term in the conservation equation.

We get an equation of the form rho u R epsilon by mu partial u r by partial t minus plus the viscous terms which basically become partial square u r by partial z square plus epsilon into 1 by r. So, this k was a scaling for the pressure if you recall, we had in this case the pressure is this term becomes order one; this term becomes order one if we define the pressure as p star is equal to p by mu u by R epsilon square, that gives me the scaling for the pressure. This term is small in the limit epsilon going to 0, and this inertial term can be neglected in the limit rho u R epsilon by mu small compared to 1; in that limit one can neglect the inertial term in the radial momentum conservation equation. So, even though the Reynolds number based upon the sphere radius maybe large, if the Reynolds number based upon the gap thickness is small, inertial terms can be neglected. As the gap becomes smaller and smaller even though rho u R by mu is large at some point before the surfaces touch rho u R epsilon by mu has to become small, and in that case the fluid entirely becomes viscous dominated.

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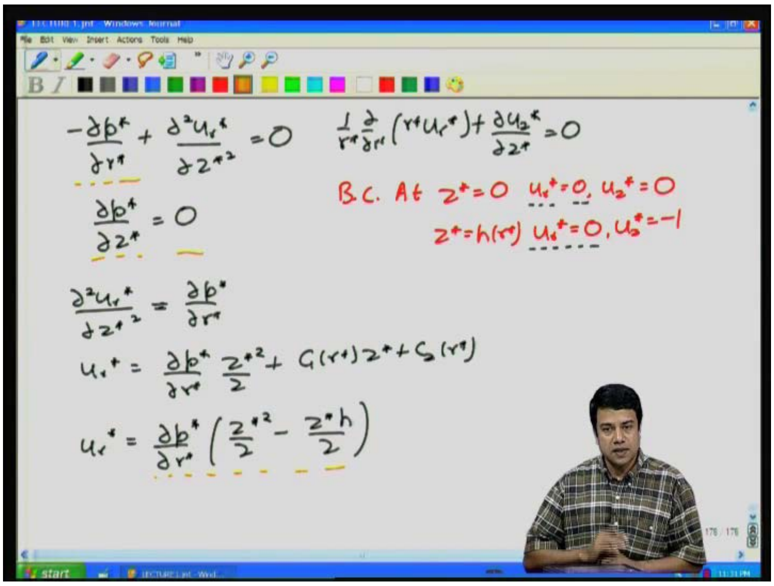

So, that gives me an equation for the R momentum equation minus partial p by partial r plus partial square u r by partial z square is equal to 0. So, that is the radial momentum conservation equation. Now so this is the momentum conservation equation in the radial direction, and from this equation I had obtained the scaled pressure. Next we go into the momentum conservation equation in the axial direction.

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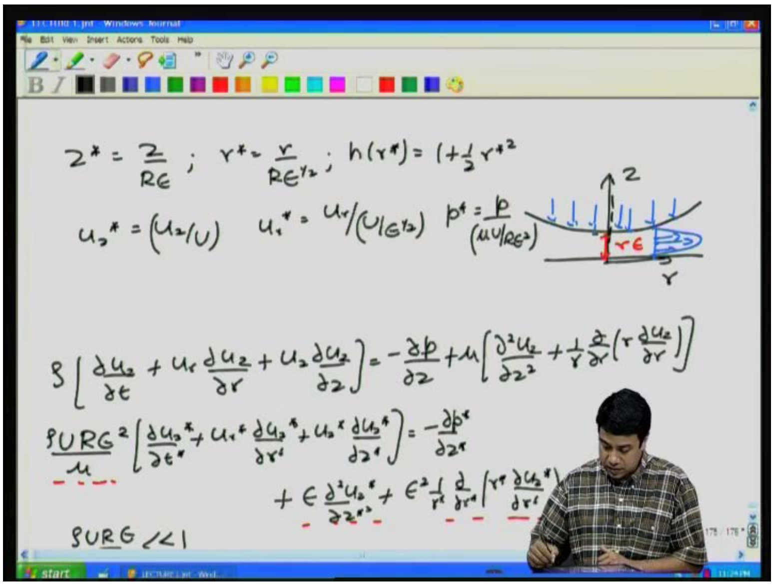

So, the momentum conservation equation in the axial direction is rho into partial u z by partial t plus u r partial u z by partial r plus u z minus partial p by partial z plus mu.

There is a momentum conservation equation in the axial direction. Once again we express everything in terms of the scaled coordinates, and the final equation that we get is rho u R epsilon square by mu into partial u z by partial t plus u r minus partial p by partial z plus epsilon partial square u z by partial z square plus epsilon square. In this case the largest term is the pressure term in the limit as rho u by R epsilon by mu going to 0, this term is very small in the limit as epsilon going to 0 these two terms are both very small, and therefore, one is left with just the pressure gradient in the z momentum conservation equation, and the z momentum conservation equation just becomes partial p by partial z is equal to 0.

Physically why is partial p by partial z equal to 0? The physical reason is because in this gap there is a radial velocity, the shear stress exerted due to the radial velocity is being balanced by the pressure gradient in the radial direction. In the axial direction the velocity is much smaller; as I said the radial velocity is much larger than the axial velocity. Therefore in the axial direction the velocity is much smaller; gradients are also much smaller, and therefore, the axial contribution to the viscous that diverges the viscous stress is much smaller than the pressure gradient in the axial direction. Therefore, in the leading approximation one has to have the pressure gradient being equal to 0. This is common to many flows where the length scale in the radial direction is large compared to the length scale in the axial direction.

In all of these cases the pressure gradient along the flow balances the viscous stresses along the flow, and because the velocity in the perpendicular direction is very small you find that the pressure gradient in the perpendicular direction cross-stream direction has to be equal to 0. So, these are the simplified equations that we have, and we have to solve them subject to boundary conditions. I should add that there is also a mass conservation equation 1 by r d by d r of r u r plus partial u z by partial z, and I have two boundary conditions at z is equal to 0 bottom surface z is equal to 0, u r is equal to 0 and u z is equal to 0, and at z is equal to h of r top surface 1 plus half r star square, u r is equal to 0, and u z is equal to minus 1, because the dimensional velocity is minus u and I have defined the scale velocity as u z by capital u.

So, it is all subject to these boundary conditions. This equation tells us the there is no pressure gradient in the vertical direction; that means that p is independent of z, p star is independent of z star; that means in this equation partial p by partial r is also independent of z, because if some function is independent of z is derivative with respect to r should also be independent of z.

So therefore, I can solve this equation quite easily to get partial square u r by partial z square is equal to partial p by partial r. The right hand side now is independent of z. So, I can integrate it straight away to get u r is equal to partial p by partial r z square by 2 plus c 1 z plus c 2, two integration constants. Note that both c 1 and c 2 could in general be functions of R; both c 1 and c 2 in general could be functions of R, because if I do an integration with respect to z the constant of integration could be a function of R. Now I have two boundary conditions that is at z is equal to 0 u r is equal to 0. If you put this and then you will find that c 2 is equal to 0, and at z is equal to h as well u r is equal to 0.

So, that will give me an equation for u R star, that will enable me to evaluate both constants, and the final solution after I put in these two boundary conditions is partial p by partial r into z square by 2 minus z into h by 2. Note that h is a function of R. With this velocity you can easily verify at z is equal to 0 the velocity is equal to 0; at z is equal to h the velocity is equal to 0. So, it appears that we have solved the problem except that we do not yet know what the pressure gradient is. How do we evaluate the pressure gradient? Physically go back to the problem. The sphere is coming downwards, because it is displacing fluid the fluid has to rush out of the gap. So, obviously in order to drive the fluid out of the gap there has to be a difference in pressure between the center of the gap and the outer flow; between the center and the outer flow there has to be difference in pressure, only then will the fluid rush out of the gap.

How do I calculate the pressure gradient across this gap or the difference in pressure between the center and the outside? That pressure gradient has to be whatever it takes for the fluid for all this the fluid that is displaced to leave the gap. So therefore, this pressure gradient has to come out somehow from a mass conservation condition that with this particular pressure gradient whatever the fluid that is being displaced as the sphere is coming downwards just leaving the gap entirely, so that the sphere can come down. Note that while calculating this velocity profile we still have not used the mass conservation condition. We will calculate with the velocity profile u r with partial p by partial z is equal to 0; we still have not evaluated the pressure gradient. What we would have to do is the mass conservation equation gives us a relation between u r and u z. Note that we have not used the mass conservation equation.

We have also not imposed the boundary conditions for u z yet. So, if I wanted to solve this systematically what I would do is to actually use the mass conservation equation to evaluate what is u z, and insert this equation this z solution for the velocity profile into the mass conservation equation integrated to get u z; that expression since the equation for u z is a first order differential equation that would contain one integration constant have two boundary conditions. So, I would use those two boundary conditions to calculate one integration constant plus the pressure gradient. So, that is the systematic way of doing it. A simpler way to do it is to just take the mass conservation equation integrated over the entire gap. So, let us do it the secondary.

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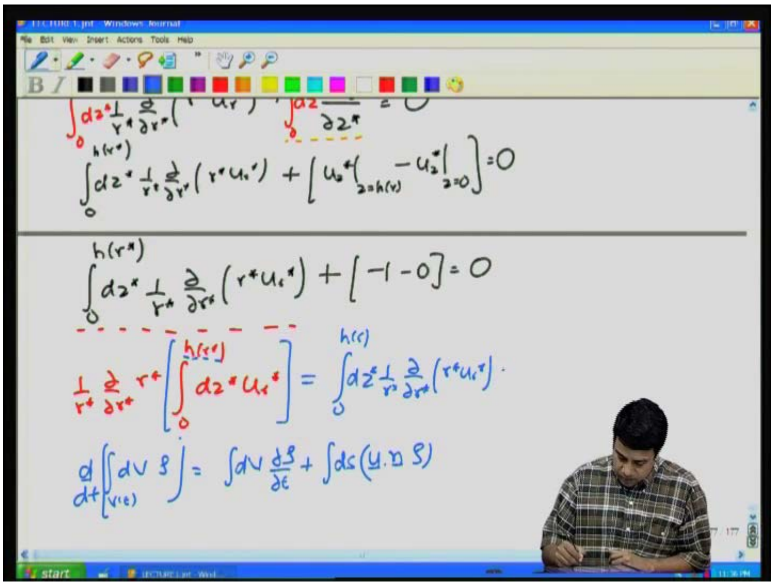

My mass conservation equation is given by 1 by d d by d r of r partial plus partial u z. So, I integrate this over the entire gap at any value of r; it does not matter what the value of r is. So, I do integral d z from 0 to h of r. I integrate two integrals d z from 0 to h of r. This second term is basically an integral of a derivative; the second term is an integral of a derivative. So, that is equal to the value of the function at the end points; the integral of the derivative is equal to the value of the function at the end points. So therefore, I will get integral d z star 1 by r d by d r plus u z at z is equal to h of r minus u z at z is equal to 0 that is equal to 0.

We know what the value of u z at h of r is; that is equal to minus 1, because the sphere is coming down with a velocity u. At z is equal to 0 of course it is equal to 0. So, I get integral 0 to h of r d z plus minus 1 minus 0 is equal to 0, because u z star at z is equal to h is equal to minus 1 whereas u z star at z is equal to 0 is equal to 0. Now this expression is the integral of a derivative in which the limit actually depends up on h of r. It would be more convenient if I could just convert it into the derivative of an integral. In this particular case it turns out to be quite simple and so I will go through that, but in general it is not as simple. In this particular case it turns out that the result is simplified.

So, let us try to examine that. So, if I had an equation of the form 1 by r d by d r of r into integral 0 to h of r d z times u r. This is the integral of a derivative in which the limit depends upon the variable of differentiation itself. This is the derivative of an integral where the limit of integration depends upon the variable which is being differentiated itself. We have seen this before; if you recall when we did the Leibnitz rule I had said that d by dt integral d v times some function rho is equal to integral d v partial rho by partial t plus integral over the surface of u dot n times rho.

In this particular case it is just an integral over a line, and the surface in that case is just the end points, the surface is just the end points. So, the equivalent of the Leibnitz rule in this case is quiet easy to see. It is going to be equal to integral 0 to h of R d z 1 by r, I will rub this off, plus a term that contains the derivative of the limits of integration the derivative of the limits of integration with respect to r plus you will get u r star into 1 by r d by d r of r star h at z is equal to h.

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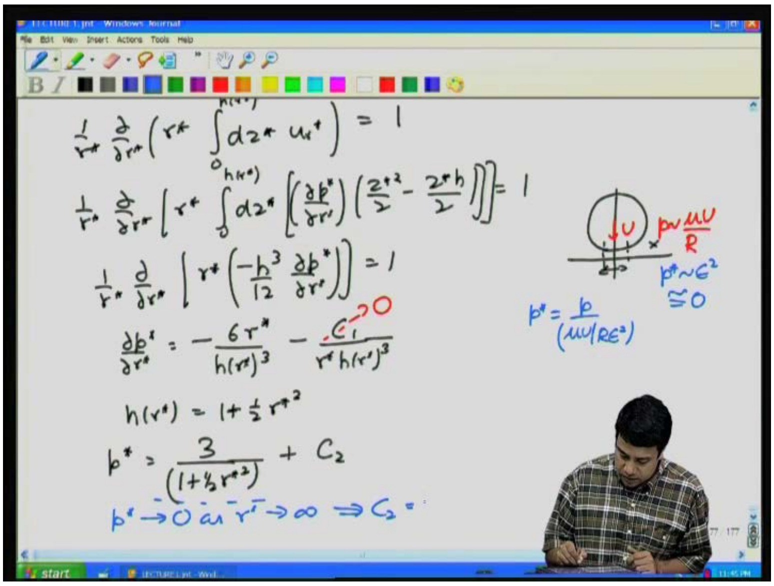

So, this basically is the function. This is the equivalent of rho in this case, and this entire thing is equal to u dot n; u dot n is equal to d by d t of the position on the surface. So, this second term is equivalent to u dot n. In this particular case since u r is equal to 0 on the top surface at z is equal to h u r is equal to 0 that is our boundary condition. Since u r is equal to 0 at z is equal to 0 h this entire term becomes equal to 0. In general when you take the derivative outside or when you take the derivative inside, there is an additional term due to the variation of the end points of integration with respect to the coordinate r. In this particular case since the velocity is 0 at z is equal to h that term ends up being 0, and therefore, in this particular case I can actually take the derivative outside. So, that is in general one has to be careful, but in this particular case it is possible to do that.

So, this basically gives me if I take the derivative outside I get 1 by r d by d r of r star into integral 0 to h of r star d z u r is equal to 1; that is the final solution, and now I have to do this integral, I have to do the integral of u R star. So, this is 1 by r d by d r of r star integral 0 to h of r d z into partial p by partial r into z square by 2 minus z h by 2 is equal to 1. You can do this integral quite easily, and you will find that the result is 1 by r d by d r of r star into minus h cubed by 12 partial p by partial r is equal to 1, and this equation can now be integrated to provide an expression for the pressure. So, if I integrate this equation I get the pressure gradient partial p by partial r is equal to minus 6 r by h of r the whole cubed plus the constant of integration I get minus c 1 by r h of r cubed.

So, this gives me the expression for the pressure gradient, minus 6 r by h of r the whole cubed minus c 1 by r into h of r the whole cubed in all non-dimensional radius. Constant c 1 determinant from the condition that the pressure gradient has to be finite as r goes to 0. R going to 0 is the axis and along the axis you require that the pressure gradient has to be finite; it cannot go to infinity. So, the requirement that the pressure gradient has to be finite implies that c 1 has to go to 0; otherwise, the pressure gradient goes to infinity right at the origin. So, this gives me partial p by partial r, not to get the pressure I have to integrate it one more time. If you integrate it one more time and use h of r is equal to 1 plus half r square, our equation for the surface that we got, use h of r is equal to 1 plus half r square and integrate it one more time; what you get is that p is equal to 3 by 1 plus half r square plus the constant of integration c 2.

What is the value of this constant of integration? We got one constant of integration from the condition that pressure has to be finite at r is equal to 0 along the axis. The second constant of integration comes out from the condition on the pressure as you go far away if the limit as r goes to infinity, because r gap extends in the radial direction from r is equal to 0 all the way to r is equal to infinity. What should the value of pressure be in the limit as r goes to infinity? As R goes to infinity you are outside the gap. Note that I have this thin region; this is the axis. I have this thin region where the pressure is large. As you go outside, if you go outside to some location here the pressure has to decrease to its value in the outer flow.

Sphere is coming down with a velocity u. If you assume the flow is viscous and the sphere is coming down with a velocity u the pressure far away just from dimensional analysis; it is sufficiently far away from the gap, that gap thickness is no longer of parameter, that epsilon is no longer of parameter, sufficiently far away. In that case the only length scale of relevance is the radius of the sphere itself. So, there only that length scale is R, velocity scale is u and therefore, the pressure far away has to go as mu u by r just from dimensional analysis. Therefore, if you go sufficiently far away in the limit as r star goes to infinity the pressure goes as mu u by r.

If you recall I had defined my scale pressure as d by mu u by r epsilon square which means that as r goes to infinity I have to write pressure goes as mu u by r; that means p star which is divided mu u by r epsilon square has to go as epsilon square, the scaled pressures, scaled by the appropriate pressure scale within the gap has to go to as epsilon square in the limit as r goes to infinity, and the limit as epsilon goes to 0. This is effectively equal to 0, because outside the gap the length and velocity scales are just capital R and u. Therefore, the pressure has to go as mu u by r; inside the gap the appropriate scaling was mu u by r epsilon square. Therefore, this gap scaled pressure has to go as epsilon square as r goes to infinity and as epsilon goes to 0 this is equal to 0. So, p star goes to 0 as r star goes to infinity; as R star goes to infinity you can easily verify that this term is equal to 0 because 1 over 1 plus half r star square. So, that implies that c 2 is going to be equal to 0. So, that has finally given as an expression for that pressure in the gap.

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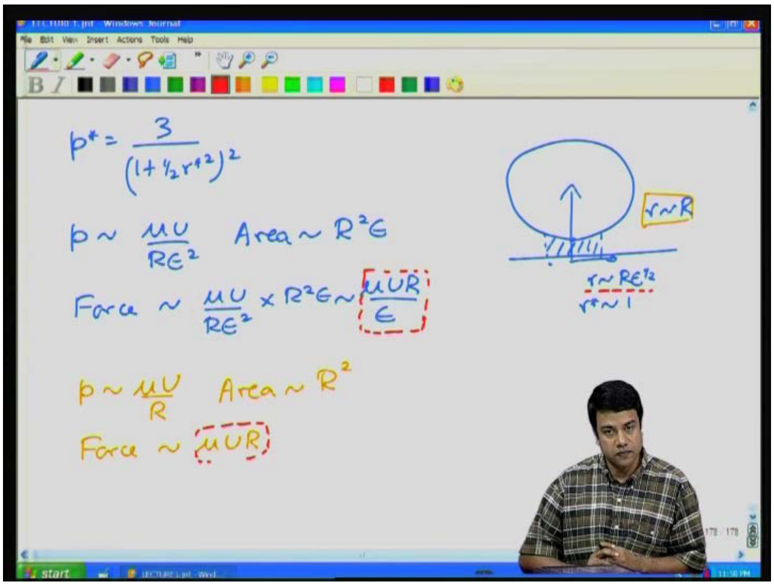

So, the pressure in the gap is given by p star is equal to 3 by 1 plus half r star square whole square. Kindly make a correction here; there should be since this term goes as 1 over h cubed 1 plus half r star square the whole cubed this has to go as 1 plus half r star square the whole square in the gap. So, that was the pressure. Our next task is to find the force acting on the sphere. Where do you expect the maximum contribution to the force to come from? Let us discus this a little bit. I have a sphere just coming towards the surface. So, I divided into two regions; one is where r goes as r epsilon power half; therefore, r star goes as 1, and the other is the outer region where r star is equal to where r goes as R.

What do I expect the force in this thin gap to be? My pressure p scales as mu u by r epsilon square; that is the pressure within the gap region. What is the area of the gap region? The area is proportional to r square projected area within the gap region. The radius is proportional to r times epsilon power half. Therefore, the projected area is proportional to R square epsilon; because r is proportional to R times epsilon power half; that means the projected area is proportional to R times epsilon. On this basis you would expect the force goes as mu u by R epsilon square into R square epsilon is equal to mu u R by epsilon. Let us be careful here; it is a numerical constant here. So, that is the force coming out of this thin gap region.

That is the force coming out of this thin gap region where z is proportional to epsilon, and the radius is proportional to R times epsilon power half. What about this outer region here; what about this outer region here? In this outer region you would expect the pressure. If the flow is viscous dominated you expect the pressure to scale as mu u by R. Reason is because there is now no flow in the outer region does not depend upon epsilon anymore because where if the outer region that small gap thickness epsilon is no longer a factor. Therefore, the flow in the outer region cannot depend upon R times epsilon; it depends on the only length scale available that is R itself. Velocity is u and therefore, the pressure goes as mu u by R.

The area goes as R square because in the outer region the area the surface area of the sphere is 4 pi r square projected area is pi r square. So, the area goes as r square. On this basis you would expect the force in the outer region goes as mu u times R. Clearly in the limit as epsilon goes to 0 this force that is coming out of this gap region mu u R by epsilon is much larger than the force exerted in the outer region; that was the rational for our focusing on this gap region, because as the particle comes down as the sphere comes down it generates large outward velocities. In order to push the fluid out you need a large pressure gradient and that could exert a large force on the sphere itself. So, that was the rational for focusing on this region; therefore, it expects the force due to the flow in this thin gap to be much to cause the dominant contribution to the force. Now this normal force is quite easy to calculate. It is just equal to the pressure times the projected area, the pressure times the area perpendicular to the surface.

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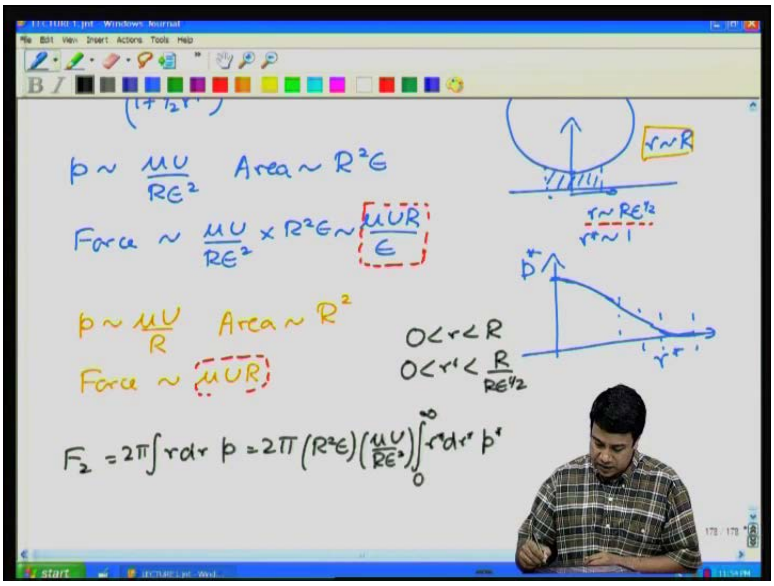

So therefore, the total force in the z direction the total force that is exerting upwards in the z direction is going to be equal to integral r d r times the pressure. So, if you look from the top, if you look at the same configuration from the top, you have one bottom surface, you have bottom surface I am looking from the top. And I have this sphere which is settling downwards, and I have this region this thin gap of thickness R epsilon at the bottom. So, the projected area is equal to 2 pi r times d r. So, I have a factor 2 pi here, 2 pi r times d r times the pressure itself, and I can now express it in terms of the scaled coordinates, because R is equal to capital R by epsilon power half times R star.

So, I will get 2 pi into R square by epsilon into R star, and the pressure scaling is equal to mu u; I am sorry mu u by R epsilon square times the integral, integral of r d r times p. This has to go from 0 to some large value. What is the upper limit of integration? Of course, the upper limit of integration has to be the radius of the sphere itself, but as the distance from the axis becomes comparable to the radius of the sphere the approximation that we had used is no longer valid, because we are concentrated only on a distance of order epsilon power R times epsilon power half along near the axis. As you go far away the approximation that we had used is no longer valid; however, that does not really matter. The reason is as follows.

If I plot the pressure profile, if I plot the pressure as function of the radius R, p star is function of R star, it has some large value; it has some value at the center and then as you go far away it goes to 0. As you go far away it goes to zero, because p scales as mu u by R which means that p star is proportional to epsilon square. So, of course one can get different results by taking different limits of integration here, but if I go sufficiently far away the pressure itself goes to 0. So, any additional contribution I get to the integral will be 0 because the pressure is already gone to 0 sufficiently far away. So therefore, I can do the integration all the way from 0 to infinity without loss of ambiguity. The reason is because if I go sufficiently far away am making an error in the limit of integration, but the pressure there is 0 anyway.

If I go from 0 to infinity I am making an error in the limit of integration, but since the pressure is 0 there anyway it has already decreased to 0 as you go far away, there will be no net contribution to the integral. So, without loss of generality I should be taking this limit of integral as of r going from 0 to capital R which means that r star would imply that 0 less than r star less than 1 by epsilon power half. It is basically 1 over epsilon power half; however, in the limit as epsilon goes to 0 this limit of integration goes to infinity, and I can do the integral without correct to leading order epsilon by just taking the upper limit as infinity. Upper limit infinity is always possible in this case because my integral is convergent. As I recall the pressure is equal to 1 plus half r star square in the denominator. So, if I integrate from 0 to infinity the denominator goes as r star power 4, the numerator goes as r star times t r star and I get a convergent integral.

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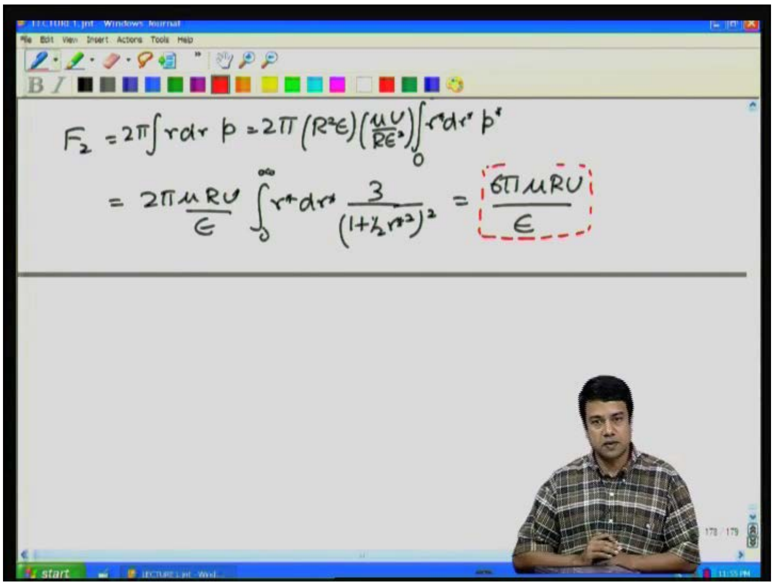

So, with that I will get 2 pi mu R u by epsilon integral 0 to infinity r star d r star into 3 by 1 plus half r square whole square. This turns out to be equal to 6 pi mu R u by epsilon. So, this is the leading order contribution to the force exerted on the surface. Note that this force was a pressure force, because as the shear came down there had to be a radial flow generated, and the pressure gradient which was generated was sufficient for all the fluid to flow, so that the sphere can come down. You can see that this force increases proportional to 1 over epsilon in the limit as epsilon goes to 0. This has an important lesson, because the force increases as 1 over epsilon the force goes to infinity as the surfaces come closer and closer. And that is why in lubrication approximations in machinery for example, you will almost never have solid to solid contact.

The reason is because the force required for that for the force required for bringing two solid surfaces towards each other with a velocity u increases as one over the gap thickness. And that is the reason that two solid surfaces in real applications with an intervening viscous fluid film will never come into contact. The force goes as 1 over epsilon. The work done to bring them together as to go as force times the distance the distance is proportional to epsilon. So, if you integrate 1 over epsilon over distance you will get a force the work done going as log of the distance between the two surfaces, 1 over x d x; the force goes as 1 over x. You integrate that over the distance x you get a logarithmic function. Therefore, the force goes to infinity as 1 over the distance, the work required to get two surfaces close to each other goes as log of the distance.

Of course, this approximation this calculation assumes that the system is always in the continuum level. In the case of liquids that is a good approximation. The reason is because the microscopic scale in a liquid is comparable to the molecular distance in simple liquids that is of the order of angstroms, and therefore, this expression for the force is valid down to that length scale, and that is why lubrication works so well. The reason is because in liquids you need to bring them down to angstrom scale distances in order for the continuum approximation to fail. In gases the length scale is the mean free path as we calculated at various stages goes between 0.1 microns to it is about between 10 power minus 2 to about 1 microns for normal gases. And in that case one could have a situation where the surface is as sufficiently close with the continuum, the approximation breaks down. In that case this calculation is no longer valid.

So, to recap physically the problem that we started off on distance between the surface is very small and therefore, the gap the lateral extent of the gap is large compared to the vertical distance. Flow is almost unidirectional. If you recall that the velocity profile that we got it was essentially a quiet flow; I am sorry the velocity profile that we got was essentially a parabolic velocity profile between two flat plates two good approximation; that is because the lateral distance was much larger than the distance between the gaps. In order to generate that velocity you have to have a pressure gradient. Just as you know in a pipe in order to generate the velocity you need a pressure gradient; that pressure gradient has to be sufficient to displace the necessary amount of fluid to ensure that the surface comes down.

That requires a large pressure gradient, and even though the pressure gradient required for that goes as 1 over epsilon square where epsilon is the gap thickness, the lateral extent goes as epsilon power half which means the theory is proportional to epsilon. Therefore you get a very large force proportional to 1 over epsilon, and that is the reason why the lubrication works real in practical applications; why we can use lubrication to prevent solid to solid contact between nearby surfaces. So, this is an example of another application of viscous flows where we neglect inertial effects, and we also find no pressure gradient perpendicular to the flow.

And as I explained to you earlier it is very important in ensuring that there is no solid contact between two surfaces. So, this completes our discussion of viscous flows. In the next lecture we will briefly look at inertial flows. First of all we will see where the viscous limit where the viscous approximation breaks down, and then we will go to the class of flows where inertia is dominant. These are called potential flows. We include pressure forces, but we still neglect the viscous diffusion, and we look at potential flows in the next lecture, will see you then.