Fundamentals of Transport Process II
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Fundamental of Transport Process II

Lecture - 22
Lubrication Flow Part I

Welcome to lecture number 22 of our course on fundamentals of transport processes. We were in the midst of discussing viscous flows, lower Reynolds number flows, where the inertial terms in the conservation equation are neglected. In the previous few lectures, I had discussed primarily viscous flows around objects suspended in a fluid, we showed how the equations for the stokes equation for mass and momentum conservation can be reduced to Laplace equations. And solved in a manner similar to the solution for the diffusion equation that we had done in fundamentals of transfer processes one.

I tried to give you a physical understanding of what this expansions mean, if the partial exerts a net force on the fluid it is like a source of momentum. And you get a velocity field that decays as 1 over r. On the other hand if there is no net force, but there is still the disturbance because the particle is located in the fluid, and there has to be no slip conditions, satisfied on the surface of this particle. In that case there is no net force, but you get what is effectively force moment, a force dipole.

Physically, it is like having two opposite forces separated by a small distance, in such a way that the net force is equal to 0, but the force moment integral of the force times the distance from the centre, over the entire surface is non zero. It is like a di-pole moment and that results in a velocity field that decays as 1 over r square. You could have more complicated situations 1 over r cubed about correspond to a force quadruple and so on. So, that was the basis of our discussion of viscose flows, a round objects I had also briefly showed you, how we can deal with viscose flow in a internal viscose flows.

In two dimensions there is a simplification that one can make by expressing, the velocity field in terms with the stream function. So, you use the mass conservation equation to reduce two velocities to just one stream function. And then the viscose the stokes equations reduced to a bi-harmonic equation for the stream function, which can be solved by the standard methods. We had seen one particular method to solve this in polar coordinate system, for the flow in a corner in the previous lecture. This lecture I would like to start on another important class of flows, they are called as lubrication flows.

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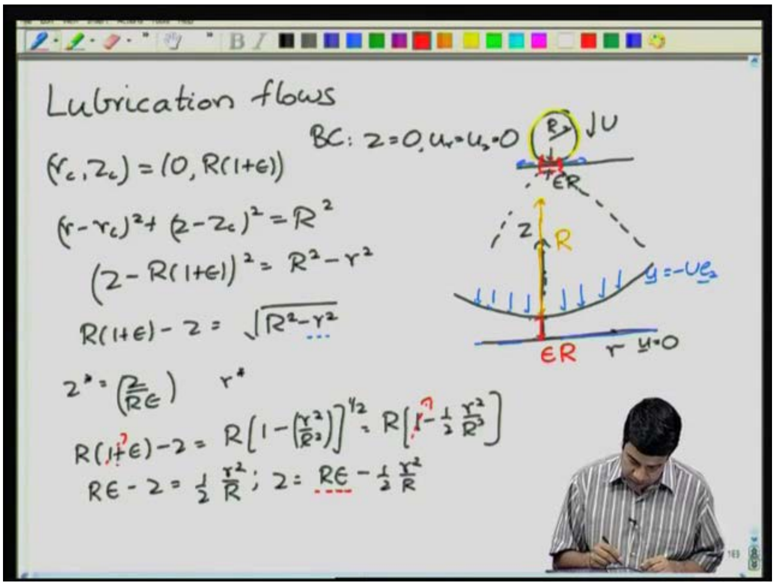

Now, these lubrication flows are between surfaces that are very close to each other, the classical example of lubrication flows is. For example, if the lubrication of a piston in a cylinder in a car for example, or in all sets of machinery where one wants to prevent solid to solid contact. Therefore, one puts in a thin viscose layer of fluid usually, an oil or a grease in between the gaps between two moving parts because the thickness of these gaps is very small the flow in these gaps is usually, at low Reynolds number. There are two things, one is the thickness is small. Secondly the viscosity of the fluid is very high.

And that is why the flow in these gaps is usually, in the very low Reynolds number regime. And one can use stokes flow to analyze, stokes flow equations to analyze these kinds of flows. There are two things that happened there, the first is that the gap is small the flow is very viscose therefore, for the Reynolds number is very low. Secondly, the flow is primarily unidirectional. Since, we have two surfaces that are in close contact with each other, the flow is primarily tangential to both of those surfaces. The component of the velocity perpendicular to the surfaces is small, compare to the component of the velocity along those surfaces, tangential to the surfaces.

However, it is not strictly speaking a unidirectional flow in the sense that there usually, this curvature and there usually, it is a small component of the velocity that is perpendicular to the flows though that component has to become 0, at the surfaces themselves. So, these two things distinguish lubrication flows from other general classes of flows, and we can use the low Reynolds number approximation in addition to scaling arguments, in order to simplify the equations and get a simplified set of equations. And solve these in order to obtain, what is the force on the objects that are very nearly in contact.

Very often the force is due to the pressure force require to drive the fluid into or outside this particular small gap. So, the fact that you are moving fluid through a small gap generates very high velocities, and very high shear stresses. And the pressure require to generate that is basically, the resistance to motion perpendicular to surfaces. So, we will see these lubrication flows in a simple example of as sphere, sphere coming towards the flat surface. A spherical partial coming towards to a flat surface in the limit, where the gap between the particle and the flat surface is very much smaller, than the radius of the particle.

So, the system is as follows I have this spherical particle of radius R, coming down with a velocity U towards a flat surface. And the distance between the particle and surface is small compare to the radius of the particle. Therefore, I will call this distance as epsilon times R in the limit as epsilon goes to 0, the distance between the two surfaces becomes much smaller than the radius of the particle. And in this limit, we would like to find out what is the force that is resisting the motion of the particle towards the surface.

As I said we will analyze this in the stokes flowerage area. That means, that the force at an instant in time is determined in only by the positions, and velocities of all the surfaces at that instant in time, recall stokes flow equations quasi study. Therefore, there is no time dependence within the equations. So, given a configuration particle coming towards a surface with specified distances, I can calculate the force without knowing what the history of the system velocity in time. So, this is all I need you know to calculate force the velocity of the particle the configuration. How do I analyze this?

Physically, I would expect at most of the force will be exerted in this little region between the particle, and the surfaces. The reason is because as the sphere is coming towards the surface whatever, fluid is there has to be displaced out of the surface. And that flow it has to come out of the surface with other high velocities. High velocity between two surfaces generates high shear stresses and therefore, that force due to the shear stress in between the two surfaces exerts a force upward on the particle, in order to exert the down word motion.

Of course, we have to first do the calculation of what is the force within this thin gap. Estimate also the force everywhere else on the surface of the particle, and verify in the end that the force in this thin gap is actually, much larger than the force on the outer surface of the particle. There is the force on the outer surface of the particle should be much smaller than the force within this thin gap between the particle, and the surface. So, first we will calculate the force and then verify that this force is actually large.

So, in order to solve this problem, I focus my attention on a thin region between the flat surface and the particle. So, I expand this out focus on the thin region between the surface and the particle. So, this is my flat surface this is the surface of the particle and this particle is coming down towards the flat surface, and it has one particular access of symmetry, it has one particular access of symmetry it is access symmetric about this access perpendicular to the flat surface passing through the centre of the particle.

So, we can use that simplification, there should be no dependence on the angle around that access. So, first thing we have to specify the configuration and the boundary condition. As I said the particle radius is R, this distance this distance between the bottom of the particle and the surface. As I said is epsilon times R that distance is epsilon times R the radius of the particle this capital R. So, this distance the radius from the centre to the surfaces. So, since this configuration is access symmetric about this access I can use in access symmetric cylindrical coordinate system, in this coordinate system the access along the plain is R we access perpendicular to the surface is z.

Of course, this is the third access theta going around this access, there is a third coordinate theta that goes around this access, but because the particle is sphere and the surface is flat there should be no dependence of the velocity on the angle theta. Therefore, we have a two dimensional problem, first thing we have to specify the boundary conditions. On the bottom surface which is stationary the velocity of course, has to be equal to 0. So, U vectors equal to 0 on bottom surface that is both u r and u z about 0 on the bottom surface. The sphere is coming down with a velocity U, this s sphere is coming down with the velocity U. Therefore, on the surface of this sphere I have U vector is equal to minus U times the unit vector in the z coordinate.

The radial component of the velocity is 0, the z component of the velocity is equal to minus U on the surface of the sphere. Therefore, my boundary conditions bottom surface just shifted, the boundary conditions at the bottom surface, this bottom surface this is at is bottom surface is at z is equal to 0, z is equal to 0, u r is equal to u z is equal to 0 both components of the velocity u r and u z are both equal to 0. How about the top surface the top surface is little complicated, it is a surface of a sphere this sphere has is radius capital R. And the centre of this sphere is located the centre of this sphere in the centre coordinate r and z of this sphere.

The r coordinate of the centre of this sphere of course, 0 because we have taken the access to a passing through the centre of the sphere. Therefore, the r coordinate is 0. The z coordinate I fixed my origin at the bottom surface. So, the distance to the top surface is epsilon times r the distance from there to the centre of the sphere is r itself. Therefore, the top surface is at r into 1 plus epsilon. So, centre of the sphere is at r c is equal to 0, z c is equal to r into 1 plus epsilon. What is the equation of the surface of this sphere?

The equation of the surface of the sphere is r minus r c the whole square plus z minus z c the whole square is equal to capital R square. Therefore, I get r c is 0 so, I get z minus R into 1 plus epsilon the whole square is equal to R square minus r square. Now, on the bottom surface of this sphere z is less than r into 1 plus epsilon. So, in order to find out what is the equation for z as a function of r, I can just take the square root of this equation.

Take the square root of the left hand side there are two roots, one is positive z minus R into 1 plus epsilon here there is negative z minus r into 1 plus epsilon. On the bottom surface z is less than r into 1 plus epsilon. So, when I take the square root for the bottom surface I should use R into 1 plus epsilon minus z is equal to R square minus r square. If I were to describe the top surface of this sphere, I would say r into 1 plus epsilon minus z because z is greater than r into 1 plus epsilon there, but my attention is focused on the bottom because that is where that gap is there thin gap is.

Now, we will start to scale the equations the appropriate a length scale for the z is of course, is R into epsilon because this gap thickness, this gap thickness is R into epsilon.

So, I can define a scaled coordinate z star is equal to z by R epsilon. Now, the question is what is R star, what is the equation for R star the equation for R star will get from the surface itself as follows. If I assume that this radius this radius small r, in this gap is small compare to capital R, I can use a series expansion Taylor series expansion for this I would expect the small r to be small compare to capital R because I am focusing in thin region, as you can see here on a thin region near the bottom and I am expanding that.

Therefore, I would expect the within the thin region small r is small compare to the total radius of the sphere itself. So, in that case I can use a binomial expansion sorry it is a Taylor series expansion 1 plus epsilon minus z is equal to r into 1 minus r square by R square w whole power half, and if I just retain only the first term in the expansion I get 1 minus half r square by R. I am sorry. So, this was assuming that r is small compare to capital R, we just retained the first term in the series.

Now, if I express this in terms of z star I can of course, cancel out this one on the right hand side, I can cancel out I have an r on the right hand side, r into 1 in the right hand side and r into 1 on the left hand side. So, those two get cancelled out and I will get R epsilon minus z is equal to half r square by R, or put another way z is equal to R epsilon minus half r square by R. So, the length scale for the z coordinate is set basically by this r epsilon, I can divide throughout by r epsilon.

I can divide throughout by R epsilon and I will get so, there is a negative sign that I am missing here. So, this becomes z is equal to so let me just work this out once again this is important because we have a negative sign that is missing, and therefore, I will work it out once again. So, the equation for the surfaces is R into 1 plus epsilon minus z is equal to square root of R square minus r square.

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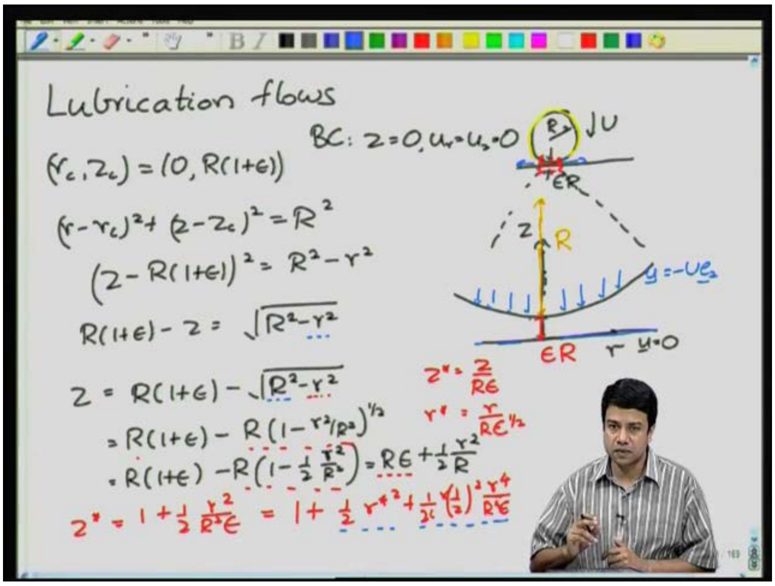

Therefore, z is equal to R into 1 plus epsilon minus square root of R square minus r square. Now, if I assume that small r is small compare to capital R of course, I am focusing my attention on a thin region near the bottom here, I am focusing my attention on a thin region near the bottom here within this thin region, I have a thin gap and that I would expect that both the vertical coordinate in this gap. As well as the radial coordinate in this gap are both much smaller than the radius of this sphere.

Therefore, I would expect that r is small compare to capital R and in that limit I can use an expansion Taylor series expansion in the limit, where this small r is small compare to this capital R. So, in that limit I can use the Taylor series expansion and I will write this as 1 plus epsilon minus R into 1 minus r square by R square w whole power half. For this 1 minus r square by capital R square w whole power half, one can use a Taylor series expansion and retain just the first term in the series.

1 minus x w whole power half in the limit x goes to 0 is 1 minus half x plus terms of order x square. So, this becomes r into 1 plus epsilon minus r into 1 minus half r square by R square. And you can see that this r into one cancels out on both sides and this equation just becomes R epsilon plus half r square by R square capital R. As you can see the scale, the length scale for the height in this case this is just equal to R times epsilon.

The length scale for the height this case is just equal to r times epsilon therefore, I can define a scaled z co-ordinate as z star is equal to z by r epsilon because I would expect in that case z star is equal to 1 at the bottom of the surface of the sphere. So, how do I get and equation for the scaled z coordinate I divide throughout by R times epsilon, I divide throughout by R times epsilon on the left hand side I get just z star is equal to 1 plus half r square by R square epsilon, by R square epsilon. This equation also provides us a scaling for the radial coordinate, I told you that in the thin gap r is small compare to capital R. How small comes out of this equation?

As you can see z star the scale z coordinate is order 1 is equal to 1 plus half r square by epsilon times capital R square. Therefore, in this thin gap region I would expect that r square by capital R square epsilon is also order 1. So, that provides me a scale radius r star is equal to r by R epsilon power half, r star is equal to r by R epsilon power half. So, with this scale coordinate the equation becomes the surface of 1 plus half r star square. So, that gives me the equation for the surface correct to leading order in an epsilon expansion, only in the region where z is comparable to R times epsilon. As you go further and further outwards of course, z will become comparable to capital R in that case this equation is no longer value.

However, we are focusing our attention on this thin gap region and in this region I make this approximation for the z and the R coordinates. Of course, I have expanded this up to second order I have expanded this term here, I have retained only the R square terms. I could have retained the r power 4 terms as well. I would have got r power 4 by capital R power 3, when I express it in terms of this scaled coordinate you can very easily verify that the next higher correction this order epsilon smaller than this one.

So, the next time that I will get is plus 1 by 2 factorial into 1 by 2 whole square times r square by sorry times r power 4 by capital R power four times epsilon. When I express this in terms of r star, this term will turn out to be order epsilon smaller than this one. So, in the limit as epsilon goes to 0 this next, higher order term will be small provided z is comparable to capital R times epsilon. So, it is in this limit that I am doing the calculation, I am neglecting all terms of order epsilon and smaller in the limit as epsilon goes to 0. So, inherent in this calculation is that we are taking the limit as epsilon is going to 0, and we are finding the largest contribution to the force in that particular limit.

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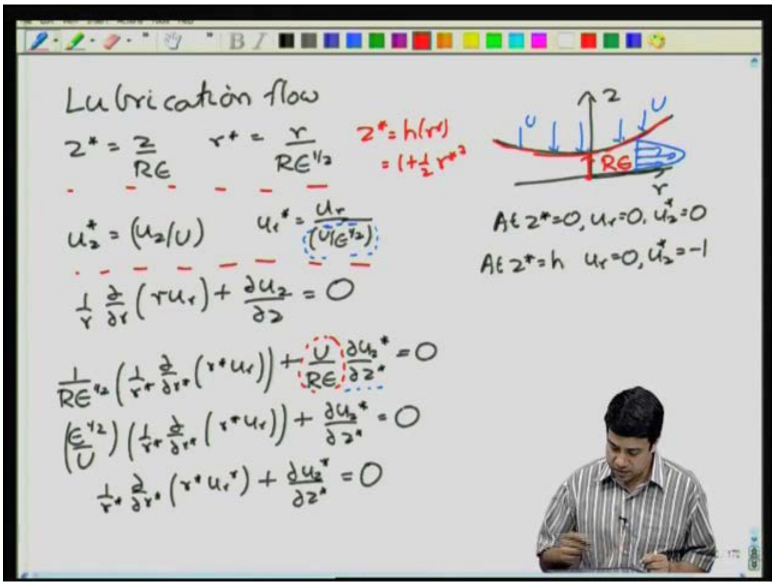

So, to recap for our lubrication flow problem, I have a bottom surface, I have a top surface. The top surface is coming down with the velocity u, the bottom surface is stationary I have used a coordinate system and r z coordinate system in which this gap thickness is r times epsilon, this gap thickness is R times epsilon, radius of this sphere is equal to r. So, I defined the scaled coordinates z star is equal to z by R epsilon, r star equal to r by r epsilon power half. In terms of this the equation for this top surface, this bottom surface of this sphere that is very close to the flat surface. Equation for this bottom surface of this sphere that is very close to the flat surface is given by z star is equal to I will call it the height this is equal to 1 plus half r star square.

So, this has given me the height function for this bottom surface and I have to apply boundary conditions on this height function. So, boundary conditions are at z star is equal to 0 u r is equal to 0, and u z is equal to 0, no slip condition. And at z star is equal to h, h is the function of r. Radial velocity is once again equal to 0 u r is equal to 0 and u z is equal to minus u because the sphere is coming downwards with the velocity u. Therefore, u z is equal to minus u good.

Now, we proceed to solve the mass and momentum conservation equations before that we can scale the velocities. The natural scaling for the z velocity, the natural scaling for the z velocity is equal to u z by capital U, the reason is because on the surface of the sphere itself velocity is u downwards at the bottom it is 0. So, your expected velocity to be varying between minus U and 0. Therefore, the scaled velocity will vary between minus 1 on the sphere surface and 0 at the bottom. So, scale this way the velocity is order 1 in the limit as epsilon goes to 0 it varies between minus 1 and 0.

So, expressed in terms of the scaled velocity u z star turns out be equal to 0 on the bottom surface, and u z star is equal to minus 1 on the top surface, on the top surface. What about the radial velocity how do we scale u r turns out the simplest way to scale u r is to go to the mass conservation equation because as the sphere is coming down whatever, fluid is being displaced because the sphere is coming down has to leave the gap radially. So, that conservation of mass gives me a relationship between the velocity with which the sphere comes down, and velocity with which the fluid has to leave the gap in order to maintain conservation of mass.

The mass conservation equation in this case is 1 by r d by d r of r u r plus partial u z by partial z this is equal to 0 so, that is the mass conservation equation in this case. I express the mass conservation equation in terms in scaled radius z coordinate as well as the velocity u z. So, u z is equal to capital u times u z star z is equal to r epsilon times z and radius radial coordinate is R epsilon power half times r star. So, if I express these in terms of the scaled coordinates, what I get is 1 by R epsilon power half into 1 by r star d by d r star of r star u r plus u by R epsilon partial u z star by partial z star is equal to 0.

Radial velocity u r is not yet scaled. So, what I do is I divide throughout by this co-efficient here divide throughout by this coefficient here because I would expect. Since, u z and z r scaled this ratio partial u z star by partial z star is going to be order 1, in the limit as epsilon goes to 0. So, this term I would expect to be order 1 because it is the ratio of 2 scaled quantities. Therefore, I divide throughout by the denominator, what I will get is a scaling for u r because both terms have to be at order 1, they have to be comparable in the mass conservation equation.

So, if I scale throughout by u by R epsilon I will get epsilon power half by u into 1 by r d by d r of r u r plus partial u z by partial z is equal to 0. In this equation u r is the only thing that is not scaled. So, if I scale u r by u divided by epsilon power half I will get an equation that is completely dimensionless, and continuous to remain independent of epsilon in the limit as epsilon goes to 0. So, if I define u r star is equal to u r by u by epsilon power half u r star is equal to u r by u by epsilon power half, then my mass conservation equation becomes 1 by r d by d r of r u r plus partial u z by partial z is equal to the 0 so, that is my scaled mass conservation equation.

Important point is note here the scaling for u r came out of the mass conservation equation. More importantly, the velocity scale for u r is u by divided by epsilon power half, the velocity scale for u z is capital U itself because this sphere surface is coming down with the velocity u. Whereas, the bottom is stationary velocity scale for u r is u r by u by epsilon power half in the limit as epsilon goes to 0, u by epsilon power half is actually much larger than capital U itself.

What that is telling you is that the velocity with which the fluid is leaving the gap, this velocity is actually, large compare to the velocity with which the sphere is coming down because the velocity scale for u r is capital U by epsilon power half. Whereas, the velocity scale for u z is just u itself which is much smaller. Therefore, the fact that this sphere is coming down, all the fluid that is below the sphere has to rush out of that small gap that is generating a velocity, which is much larger than the velocity with which the sphere is coming down.

Of course, at the two surfaces the velocity has to come back down to 0 because we have a no slip condition at these surfaces. Therefore, you have large velocity that is coming out which is reducing to 0 at the two surfaces, where its generating large shear stresses and this is it is these stresses which are going to generate the large pressure force, which resists the surfaces from coming towards each other. So, that we have scaled u r and u z from the mass conservation equation, in order to solve the equation we have now got to go to the momentum conservation equations.

So, let us do that of course, when we go to the momentum conservation equations we will have to scale the pressure itself, and we will see a little later how to do that. Kindly keep in mind these two scaling for the velocities and these two scaling for the positions, when we go to the momentum conservation equations. So, let me writhe the scaling once again is equal to z by R epsilon.

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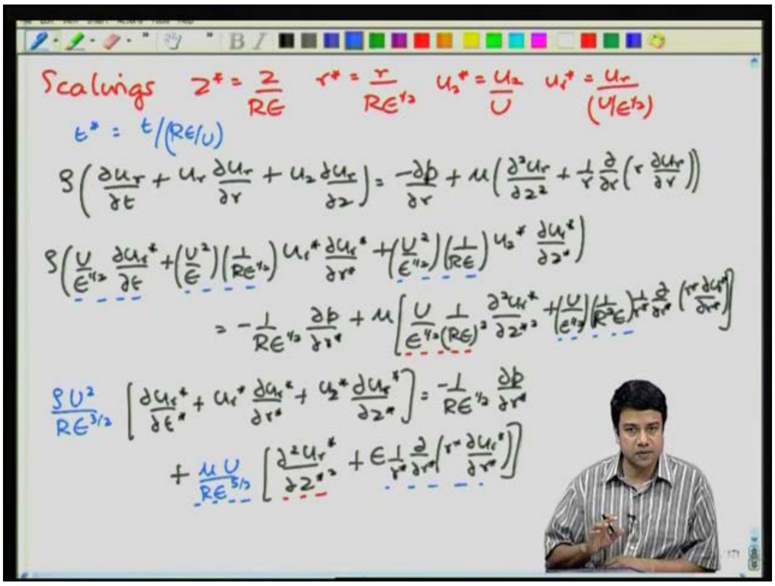

R star is equal to r by R epsilon power half u z star is equal to u z by capital U, u z star is equal to u r by u by epsilon power half. So, those for the scaling momentum conservation equation in the radial direction, I will take that first because I would expect the momentum conservation equation, the radial direction to generate the pressure force required to push the fluid through physically that is the reason. So, the momentum conservation equation in the radial direction, access symmetric and for the moment well I can include the that the time dependent term as well.

Partial u r by partial t plus u r be careful in cylindrical coordinate system. So, that is the momentum conservation equation in a cylindrical coordinate system. I scale, I express all velocities in terms of the scale velocities. So, u r is equal to u r star times u by epsilon power half r is equal to r star times r epsilon power half and u z is equal to u z star times u. So, putting all of those into this conservation equation, I will get rho into u by epsilon power half partial u r by partial t plus I have two u r's in the numerator and one r in the denominator.

The two u r's in the numerator will give me u square by epsilon, and 1 over R will give me 1 by R epsilon power half u r partial u r by partial r plus u z scales as u, u r scales as u by epsilon power half. So, I will get u square by epsilon power half into 1 by R epsilon into u z partial u r by partial z we have not scaled the pressure yet. So, we will just leave at as such so, this equal to minus 1 by R epsilon power half partial p by partial, partial r plus the viscosity into u by epsilon power half 1 by r epsilon whole square d square by d z square is 1 over R epsilon the whole square times partial by partial z star square because there are two z's in the denominator.

So, this is partial square u r by partial z square plus in the second term, I have one u in the numerator and two r's in the denominator. So, let me shift this a little bit plus mu into this is u by epsilon power half 1 over r epsilon the whole square partial square u r by partial z star square plus u by epsilon power half 1 by I have two r's in the denominator. So, this gives me 1 over r square times epsilon for each r, I have for each small r to be I have capital R times epsilon power half.

So, this becomes r square epsilon two r's in denominator. In the inertial terms you can see that both of these contributions, go as u square by R epsilon power three half both of these terms go as u square by R epsilon power three half. I would get the same thing here if I scaled t star, if I scale t star is equal to t by r epsilon by u r epsilon is a distance u is a velocity, r epsilon is the distance from the bottom of the sphere to the bottom plate, u is the velocity.

So, r epsilon by u is the time scale to take for this spheres surface to touch the bottom surface, if it was moving at a constant velocity. So, that gives me the scaling for t star. Once I have that I get have the key factor rho u square by R epsilon power 3 by 2 into partial u r by partial t plus u r partial u r by partial r plus u z partial u r by partial z is equal to minus 1 by R epsilon power half partial p by partial r star have not scaled the pressure yet.

Now, the second term I have two contributions, I have one term here which goes as 1 over R square epsilon power 5 by 2 it goes as 1 over r square epsilon power 5 by 2. I have a second term here which goes as 1 over r square epsilon power 3 by 2, 1 over r square epsilon power 3 by 2. In the limit as epsilon goes to 0 obviously, the higher power in the denominator is larger the limit as epsilon goes to 0, the higher power in the denominator is larger.

Therefore, the red term is larger than the blue term because it has a higher power of epsilon in the denominator. So, I take that out as the common factor because that is what I expect to be the largest term, and I get mu u by R epsilon power 5 by 2 into partial square u r by partial z square. And if I have scaled by r square epsilon power 5 by 2, if I scaled by r square epsilon power 5 by 2 the second blue term had only an r square epsilon power 3 by 2 in denominator.

Therefore, I should have epsilon times 1 by r d by d r of r partial u r. So, what this is telling me is that in the limit of epsilon going to 0, this term here the viscous stress due to the gradient in the stream wise direction, a small compare to this term is just the divergence of the stress due to the gradients in the z coordinate smaller. So, we have sharper gradients there and that gives me the largest contribution to the stress, in the limit epsilon going to 0 the blue term can be neglected in comparison to the red term.

Now, to get the scaled equation I divide throughout by this pre factor, I prefer to divide by the viscose forces because I expect the viscose forces to be dominant in this case. Of course, once I divide I will end up with the Reynolds number and that Reynolds number will tell me actually, if the viscose forces are dominant or not, but anyway we will divide by this and then find out what the result is. So, divide throughout by mu u by R epsilon power five half's once, you do that you are going to get on left hand side.

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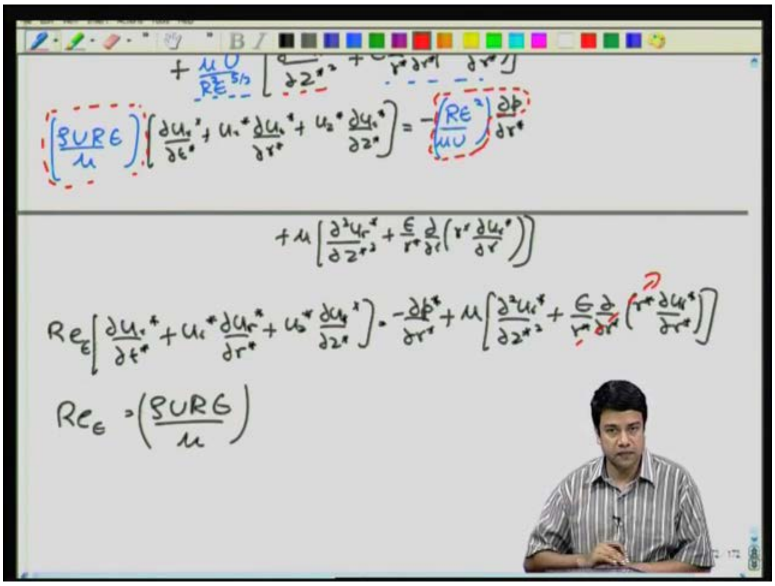

Rho u r epsilon by mu into divide throughout by mu u by R epsilon power three half's that gives me, where to scale the pressure. So, this is equal to minus sorry this should be r square cannot you make that the corrections is to be r square epsilon power 5 by 2 because I have two factors of r in the denominator here we can so r square. So, this gives me minus R epsilon square by mu u into partial p by partial r plus the terms on the right hand side plus mu into partial square u r by partial z square plus 1 by r d by d r of r partial u r by partial r.

With an epsilon sitting there as you recall this factor of epsilon sitting in front of this term epsilon by r. So, there is an epsilon sitting front of the second term there. So, this has given as two things firstly, it has given us a scaling for the pressure ok, this thing is dimension less because everything else in the equations is dimensionless. Therefore, this has given me how the pressure varies what is the scaling for the pressure.

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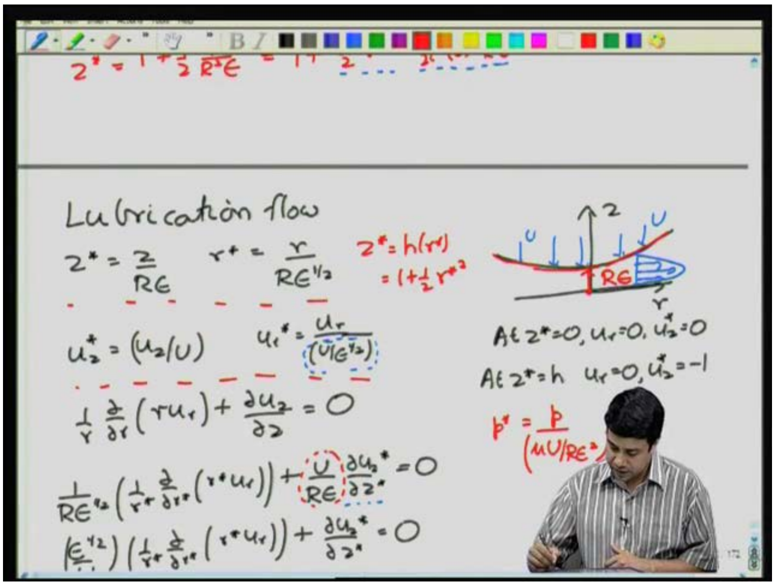

So, it is appropriate to define a scaled pressure as p star is equal to p by mu u by R epsilon square, this is given as a scaling for the pressure. And the other thing is it has given us what is the appropriate Reynolds number for this problem. Of course, the appropriate Reynolds number is this one because this is what, gives me the ratio of the inertial and the viscose terms. So, finally, from this equation I will get the Reynolds number into I will call it as R e epsilon, the subscript epsilon into partial u r by partial t plus u r is equal to minus partial p by partial r plus mu.

And now, I have two terms here partial square u r by partial z square plus epsilon by r partial by partial r of partial u r by partial r. So, these are the two terms, where R e epsilon is equal to rho u r epsilon by mu. Note that the Reynolds number based up on this sphere radius, and the velocity is rho u r by mu. This is telling us that the appropriate

Reynolds number for this problem is the velocity times the gap thickness divided by the kinematic viscosity.

So, even if the Reynolds number based up on the sphere radius is large, as epsilon becomes smaller and smaller. This is going to come some stage at which rho u r epsilon by mu becomes small because sitting the limit as the sphere coming to very close to the surface. Once that happens the Reynolds number, based up on the gap thickness is small even with the Reynolds number based up on the sphere radius may not be small. And it is appropriate to use the stokes flow equations in the gap between the two surfaces.

So, as the gap becomes smaller and smaller the appropriate Reynolds number to use is the Reynolds number based up on gap. And the other thing of course, limit epsilon going to 0, this term due to the variation of the velocity along the gap is small compare to the divergence of the stress due to the velocity across the gap. Therefore, I can neglect the second term on the right side in comparison to the first term.

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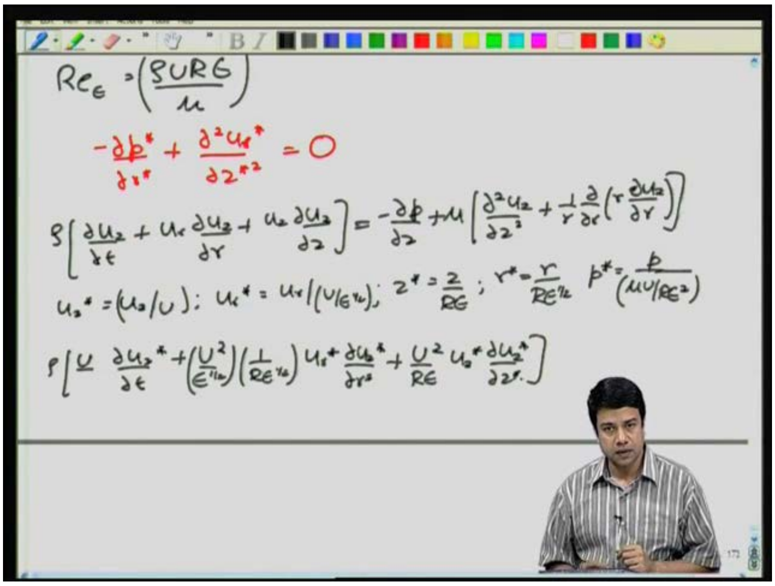

And in that case my simplified equation for the radial momentum conservation equation reduces to minus partial p by partial r plus partial square u r by partial z square equal to 0. So, that is my simplified momentum conservation equation in the radial direction, small gap thickness and limit as the Reynolds number based up on the gap thickness goes to 0. So, next we have to solve the momentum conservation equation in the z coordinate, in the cross stream direction, let us go through that rho into partial u z by partial t plus u r partial u z by partial r plus u z partial u z partial z is equal to minus partial p by partial z plus mu.

That is the complete momentum conservation equation in the z direction, once again I substitute u z star is equal to u z by u u r star is equal to u r by u by epsilon power half then z star is equal to z by R epsilon r star is equal to r by R epsilon power half. And we had just got the scaling for the pressure p star is equal to p by mu u by R epsilon square. Note the pressure is very large is going as mu u by R epsilon square. So, it is a very large pressure in the scale substitute these into this equation. So, I will get rho into u by u partial u z star by partial t plus I have u r, which is u by epsilon power half and u z is u.

So, I get u square by epsilon power half into r gives me 1 by r epsilon power half u r partial u z by partial r plus the second term I have two u's on the numerator u z. So, I get u square by r epsilon u z partial u z by partial z. The pressure gradient so let us write that here pressure gradient has mu u by R epsilon square for the pressure, and there is a z in the denominator which is gives me 1 over r epsilon.

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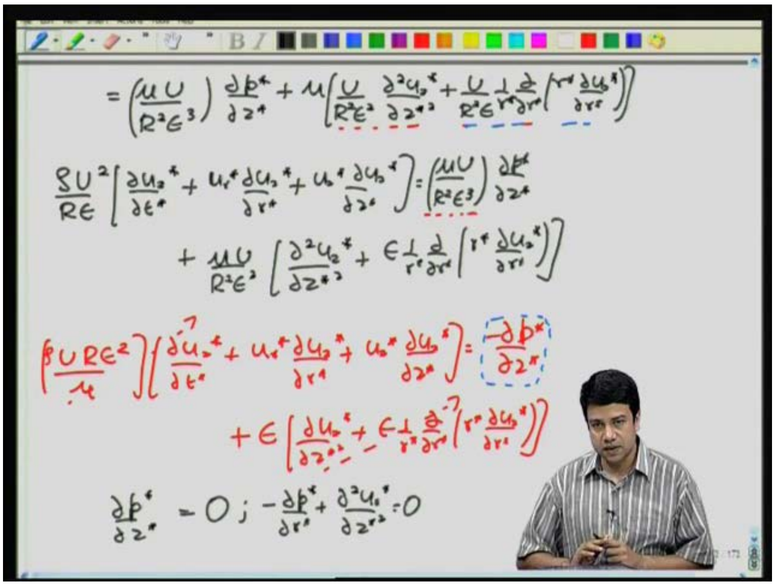

So, I get mu u by R square epsilon cubed partial p by partial z plus u z goes as capital U and z goes as 1 over r epsilon. So, I will get mu into u by R square epsilon square partial square u z by partial z square plus u by R square epsilon because there are two derivatives with respect to r and each one has a r times epsilon power half, this gives me 1 by r d by d r of r power. So, on the left hand side, you can easily see there is a factor of u square by R epsilon, which is common to all the terns if I express the scale time as well I will get u square by R epsilon times t star because u square by R epsilon which is common.

So, the equation becomes rho u square by R epsilon into is equal to mu u by R square epsilon cubed plus in the second term here as usual, the derivative with respect to z is much larger than the derivative with respect to r then the derivative with respect to r. So, I can take that out as a common factor, and I will get mu u by R square epsilon square plus epsilon into so that is my final expression. So, this is the scaled equation however, when I want to get an equation in which all terms the largest term is order 1 in the limit as epsilon goes to 0 that I want an equation, in which the largest term does not go to infinity largest term has to be r over 1 as epsilon goes to 0.

The other terms may be smaller, but at least the largest term has to be at maximum order 1, we should not go to infinity otherwise it does not make sense to have an equation by one of the terms goes to infinity, which is largest term the one with the highest power in the denominator is the largest term. And you can see here that the largest term here is actually the pressure gradient. So, I have to divide throughout by the pre factor of that in order to get an equation in which the largest term is order 1 in the limit, as epsilon goes to 0. Divide throughout by that and you will get rho u R epsilon square by mu into partial u r by partial t.

I am sorry partial u z plus u r is equal to minus partial p by partial z the viscose term actually has a factor of epsilon in front of it because I divided throughout by mu u by R epsilon square. So, I will get epsilon to partial u z by partial z in the limit as epsilon goes to 0, you can very easily verify rho u r square r epsilon square by mu this has to go to 0. So, this goes to 0 so, the entire inertial terms can be neglected also in the z momentum conservation equation. The viscose stress is order epsilon smaller than the pressure in the in the z direction. Therefore, this can also be neglected in the limit as epsilon goes to 0 this term is small compare to the pressure gradient.

Therefore, this is the only term that is remaining in the equation in the limit as epsilon goes 0 at my momentum conservation equation in the z directions becomes partial p by partial z is equal to 0. No pressure gradient perpendicular to the flow, we will see little later this is common feature in all cases, where the velocity in the stream wise direction is large compare to the velocity in the perpendicular direction. You will find that the pressure gradient perpendicular to the flow to the to the largest velocity component as always 0.

So, this gives as the three scaled mass momentum conservation equations, this was for the z direction for the r direction, I have minus partial p by partial r plus by partial z square is equal to 0 then I have a mass conservation equation. Next, class we will see how to solve this equations in order to get the force, I will briefly discuss once again the scaling how the rho’s, and then I will go on to discuss how to solve this in order to get the actual force. We will continue this in the next lecture, we will see you then.