Fundamentals of Transport Processes II
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Lecture - 20
Effective Viscosity of a Suspension

So, welcome to this lecture number 20. In our discussion of fluid mechanics, a lower Ryland's number flows. If you recall in the last class, we had a little bit to complete on the torque exerted on a particle, which is rotating in torques flow. So, going back to our discussion. Then, we had this spherical particle with center at the origin, which is rotating with an angle of velocity omega.

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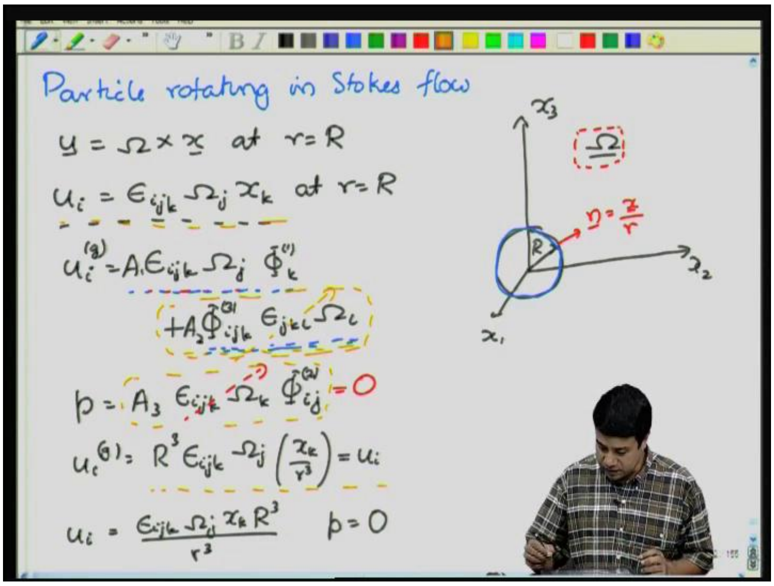

Now, the velocity field at the surface of the sphere is equal to omega cross r vector. So, this was our spherical particle, which was rotating with angle of velocity omega cross x; is the linear velocity, at points on the surface of the particle. So, we are suppose to solve, the stokes equations with this boundary condition. As, you recall the way, that we solved; it was that. Since, these equations; the velocity solution for the velocity field, satisfies stokes flow equations.

The solution has to be linear in these velocity omega, as well as linear in one of the spherical harmonics; it cannot be just linear function of omega, because omega is a soda vector, where as the velocity itself is a real vector. Therefore, it can only be a linear function of epsilon dot omega, epsilon anti-symmetric tenser, epsilon dot omega gives you a second order tenser, because epsilon is third order and omega is a first order vector. Therefore, we had taken a general solution; that had three components, a one is due to the dotting of dot product of epsilon dot omega, which is second order with a first order vector, and then dot product of that to the second order with the third order tenser, so third order tenser double dot product with the second order tenser; that is the epsilon dot omega, will give you a vector.

Similarly, pressure is a scalar. Epsilon dot omega second order tenser, say to take two dot products of that with the second order tenser, to get the pressure; however both of these turn out to be 0, both of these terms say turn out to be 0, because as I just showed you; these are products of an anti-symmetric in the symmetric tenser; both of these are products of symmetric and anti-symmetric tenser.

So, for the velocity field, you would like to just one solution, which is the dot product of a epsilon dot omega with the first order vector solution of the stokes of the Laplace equation; and so quite easy, to get a solution; that has fetch the boundary condition; that was our solution is simple. However, in order to find out the torque; that is exerted on the sphere.

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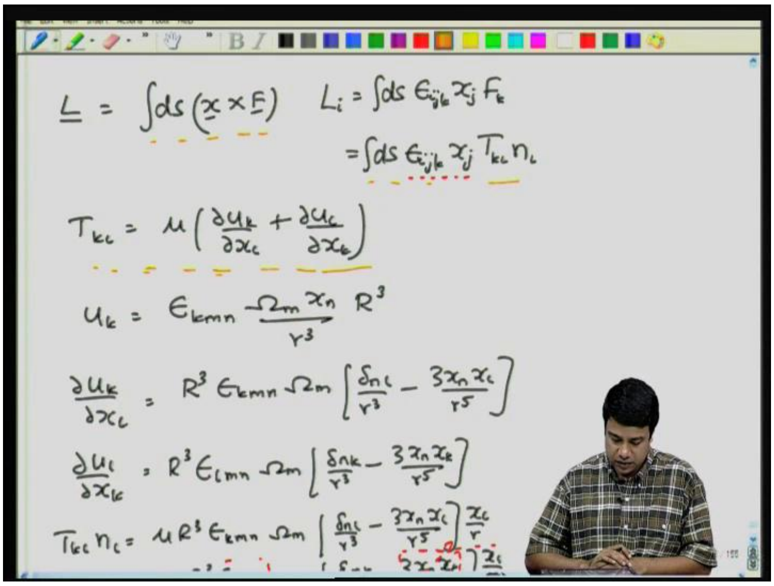

We have to find out, the stress tenser ok. The torque can be written as integral of the surface of x cross f, f itself is written in terms of the stress tenser of the surface. The stress tenser is related to the velocity gradients. As, you recall in the last lecture, we do not have a pressure here, because the pressure is equal to 0, but still we have a velocity field and we need to take its gradient. And, the stress tenser related to the gradient of the velocity field.

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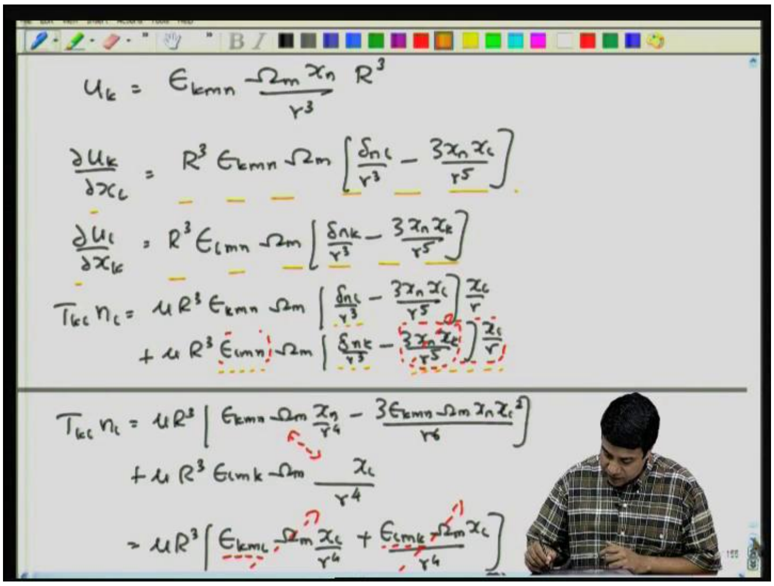

So, it calculated the two gradient of a velocity field. We had take the gradients of the velocity field, here multiply that by the viscosity and then dotted with the unit normal. Unit normal n l is equal to x l by r, where x l is the position vector, and r is the distance from the origin. And, I showed you that, when you take this dot product, there is one term, which vanishes, because it contains a symmetric types a symmetric tense. If you recall, here epsilon l, m, n is anti-symmetric; in n and l in other words, if you interchange n and l, you will get the negative result, where as this one x n, x k, x l is symmetric.

So, when you multiple those two, you get 0 and then, when you looked at the other two terms here; these first two terms here; this one and this one, I had showed you that; they are opposite of each other, because I have epsilon k m l and I have epsilon l m k, epsilon l m k is equal to epsilon k l m, that is when you do the permutation of the indexes; this is minus epsilon k m l and because of that, these two cancel out and you get 0.

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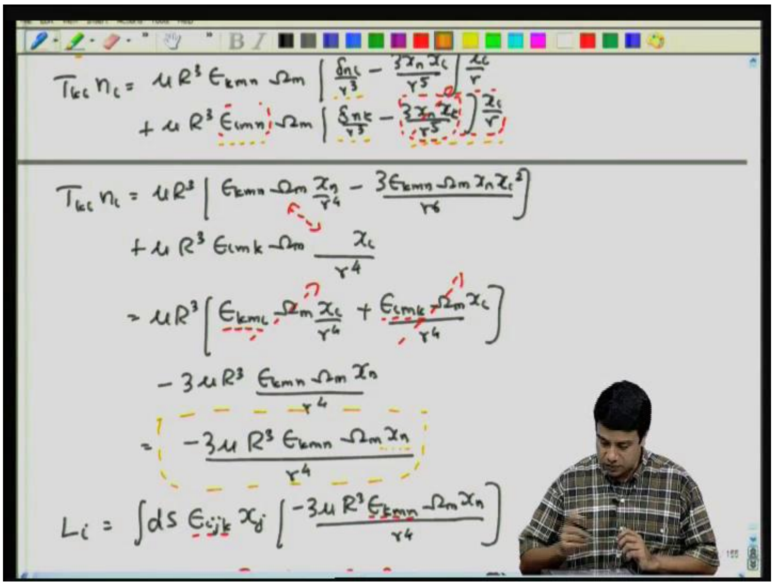

And, I get just one expression, which survives for the force acting at a given location of the surface, that is of this form. There is the force exerted per unit area on the surface. If I just take this force, and I integrate to the entire surface; I will get 0. The reason is simple, this force is an odd function of the position vector; it is an odd function of the position vector.

So, it is proportional to x 1, x 2, x 3 components of that position vector. When you take an odd function of the position vector, integrated over a close surface, you will end up getting 0, because for each value of plus x 1; that surface there is a value of minus x 1, as well it is an odd function, when integrate an odd function over the entire surface, you will get 0; this is true, for all odd functions ah function; that is linear in x vector will be 0, 1; that is cubic will also be 0, 5th power will be 0 and so on. So, if you integrate, take this expression for the force and just integrate it. Simplistically, over the entire surface, I will get 0; that is because rotating sphere should not experience a force in any particular direction at stokes flow ok.

Because, it is a linear function, different high reylonds numbers, because its high reynold as numbers; these not linear functions. So, rotating sphere at high reynolds number can experiences the force. However, at low reynolds numbers, because the force and the angle of velocity have to be related in a linear fashion.

Angle of velocity is a soda vector. Therefore, the force has to related to epsilon i j k times omega k epsilon i j k times omega k is a second order tenser. There is no way to get a real vector out of that, therefore the net force has to be equal to 0. However, the torque is also a pseudo vector, the direction of the torque also changes sign, when we go from right, left handed quadratic system. Therefore, you can get a net torque due to a rotating sphere.

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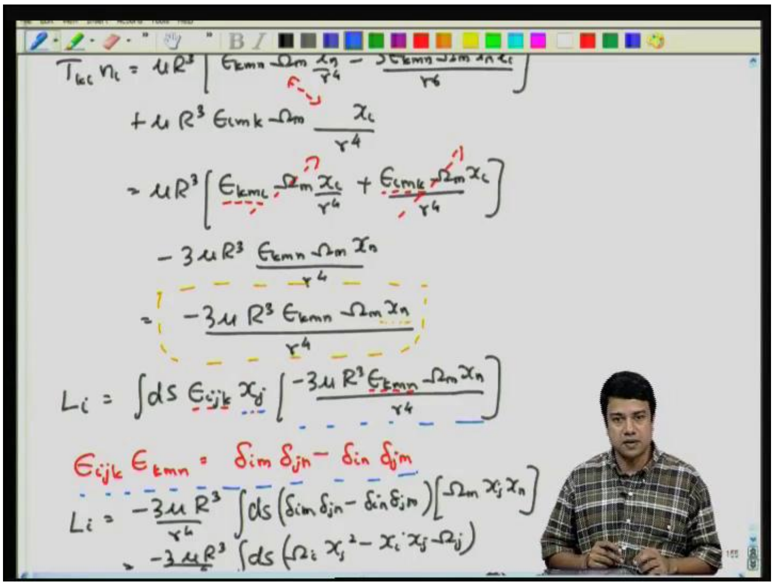

And that, net torque is obtaining by, taking the cross product of the position vector on the surface, the cross product of the position vector of the surface, and the force on the surface. And, when you take the cross product force; has a epsilon k m n times omega m x n, then the cross product of that with x, has another epsilon. I told, and I gave you a formal in the privies class, which will simplify this calculation; product of two epsilons, two anti-symmetric tensors has to be am sorry, the product of two pseudo tensors has to a real tenser, because each one reverses sign, when you go from right handed to left handed coordinate system.

Therefore, the product of these two has to something; that does not change sign. Forth order tenser, which is real and it is also interdependent of coordinate systems. Epsilons are independent of coordinate systems; they are always equal to plus 1, minus 1 or 0. Therefore, the multiple; the product of two also has to be independent of coordinate systems. And, as I said, you can construct a fourth order tenser by taking, the tenser product of two second order tenser; that is the identity tenser and there is an exact expression, which will relates the product of two anti-symmetric tenser with two identity tenser.

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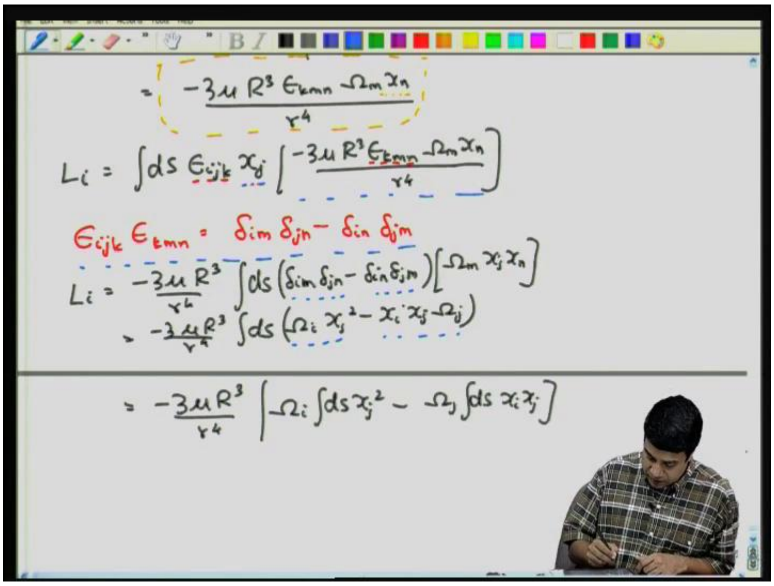

And so, we had used this, in order to find out, what is the expression for that torque? So, when I expressed in term of delta i m, delta j n, and delta i n, delta j n, I get one; that is equal to omega i times x j square minus x i x j omega j

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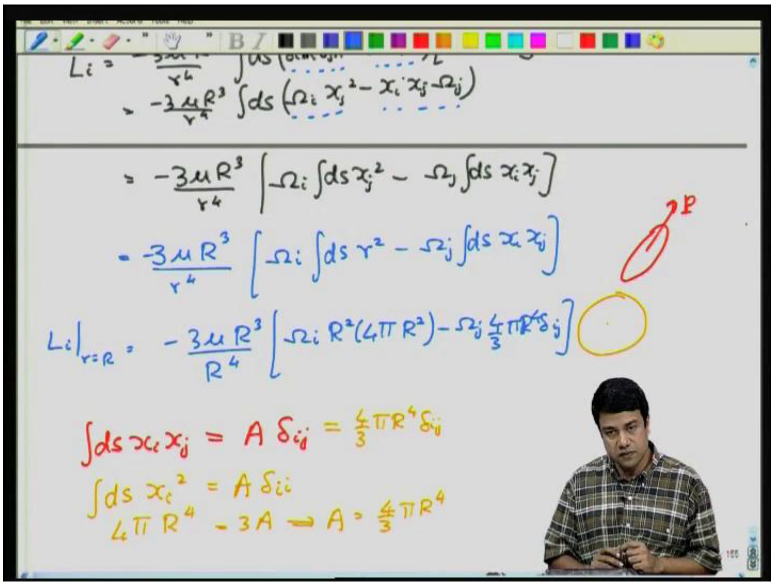

So, let us just write this expression, once again for clarity; minus 3 mu R cubed by r power 4. We should have a surface at the integral, here into omega i integral d s x j square is just r square, just equal to r square, itself minus omega j integral d s x i x j integral d s of x j of r square is just equal to the surface area times r square, because r is just the distance, from the origin to the surface of the sphere so on.

The surface of the sphere, the torque at r is equal to capital R minus 3 mu r cubed by r upon 4, r is equal to capital R on the surface of the sphere. I will get omega i times r is equal to capital R on the surface of the sphere times the surface integral; so this just equal to r square times 4 by r square. And, I have the second term integral d s of x i x j; this is an integral over the entire surface of the sphere x i x j now, there is no direction in this field, because the sphere surface itself is isotropic. Once, I integrate over the entire surface of the sphere, I cannot get result, which is which depends up on any particular direction, because this integral itself is isotropic, second order tensor isotropic.

Therefore, it has to be some constant times the identical tensor, because it cannot be depend up on any particular direction. If for example: I will integrate something over an object like this; that object has a net direction p, the orientation of the of the axis of this object. If I integrate x i, x j over this object, I could get something; that depends upon this vector p, so for example I would get something; that could be written as p i p j, but a sphere itself is an isotropic object, a sphere itself has an is an isotropic object, there is no net direction here.

Since, there is no vector direction, the result that I should get will depend only up on the identity tensor; is an isotropic tensor in three dimensions. So, how do I find out, what is value of A? The answer is quite simple, I can multiply both sides by delta i j, I multiply both sides by delta i j integral d s x i x j times delta i j, is just x i square is equal to a times delta i j times delta, i j is one of those, I replace j by i, and I just get delta i i; what is the value delta i i is equal to delta 1 1 plus delta 2 2 plus delta 3 3, one repeated index; that means the summation, no unit vectors, so delta 1 1 plus delta 2 2 plus delta 3 3 in three dimensions, is just equal to their.

And, on the left hand side I have integral d s of x i square, x i square is capital R square on the surface, therefore integral d s of x i square is just equal to the surface area 4 pi R square times r square, so this becomes 4 pi R power 4 ok. So, that is equal to 3 A, which means that A is equal to 4 by 3 pi R power 4 times delta i j, so this is equal to 4 by 3 pi R power 4 delta i j. So, use that in this expression: minus omega j 4 by 3 pi r cube delta i j omega j times delta i j, is just omega i omega j times delta i j, is just omega i, this 4 pi R power 4 times R power 4. So, 4 pi R power 4 times delta i j minus 4 by 3 pi R power 4 delta i j; this is gives me 8 by 3.

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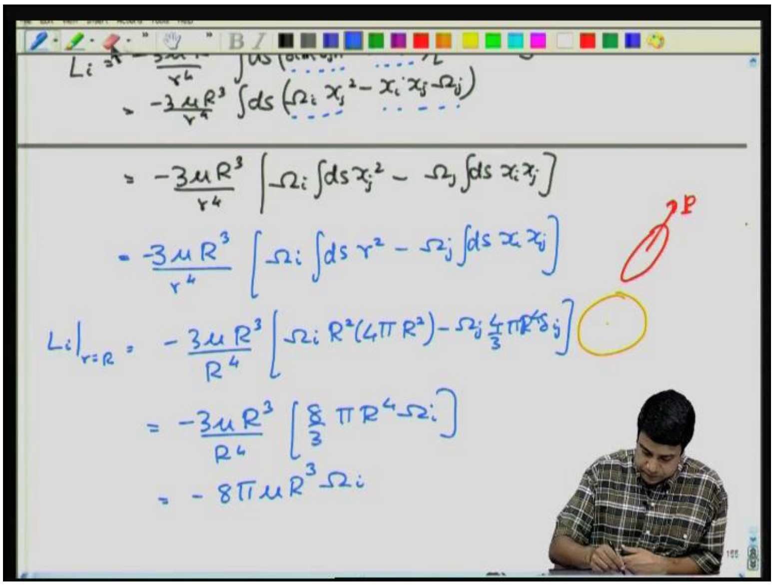

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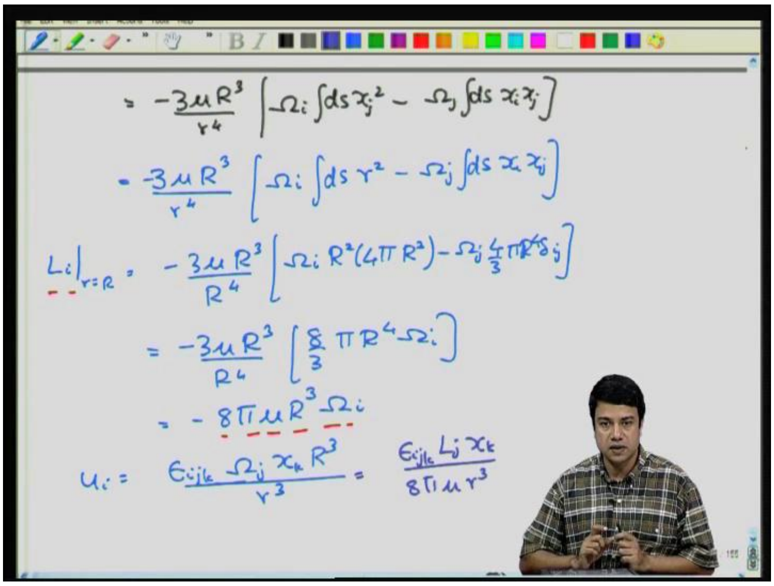

This, because minus 8 by mu R cubed omega i. So, this is the expression for the torque acting on a spherical particle. So, torque acting on the sphere, due to the fluid is in the direction, opposite to the direction of rotation of the sphere, where as the torque; that the sphere exerts on the fluid will be in the same direction as the rotation of the sphere. So recall, the velocity field; that I have, when I, when I solved for this expression, the velocity field; that I had was u i is equal to epsilon i j k omega k R cubed omega j x k R cubed by r cubed; that was my expression for the velocity field. And, instead of expressing this in terms of the angular velocity, I could also expressed in terms of the torque.

So, I just used this relation, to rewrite the angular velocity in terms of the torque. So, the angular velocity becomes 1 by 8 by mu r cube ok, so this if I write it in terms of torque, I will get epsilon i j k omega j l k by 8 by I am sorry, epsilon i j k l j x k by 8 by mu r cubed. Once again, an important point to note, when the velocity field is expressed in terms of the torque, rather than the angular velocity of the sphere. The final result is independent of the radius of the sphere; it depends only upon the torque exerted by the sphere.

So, this is like the solution; that we had the (()) seen tenser for the fluid, due to a linear velocity. In that case, the velocity the velocity when express in terms of the force, the slowest decaying component; the component proportional to 1 over r was independent of the radius of the sphere.

So, it depended only upon the force, and the distance; from that force for that velocity disturbance; that velocity disturbance decreased as 1 over r, and is proportion to the force for a rotating sphere. The velocity disturbance decays as 1 over r square, so this is the dipole a dipole disturbance. As we will see, it is an anti-symmetric part of the dipole momentum; it is proportional to the torque and it does not depend up on the radius of the sphere. So, in a sense this is a solution; that is that this solution for a point particle; that is rotating in the limit as r going to 0 provided the torque means constant. The velocity disturbance will depend only upon the torque, so this is rotating sphere.

So, let us go on to, our next fundamental problem. So, we looked at a sphere; that was translating in the in a fluid a sphere; that was rotating ok. So, there are three different kinds of deformation; that we had seen earlier at a point. One is radial expansion of compression, anti-symmetric part of the deformation tensor, which related to the local rotation; that was the symmetric part, which was due to pure extensional share. So, firstly we discussed net force exerted due to a sphere on the particle, and for a rotating particle a rotation, the disturbance to the velocity field due to the rotation of the of the sphere.

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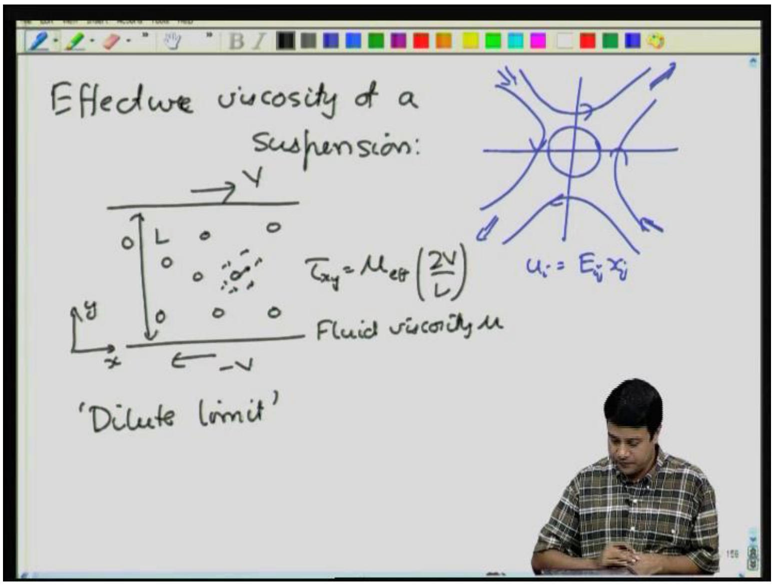

The next example relates to a sphere, placed in extensional field. So, what is the disturbance due to a particle? Which is placed in a velocity field? Which under goes extensional flow far away from the particle? So, this example relates to having a particle in a velocity field, where far away from the particle. You have an extensional force, so if the particle were not present, the stream lines will all go like this, it have pure extensional flow, in which the velocity stream lines are u i is equal to this tensor times the vector x j.

So, this was what we had analysis for extensional flow, and told you that, this is the only a part of the rate of deformation tensor; that results in stresses. The question, we are going to ask here is, if I have a sphere, we know that, the net velocity pass that, sphere is going to cause a net force, net rotation of that sphere or the net rotation of the fluid relative to that sphere, at that location; this going to cause a torque. What does this type of deformation? What is the velocity disturbance caused due to this type of deformation? What is the practical application there it can used for? Ok.

So, let us just discuss that in little, this is for the effective viscosity of the suspension is effective viscosity of a suspension. Let us say, I had a suspension of solid particles, and I imposed a velocity field, and I calculated the stress as a function of the strain rate, so the stress will be equal to or in this case, since it is two dimensional, let me just take an x y coordinate system, taw x y is equal to some viscosity. I will call it as effective viscosity, partial u x by partial y.

If I had a this distance, was l into the macroscopic strain rate 2 v by l. So, since I have placed moving with plus and minus v, the net relative velocity is 2 v separated by distance l, therefore the macroscopic strain rate is 2 v by l. Of course, if I had no particles, then the viscosity would be just a fluid viscosity mu means, if the no particles present, the viscosity would be just the fluid viscosity mu. However, I have this particle; that are present in the fluid, therefore the viscosity is going to be different from the fluid viscosity mu, how does the presences of the particles effects the viscosity of the fluid? So, that is the question. So, how is mu effective related to the number of particles, their mass density and so on.

So, that that is the question, we will consider this only in, what is called the dilute limit? only in the dilute limit, the assumption here is that, the velocity field, the velocity disturbance due to one particle; so if you have one particle, that is a moving some direction; that is causing a velocity disturbance in the fluid, this is small velocity disturbance to one particle in the fluid, the dilute limit.

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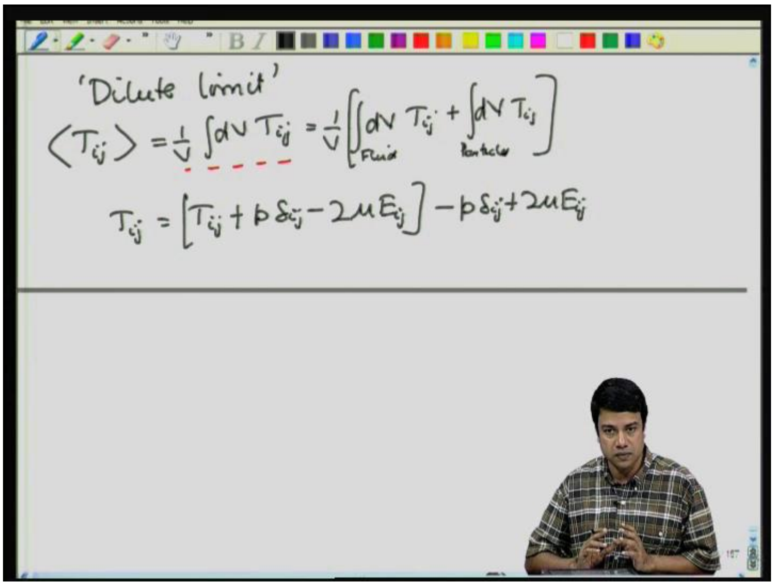

The assumption is that, this velocity disturbance does not affect another particle; the particle is sufficient well separated; that the velocity disturbance around one particle, does not affect another particle; that is what, I meant by dilute limit; it is like the ideal gas limit, the expansion for the gases. For example: we assume that, there is no interaction between molecules in that case; similarly, in this case we assume, there is no interaction between particles this dilute limit; so in that limit the, we are trying to analysis the problem.

So, here we need to find out, the ratio and stress and the strain rate. Now, I will define the stress T i j; the average stress, the total stress of this entire fluid plus particles. The entire system, which consist of the both fluid and particles; there are different ways to define it; the most convenient way is take that as volumetrical of the total stress, the most convenient way is to assume, that is the volumetrical, the average stress is a volumetrical over the entire domain of the local stress at each point within that domain.

So, that is a total stress. Now, the simplistic way to separate out, this stress is to write, this as 1 over v integral, separated it out to the into two parts; one is over the fluid T i j plus integral of the particles, one is to separate out the one part over the fluid; the other part over the particles. However, we will use a different method to do this, a method that makes the particle contribution clear in this in this entire thing, so what I will do is? that I will write down ok.

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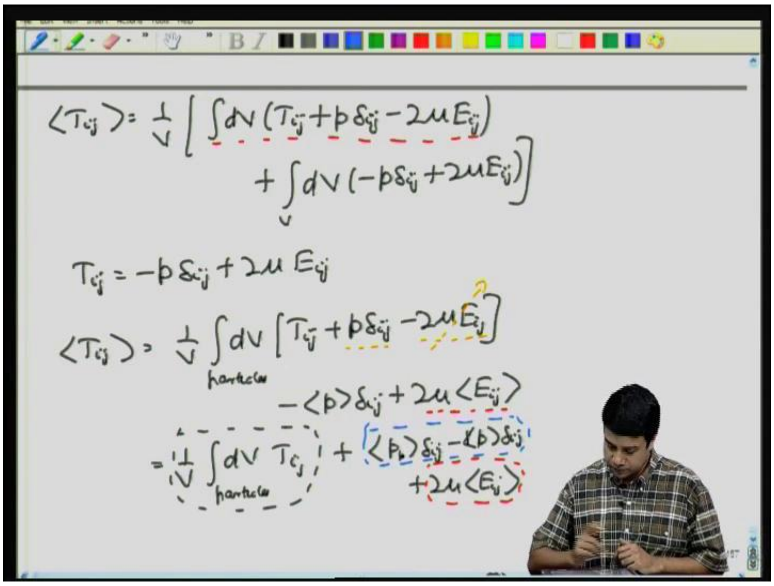

In this expression for T i j i will write down as, T i j plus p delta i j minus 2 mu E i j, where E i j is the rate of deformation tensor locally. E i j is the local rate of deformation tensor. I made a simplification here; the simplification is that the rate of deformation tensor consists of two part; one is the rotational part, the anti-symmetric, and the other is symmetric trace less. There is no isotropic part, because the flow is incompressible, the densities are constant.

So, there is no radial component, there is an anti-symmetric and a symmetric trace less part. Here, my assumption is that, the stress is not affected by the anti-symmetric part. As, I just said anti-symmetric part causes only rotation locally; it does not change the distance between nearby points in the field. So, I will write down, the stress tensor in this manner and will see, how it works out integral of the entire volume plus integral over the entire volume of d v? So, there is the final, there is the expression for the stress tensor. Note that, here I have taken mu as the fluid viscosity, so I am adding and subtracting over the fluid viscosity times, the symmetric trace less part of the rate of the deformation tensor as also the pressure.

Now, we know that the equation, the the constitutive relation for the fluid stress tensor is, T i j is equal to minus p delta i j plus 2 mu E i j. Therefore, what has happened is that in this expression, the integral is 0 in the fluid integral is 0 in the fluid, because in the fluid we know that, the T i j is minus p delta i j plus 2 mu times E i j, so integral is 0 in the fluid; that means that this integral contains, only the integral over the particles, in the originally at posted as an integral over the entire system integral is 0 everywhere in the fluid. And therefore, the integral only over the particles, therefore the average T i j is equal to 1 over v integral over the particles.

And, the second term; here is a volumetrical of the pressure in the symmetric part of the rate of deformation tensor, averaged to the entire fluid, so this is just the volume averaged pressure, and the rate of deformation tensor in the fluid. So, this is just equal to minus average p delta i j plus 2 mu times E i j. Of course, this average symmetric part of the rate of deformation, symmetric trace less part of the rate of the deformation tensor, the average value; that has to be equal to the macroscopic applied rate of deformation tensor.

So, this average value is equal to, what is the symmetric trace less part of the rate of deformation tensor? which is applied by the sharing of the two plates, because what is averaged, over the entire system has to be, what is obtained by the macroscopic shell?

So, there is the… now we have to…so, this is so this is the viscosity; that ever got if there were no particles, because the integral over the particles would have been identically equal to 0 but, since I have particles; these are causing the disturbance in the flow. Therefore, I get a integral, which is in general non 0, and the task is to evaluate, what this integral does?

Now, first thing is first this is the rate of deformation tensor within the particle; this is rate of deformation tensor within the particles; particles are rigid. Therefore, when you apply a stress on them; they deform and then stop, there is no continuous deformation up on application of stress.

Therefore, this deformation tensor rate of deformation tensor within the particles has to be identically equal to 0, because particles do not deform, continuously upon application of sharing, of course there could be a compression of the particles but, that will not, that will not result in a the symmetric trace less part of this volume, average stress tensor therefore, there could be some compression, expansion, but on average, that will not result in the contribution to the symmetric trace less part of the rate deformation tensor plus there could be some compression expression to due to the particles.

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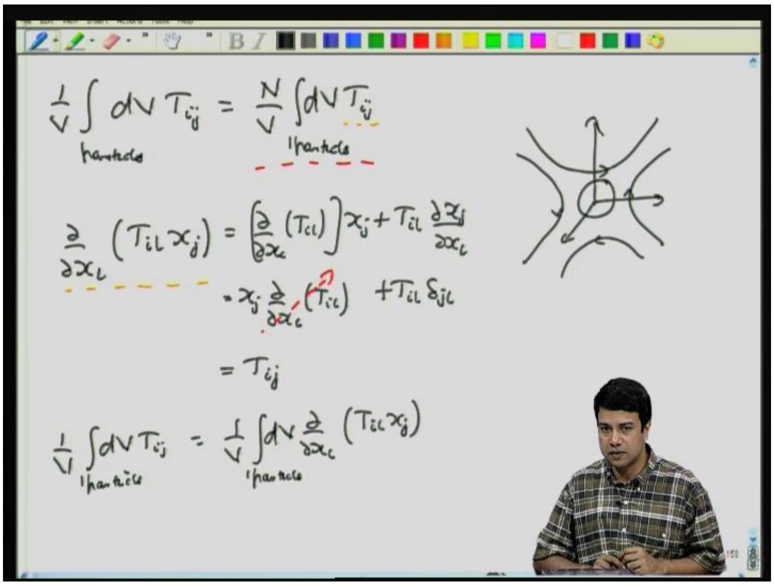

So, I just leave to that, as particle pressure delta i j minus the fluid pressure delta i j plus 2 mu times E i j. So, in order to calculate the effective viscosity, I have to find out, what is the contribution of the particles stress? I have to find out, what is the contribution due to the particle stress to the total stress? Which is proportional to the average rate of deformation tensor? Total stress contribution proportional to the average rate of deformation tensor, these two parts are the isotropic parts, which will not contribute of the viscosity of the system. So, I have to calculate, what is integral over the particles integral d v of T i j?

So first thing is first I have to calculate one over v integral over all the particles d v of t i j assumed that the solution that the suspension was dilute the particles do not interact with each other that means the integral over all n particle of d v times t i j is going to equal to the number of the particle dived by volume integral over one particle integral over the one particle d v t i j of times. The number of particles will be flowing these all of this particle are in a macroscopic flow field in which there is some deformation that is taking place there is symmetric stress.

Deformation that is taking place because of which we are getting a net contribution to this stress within the particles so this one particle is now being placed in extensional flow as I said the rotational part of the velocity field does not contribute to the stress its only the extensional part so this particles is being placed in extensional flow and here to calculate this integral over the particle volume of d v times t i j. Now we do not have the constitutive relation for the elastic deformation within the particle.

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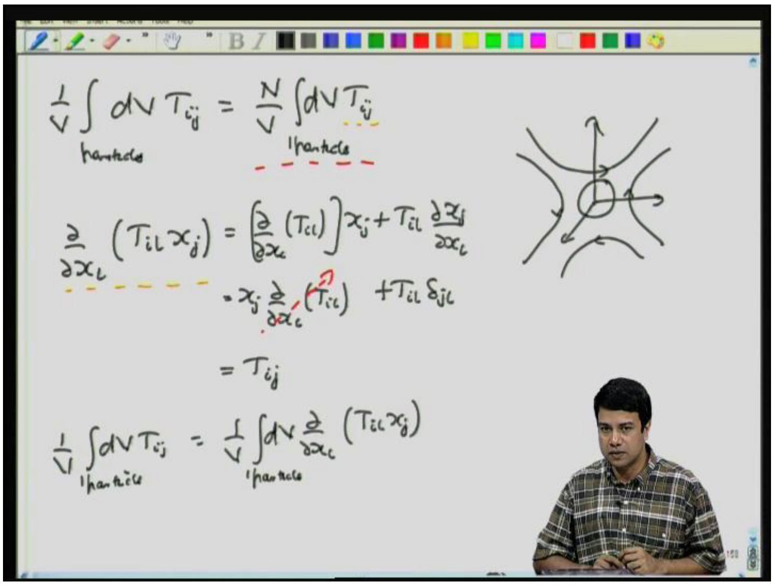

So we cannot just calculate this integral straight away because we do not have a constitutive relation for the elastic deformation within the particle rather what we can do is to rewrite this t i j so let me just rewrite it of t i if you take if you take this derivate partial by partial x l of t i l x j ok. If you take this derivate partial by partial x l of t i l x j this will be equal to partial by partial x l of t i l times x j plus t i l partial x j by partial x l the second part is just delta j l t i l delta j l the first part is x j times the divergence.

Now for the solid if you assume that the inertial effects in the solid are negligible then the divergence of the stress within the solid has to be equal to zero because the divergence of the stress within the solid is also balanced by inertia and by body forces the viscous forces in the fluid are comparable to the viscous forces in the solid because you have to have continuity of stress boundary condition at the surface, if the solid inertia is small then the viscose then elastic forces in the solid are large compare to inertial forces in the solid which means that the divergence of the stress within the solid has to be equal to zero, so this is an additional piece of.

Information that I will use that because inertia is neglected in the flow fluid the density of fluid and solid are comparable then the inertia can be neglected in the solid as well and even though the constitutive relations are very different for a solid elasticity and fluid viscosity viscous flow of a fluid and the elastic deformation of a solid the constitutive relations are very different but, the momentum conservation equation will still tell you that the divergences of the stress is balanced by the body force and inertial forces and if those are negligible the divergence of the stress in the solid also has to be equal to zero.

So if this is equal zero I just get the partial by partial x l of t i l x j is equal t i j that is the stress tensor itself so this are little bit additional information that I have used about the solid itself just not clear a but, provided i can neglect inertia in the solid the mass is not too high, I can always neglect the divergence of the stress tensor within the solid now for this t i j i can substitute this divergence therefore, one by v integral d v of t i j over one particle is equal to one over v integral d v of partial by partial x l, T i l x j over one particle.

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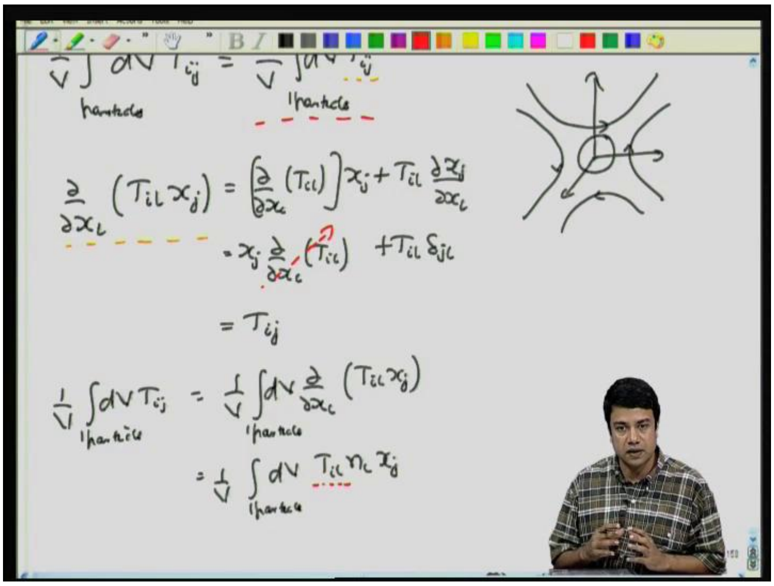

Now this divergence of something integrated over a volume is equal to that dotted with the unit normal integrate over the surface divergence theorem so this is equal to one over v integral over one particle d v t i l n l times x j the unit normal because I had a divergence with respect to the index l therefore, i have to substitute the unit normal also with index l. So this is an integral over one particle of the stress unit normal times the position vector integrated over that particle of course, I still do not have the constitutive relation for the stress for the particle.

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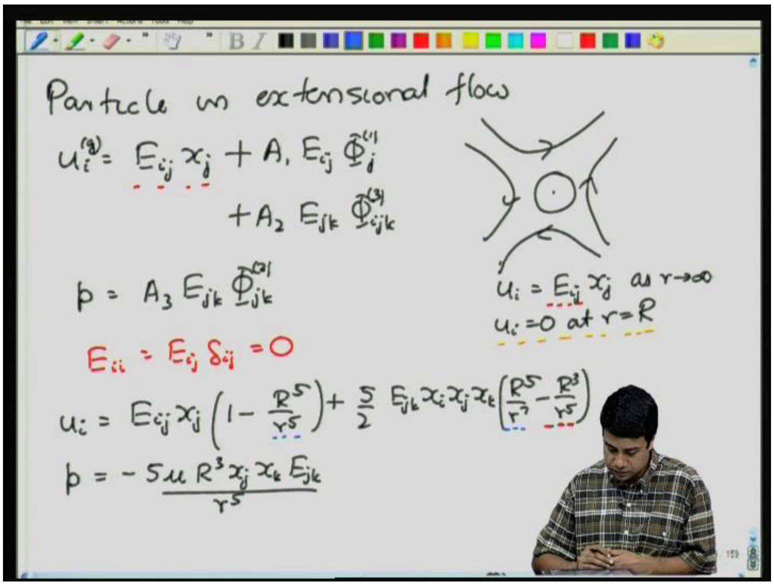

However, I know that the particle stress on the surface has to be equal to the fluid stress on the surface therefore, in this expression for the stress rather than using the fluid stress I can use the particle, I am sorry rather using the particle stress I can use the fluid stress here and that is the advantage of this I have reduced it to a surface integral where I have managed to get the fluid stress on the surface in the integral and once I have that I can actually solve that how do you solve this so this is solution for in extensional flow.

So I have particle which in extensional flow field far away a particle extensional flow field far away and the velocity u i is equal to e i j times x j that is the extensional flow field in the limit as you go far away from the sphere as r goes to infinity and at the surface itself because it is rigid, there is no rotational part. So it is not rotating therefore, the velocity field itself has to be equal to zero at the surface at r is equal to capital r. Now we have to solve for the velocity profile for an extensional flow far away from the sphere and velocity equal to zero on the surface in order to find out what is the velocity disturbance due to the sphere. So the velocity the fluid velocity field of course, it will contain one component.

Ok it will contain one component which is just the extensional flow very far away plus a disturbance due to the sphere because the sphere is imposing zero velocity boundary conditions on its surface there is going to be a at disturbance to the velocity field due to the sphere how do you get the disturbance both the pressure and the velocity field the general part of the general solution for the velocity field and the pressure have to satisfy Laplace equations they have to be linear and one of the harmonics. We have to take only the decaying harmonics because the velocity field disturbance has decreased as we go far away from the sphere is imposed velocity is already here.

So the other part which is the disturbance has to be decrease as you go far away it has to be linear in one of the decaying harmonics and it has to be linear in this tensor e i j not an e i j, x j, x j is field quantity depends upon position what is forcing this flow is the symmetric trace less part of the deformation tensor e i j itself that is a constant sphere just as the velocity of the sphere for the translating sphere was a constant angular velocity of the rotating sphere was a constant. So we required that the velocity should be linear in that similarly, in this case e i j is constant e i j k j is depends upon position the forcing function here is the symmetric trace less second order tensor e i j so using these I have now got to construct solutions.

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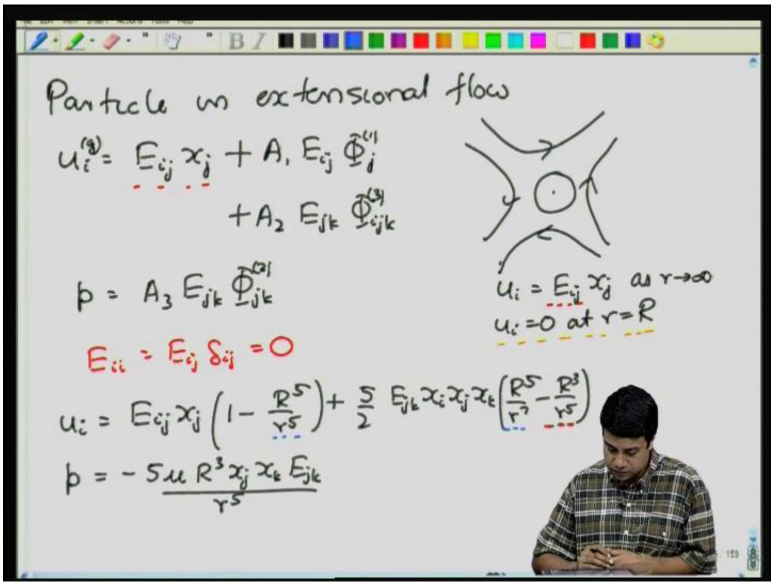

So from the second order tensor I can have the velocity vector created in two ways e i j dotted with a vector that is the decaying vector decaying spherical harmonics plus a 2 e j k phi 3 i j k. So that is the general part of the velocity field the pressure is equal to a 3 it has to be linear in e then only we will get a scalar that is to double dotted with the second order tensor decaying spherical harmonics and then you have to determine constants a 1, a 2 and a 3 from the incompressible condition the divergence of the velocity s equal to zero as well as these boundary condition.

Here I would not go into details of how that calculation is done the only thing that I would note is that since this is symmetric trace less rate of deformation tensor I can use the fact that e i i that is e 1 1, e 2 2, e 3 3 is equal e i j times delta i j that has to be is equal to zero that is the simplification that can be used and once. You use that simplification you will get the solution for the velocity field I will just give you the final solution for the velocity field the final velocity field is u i is equal to e i j, x j into one minus r power five by r power five plus e j k must be five by two x i x j x k into r power five by r power seven minus r cubed by r power five.

So that is the final solution for the velocity field r after using the simplification that the trace of e has to be equal to zero the final expression for the pressure is equal to minus five mu r cubed x j x k e j k by r power five these are the two velocity and the pressure fields note that the disturbance to the velocity field the slowest decaying part of the disturbance to the velocity field actually is this one that is because that disturbance this this this part of the disturbance goes as r cubed by r power five times x i x j x k x i x j x k goes as r cubed x one is r sin theta cos pi x two is r sin theta sin pi and x three is r cos theta.

If I have a R cubed on the top R power 5 at the bottom therefore, the slowest decaying term goes as one over r square, these other terms here go as 1 over r to the fourth, that is these two terms actually decay as 1 over R to the fourth, you can easily verify that so this is a dipole contribution the source due to a net force was 1 over r, this is the dipole conurbation, so using these expressions I can now go ahead and calculate the stress tensor minus p delta i j plus mu into partial u i by partial x j plus partial u j by partial x i and also little bit of algebra, it may be worth a while to try and go through it I will just give you the final result here.

If you actually go ahead and calculate, the final result integral of the surface integral of the surface of of d s of taw T i l n l x j turns out to be equal to 20 phi R cubed mu E i j by 3, so that is the final expression; for the integral of the stress dotted with the unit normal. So, this is the force times the position vector times the position vector.

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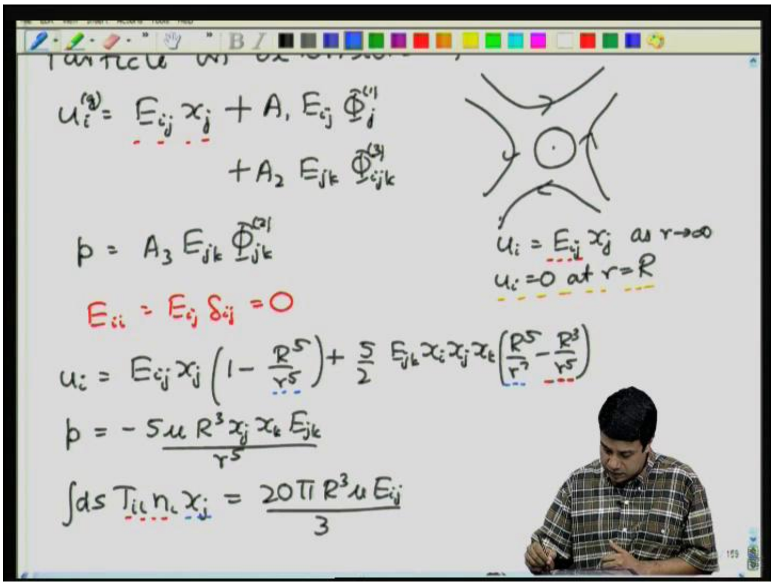

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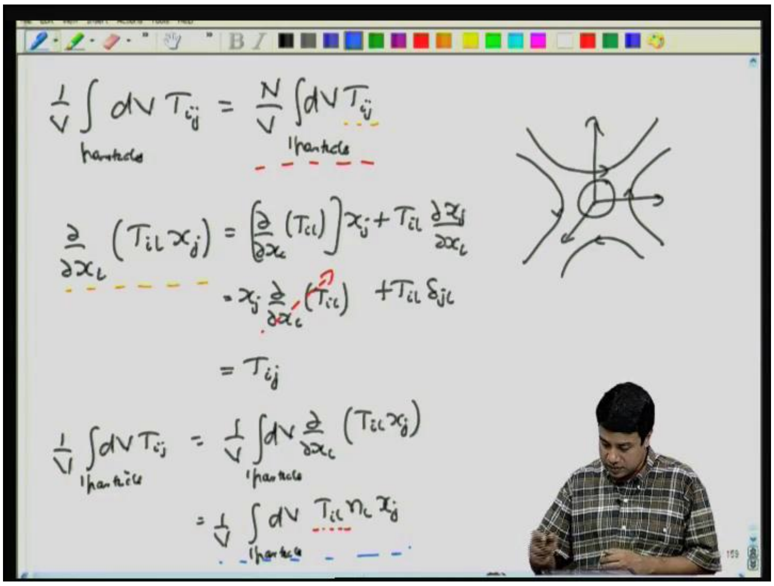

If you recall, the expression for the average stress, the expression for the average stress; that we had was N by V times integral over one particle of d v times T i j. So therefore, this d v times, T i j is substitute as 1 over one particle of T i L and l x j. So, put all of these together is equal to N by V integral over 1 particle d s T i L n L x j, then I had this minus a particle pressure, delta i j plus the fluid contribution delta i j plus 2 mu times E i j, where E i j is the average stress; that is applied on the entire system; that is also this sorry, the average rate of deformation applied on the entire system, that is also, what is driving in the particle stress here.

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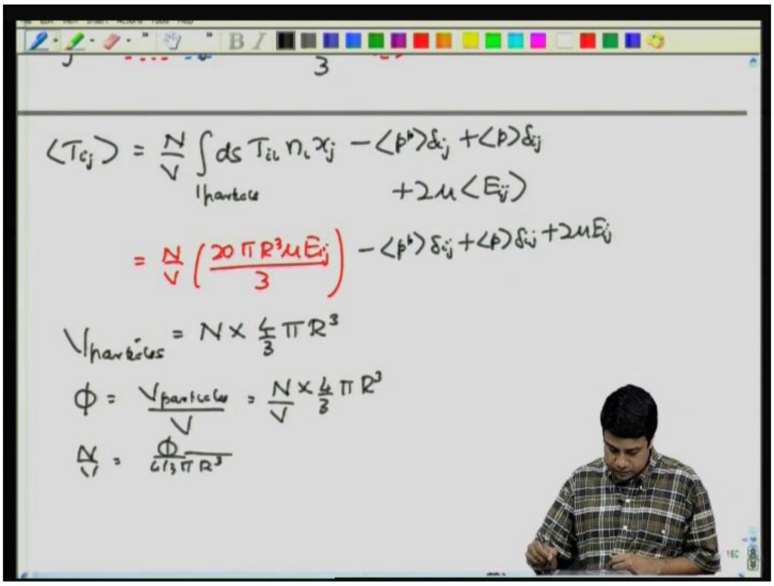

So, this is equal to N by V into 20 pi R cubed mu E i j by 3; that is the integral over one particle; this integral over one particle, N by V number per unit volume, it will be equal to the volume of a, the total the number of particles per unit, volume is going to equal to the volume fraction divided by the volume of 1 particle, because the total volume of all the particles; the volume of all the particle is equal to number of particles times the volume of one particle ok.

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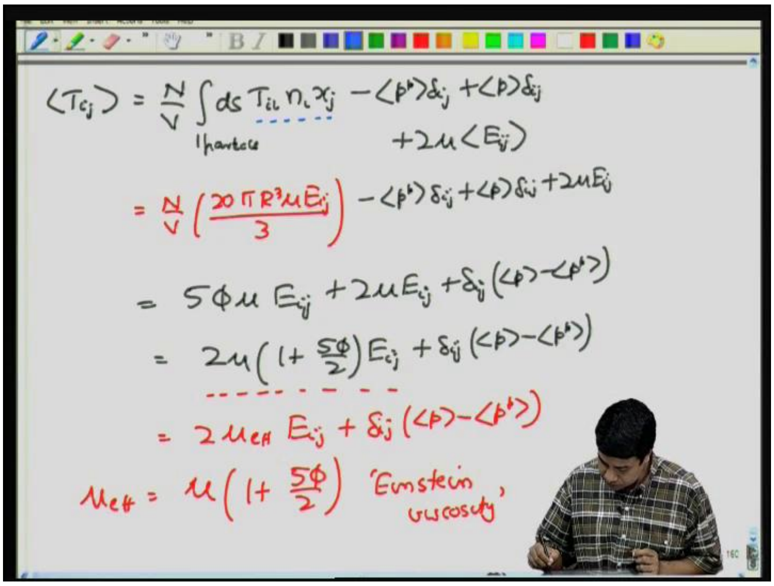

So, V particle is equal to number of particles into the volume of one particle 4 by 3 pi R cubed. Therefore, the volume fraction, which I will call here as pi; is equal to the volume of all the particles. The volume of all the particles divided by the total volume is equal to N into 4 by 3 pi R cubed, which means that N by V is equal to the volume fraction by 4 by 3 pi R cubed. Let me, just write that more clearly for you, N by V is equal to phi by 4 by 3 pi R cubed, where pi is the volume fraction.

So, I can substitute for N by V in this expression, 5 pi by 4 by 3 pi R cubed. If I substitute N by V is pi by 4 by 3 pi R cubed; what you get is 5 phi mu E i j, because 20 by 3 divided by 4 by 3 is gives me factor of phi plus 2 mu E i j plus the isotropic part, and I can add these two to give 2 mu into 1 plus 5 phi by 2 E i j. This can be written as 2 mu effective E i j, so therefore, we have got the contribution to the viscosity due to the presence of all of these particles, it does not depend upon the particle radius, it does not depend upon the number of particles, it depends only upon the particle volume fraction.

So, the effective viscosity of a suspension of particles; mu effective is equal to mu, the viscosity of the fluid times 1 plus 5 phi by 2; this is called the Einstein viscosity.

The reiterate, this effective viscosity of a suspension is only the limit of dilute small volume fraction, so that the particles are non interactive; that is the velocity field around one particle is not disturbed by the presence of the motion of the other particle. In that case, the effective viscosity of suspension of particles in the limit of low Reynolds number is equal to the fluid viscosity times 1 plus 5 phi by 2. So, this is an a significant result; this tells you, how the effective viscosity for suspension is related to the volume fraction of particles in the that suspension.

If you recall, in fundamentals of transport processes; one we had derived the similar result for the effective thermal conductivity of a suspension, it follow the similar procedure except that in that case, we have done the calculation, sometimes the legend of polynomial expansion for the temperature field.

In this case, we had solutions for the velocity and the pressure field and from the movement, the force moment integration of this term is T i L times n L is the force acting on the surface times x j T i l n l is just the f i, the force acting on the surface times x j, so it is a force moment, it is a second order tensor; it is not the cross product as in the case of the torque. So, based up on this force moment on the surface, we have calculated the correction to the viscosity.

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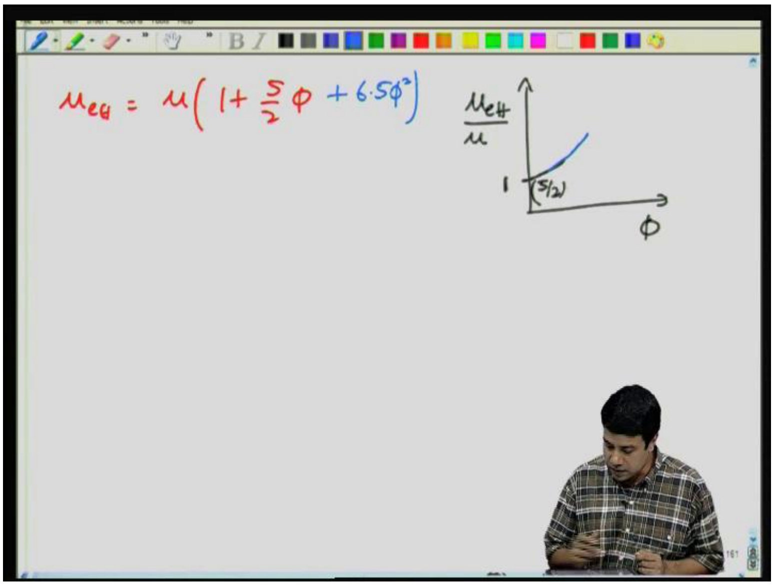

So, this result is equal to the fluid viscosity times 1 plus 5 by 2 pi predicts; that the viscosity of the suspension increases linearly towards the volume fraction of particles in the dilute limit in the limit, where there is no interaction between the pares of particles.

So, if I plot the volume fraction, verses the viscosity effective. I can just plot it as, a ratio of effective viscosity by the fluid viscosity itself. There are no particles volume fraction is 0, the fluid viscosity is equal to the effective viscosity at the ratio is just becomes 1, as I add a small number of particles, I get these linear relationship but, the slope that is given by 5 by 2 1 plus 5 by 2 pi, that is the slope very close to the origin of course, as the volume fraction increases there is going to start to be interaction between particles.

And, the calculation of the viscosity due to interaction between the particles is complicated one, it turns out that the next correction. Turns out to be 6.5 phi square, so the correction of the quadratic, the viscosity due to the quadratic turns out to be 6.5 phi square. This calculation of the next term takes into account the interaction between pares of particles. If your particle at one location, you find out what is the contribution due to all other particles. The disturbance caused due to the those on these particle, that is located at one particular location, average that out to find out, what is the effective viscosity, turns out is not a simple as imagined.

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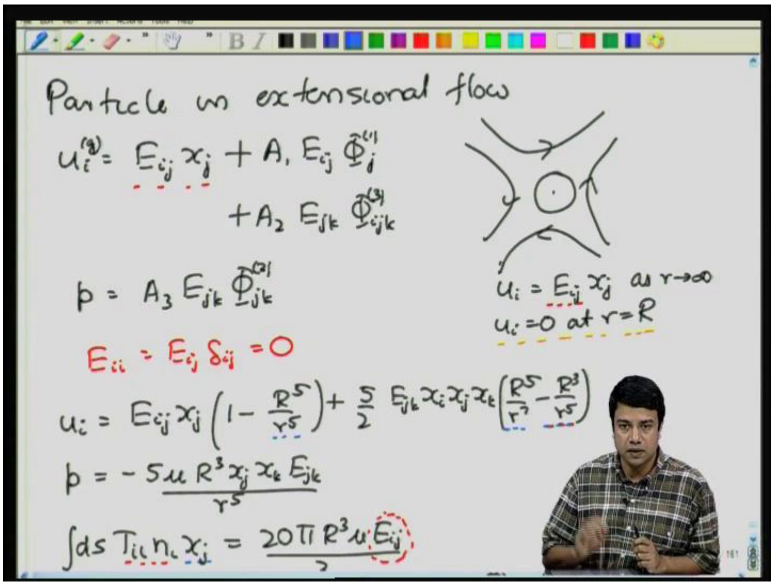

And, the reason is because of the decay of this of the disturbance; that that I talk to you briefly, about the slowest decaying contribution to the velocity. Here goes as 1 over R square, as I said its dipole term, if I try to integrate over the entire volume, that is at given location x. The decay, due the velocity disturbance due to another particle located at distance R, decreases as 1 over r square, I try to integrate over all space.

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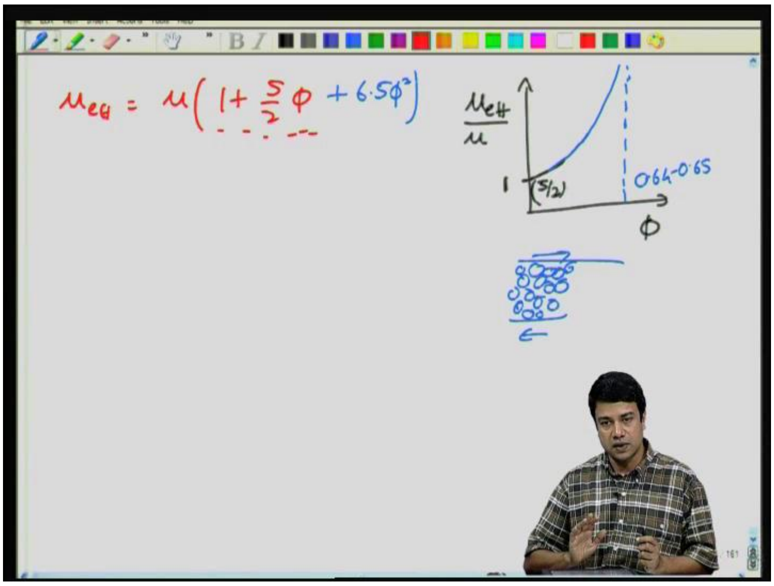

The volume in space increases proportional to R cubed, and therefore, I will get something; that increases the diverges as R goes to infinity turns out, that you have do the calculation carefully, take into account the renormalization of the forces effectively. That, we have take in to account the force exerted on a single particle due to all other particles, it also take into account the fact that, because of this interaction there is going to be a screening of the force a particle near the test particle is going to screen the effect of particles; that are far away.

Once, you do that you get a coefficient of 6.5 phi square, just this was first calculated by bachelor in the context of viscose flow. Of course, as you start putting in more and more particles, the viscosity is going to keep increasing, there is a limit, you cannot if the volume fraction goes beyond about 64 or 65 percent at this limit, the entire suspension is full of particles, and because the particles are rigid. If you just have large numbers of particles in between, and if these are rigid particles, you will find that you would not able to deform all, that means the viscosity has to go to infinite at this point, and it has to go to infinity, at the volume fraction of about 64 or 65 percent, because you cannot deform fluid, which is filled with so many particles; if the particles cannot go past each other.

So, that is how the viscosity of the suspension varies as function of volume fraction. In this lecture, we have calculated only this one term, calculated only this one term for a dilute limit but, you can experimentally determine those even vary dense limit, and it will diverge at 64 percent.

So, hope you found this interesting, this Einstein viscosity is actually quite a significant result, which was evaluated even without using many of the techniques, we used here so pretty format table calculation. We have done the disturbance due to an extensional flow, and rotational flow, as well as translating sphere. These three can now be put together, we will look at that in to the details in the next lecture.

We will see you then.