Fundamentals of Transport Processes II
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Lecture - 19
Torque on Rotating Sphere

So this is lecture number 19 in our course on fundamentals of transport processes two, where we are disusing fluid mechanics. We had first gone through some fundamental theorems of integral calculus, as you recall, and then we had derived the conservation equations, in terms of for the vector velocity field. What are called the navies stokes equations, which basically give you equations for the velocity and the pressure.

And we had discussed how to formulate the boundary conditions, and as I said these equations are in general difficult to solve, because they are partial differential equations and they are non-linear. The inertial terms in the conservation equation contains a non-linear product u dot grad u, due to which obtaining the general solution is difficult, because when an equation is non-linear, there is no guarantee that a solution exists, and if it exists there is no guarantee that it is unique.

Then we had restricted ourselves to the limit of lower Reynolds number, in the limit of lower Reynolds number the inertial terms in the conservation equations are neglected, and you get the stokes equations for the velocity and the pressure. The stokes equations are liner equations in both the velocity and the pressure and because of that; there is a considerable simplification, in the methods using first used for solving these.

The stokes equations as in the case of the diffusion equation for the concentration and temperature field, the stokes equations for the velocity and the pressure can be reduced to two Laplace equations, and these can be solved, because we have formulated ways of solving Laplace equations in different coordinate systems.

In fundamentals of transport processes one, I had shown you how to solve the Laplace equation by two methods; one was by the separation of variables separation of variables works in both Cartesian, as well as in Spherical cylindrical coordinates. Specifically, in cylindrical coordinates it gives rise to the legendary polynomial expansion and we had gone through it in some detail. And I have also shown you that the terms in the legendary polynomial expansion can be written as the source term, which decreases as one over r the dipole that goes as 1 over r square, and so on in higher order terms. In the present course, we looked at a different way to solve that and that was using vector notation and its gradients. So, let us just briefly review that before we proceed without discussion of stokes flow.

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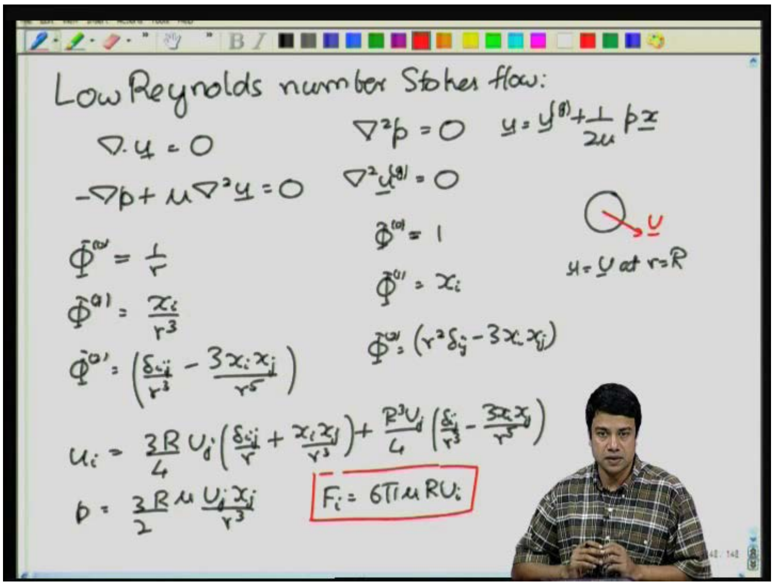

So, low Reynolds number stokes flow; the equations are the divergence of the velocity is equal to 0 and minus grad p plus mu del square u is equal to 0. And as I showed you this can be reduced to two Laplace equations by taking the divergence of the momentum conservation equation, del square p is equal to 0, that is for the pressure, a Laplace equation for the general solution for the velocity field. Note that this is the laplacian of a vector is equal to 0 and then my total solution for the velocity field just becomes u is equal to the general solution plus 1 by 2 mu p x I, this should be written as x vector in vector notation. So, that is the solution for the velocity field, if I can find the solutions for the pressure and the general part of the velocity field.

And as you recall, we found out solutions for the Laplace equation, if the first solution was 1 over r, the second solution phi 1 is equal to x i by r cube obtained by taking the gradient of phi naught, phi 2 is equal to delta i j by r cube minus 3 x i x j by r power 5, and so on and so forth.

So, these are the decaying solutions, which you would use for external flows, where the velocity decreases to 0 far away from the object or the source of the disturbance, corresponding to these are the growing solutions, which would you would use for internal flows, where r is equal to 0 is within the domain, but r is equal to infinity is not in the domain.

So, these decaying harmonics are used for external flows, where r is equal to 0 is not in the domain the flow around a particle for example, where s we except the solutions to decay as r goes to infinity, corresponding to this the growing harmonics are… this is just a constant the first one is just equal to the vector x i itself you take two gradients of this you get 0 I am sorry you take the laplacian of this you get 0, and the second one is equal to r square delta i j minus 3 x i x j, and so forth. You get higher and higher order terms.

The decaying harmonics decay as 1 over r to the n plus 1, so the first one for n is equal to 0 is just 1 over r, where as the growing harmonics increases r power plus n. So, phi naught is just a constant, phi 1 increases proportional to r, phi 2 increases proportional to r square, and so on. So, these are identical to the solutions you get by separation of variables that is the legendary polynomial expansion, and these are all orthogonal to each other, and therefore any general solution can be expressed as a linear combination of these solutions. So, one requirement of the solution of these Laplace equations is that the solution has be some combination of these legendary polynomial expansions.

Second requirement that if there is a forcing, if I have a sphere for example; the specific problem that we considered, a sphere with some velocity u in some direction, then the solutions have to be linear in the velocity of this sphere capital U, because you know that the stokes equations are linear in pressure and velocity, the stress is linear in pressure and velocity, and therefore if I have a forcing which some velocity u, and I get a certain solution for the velocity and the pressure field, if I reverse the velocity, the velocity at every pointing in the fluid reverses, if I change the velocity to some other direction, the velocity solution around that axis is identical to the solution that I had around the original axis, if I just change the coordinate in my coordinate reference frame to coincide with that particular velocity.

So, therefore the solutions for the stokes flow, velocity and pressure have to be linear in the spherical harmonics. They have also got to be linear in this velocity u and of course they stratify the incomparability condition, and they satisfy the boundary condition that u is equal to capital U at the surface of the sphere, where R is the radius of the sphere. Based up on these simple considerations, we had derived the velocity and the pressure field for this spherical particle.

Spherical particle u i is equal to 3 R by 4 U j to delta i j by r plus plus r cubed, so that was the velocity that we had obtained for the that was the solution for the velocity field, and the pressure field was equal to 3 R by 2 mu U j x j by r cubed around this basis we had actually calculated the stress on the surface of the sphere, integrated it over the entire surface of the sphere to get stokes flow. So, that was the stokes flow for the force, exerted due to the sphere moving at the velocity u in a fluid, that is stationary far away from the sphere.

Let us just take a little bit of time off in order to see, what exactly these velocities mean because we have calculated these velocities without making any reference to an underlying coordinate system; however, when you want to in practice plot, the velocity field you have to be able to determine it is components for example, u x, u y, u z in the Cartesian coordinate system, or u r, u theta, u phi in a spherical coordinate system.

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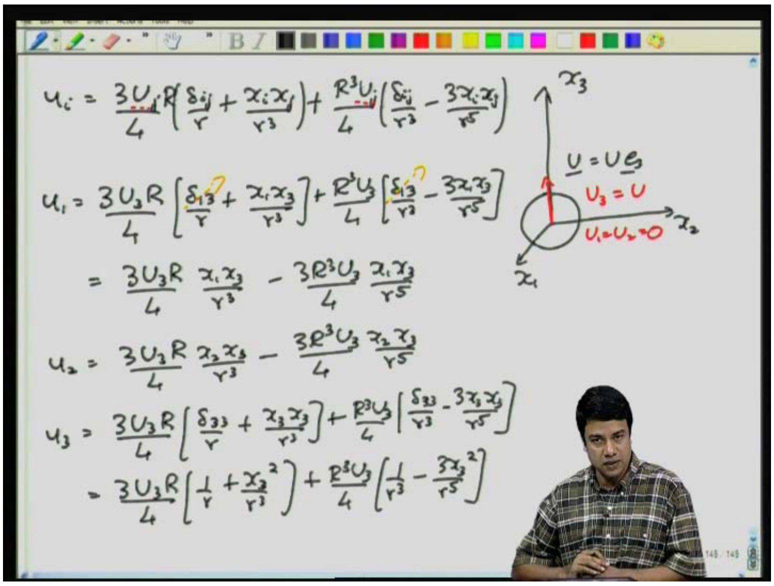

So, let us just briefly look at how that is done. So, let us say I have a Cartesian coordinate system x 1 x 2 x 3 a sphere with center at the origin, and I have my velocity field u i is equal to 3 U j by 4 delta i j by r plus x i x j by r cubed plus R cubed, so an R here. How do I find out the components of the velocity in this coordinate system? Firstly in order to make the problem simple for ourselves, we can align the sphere velocity with along the direction of one of the coordinates without loss of generality, because the sphere is moving in some direction I can always consider, that direction to be one of the coordinates in my coordinate system.

So, in this case I can consider this sphere velocity to be in the x 3 direction without loss of generality. So, that the velocity vector U vector is equal to U times e 3, that is x along the x 3 direction. Now I have u i, so u i is on the left hand side, there are three components; u 1, u 2 and u 3. Note that the only component of capital U that is non zero is u 3 itself, the only component of capital U that is non zero is u 3 itself, so whenever I have summations over that the only non zero component is going to be capital U in the x 3 direction.

So, I have U 3 is equal to U, and both u 1 is equal to u 2 is equal to 0, because the sphere itself have no velocity along the x y plan, because I have aligned the x 3 coordinate with the velocity vector. So, from this I can easily calculate what is u 1, u 2 and u 3? So, u 1 for example, is equal to 3 U 3 R by 4, I can take j as just 3, because that is the only non zero component, j is equal to 1 and j is equal to 2, u 1 and u 2 are 0.

So, I will get delta i j, i is 1 and j is 3 delta 1 3 by r plus x 1 x 3 by r cubed plus R cubed U 3 by 4 into delta 1 3 by r cubed minus 3 x 1 x 3 by r power 5, delta 1 3 is 0, because when i is not equal to j delta i j is equal to 0. So, this just becomes equal to 3 U 3 R by 4 x 1 x 3 by r cubed plus R cubed I am sorry should take a minus here, minus 3 R cubed U 3 by 4 x 1 x 3 by r power 5, because both of these deltas end up being 0, delta 1 3 is identically equal to 0.

So, that is how you would convert from the u i from the notation to the actual vector components, you can get x 2 I mean u 2 in a similar manner I would not go through the details just say that u 2 is equal to 3 U 3 R by 4 x 2 x 3 by r cubed minus 3 R power 5 U 3 of course, is a little different, because I have there delta 3 3, U 3 is equal to 3 U 3 R by 4 into delta 3 3 by r my plus x 3 into x 3 by r cubed plus R cubed u 4 by 4 delta 3 3 by r cubed minus 3 x 3 x 3 by r power 5, delta 3 3 is of course, one and therefore, I get 3 u 3 R by 4 into 1 by r plus x 3 square by r cubed plus R cubed U 3 by 4 1 by r cubed minus 3 x 3 square by r power 5. So, this tells you how to get the velocity components from r n d c l notation as excepted, on the surface of the sphere you can verify at r is equal to capital R both u 1 and u 2 will be equal to 0 and u 3 will be equal to capital U 3.

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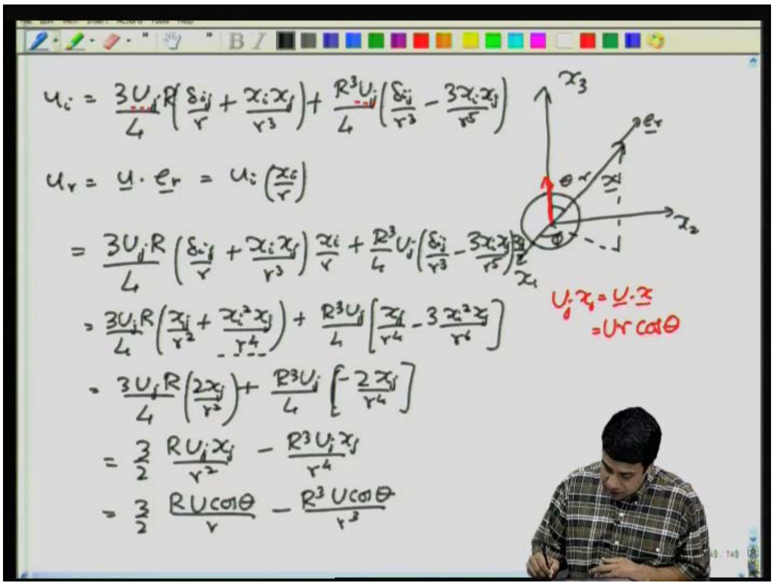

So, as excepted one could also convert into a spherical coordinate system, let me just show you how one converts this into to spherical coordinate system. Let us just get rid of this now for present in the spherical coordinate system of course, the unit vectors are e r e theta and e phi, once again I can align my coordinate system with the x 3 axis, and once I have aligned my coordinate system with the x 3 axis, then the radius vector in this case is just the radius vector to some location r vector, this is just the position vector in this spherical coordinate system to any location x vector.

This the distance from the origin is the radius r, the angle made by this distance from the x 3 axis is theta, and the angle made in the x 1 and x 2 plane is the angle phi. Now simply because of symmetry you except no velocity in the phi direction. The velocity is only in the r direction and the theta direction, you except no velocity in the phi direction just from symmetry.

So, what is the velocity in the r direction, u r will be equal to u dot e r, e r is the unit vector in the radial direction. This is also equal to x vector by r, the position vector by its magnitude, because e r is in the same direction as the position vector e r. So, this is equal to u i into x i by r, the velocity vector times the unit vector in the r direction, which is equal to the velocity vector times the position vector divide by the magnitude of the position vector. So, how do I get u i times e x i by r, you just take the expression for u i and you multiply it 3 U j R by 4 into delta i j by r plus x i x j by r cubed x i by r plus R cube by 4 U j by r power 5 into x j by r.

We multiply both by x j by r delta i j into x i is just x j. So, I will get 3 U j R by 4, this is just x j by r plus x i square x j by r cubed, x i square x j by r cubed x i square is x 1 square plus x 2 square plus x 3 square, which is just r square I am sorry I have multiplied by x by x i by r. So, I should have an r square here, and I will have an r power 4 here, plus R cubes u j by 4 into x j by r power 4 minus 3 x i square x j square by I am sorry 3 x i square x j by r power 6, and as I said x i square is equal to r square.

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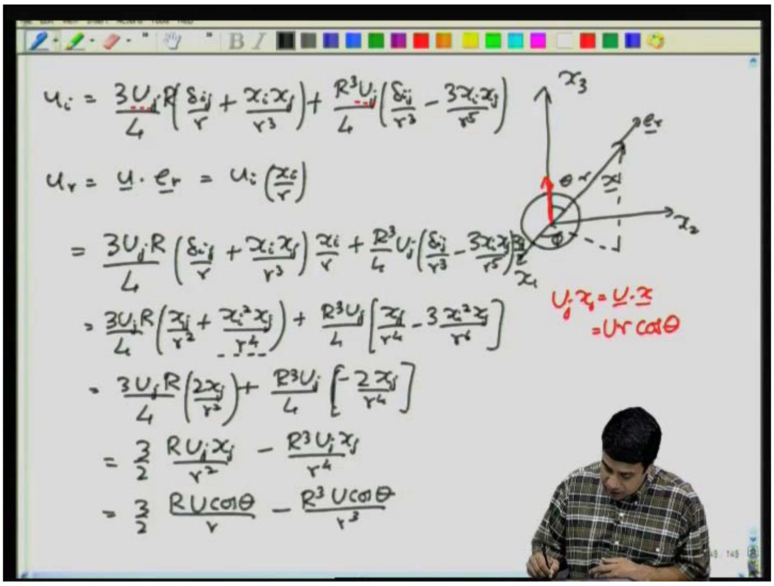

So, the first term jus becomes 3 U j R by 4 into x j by r square plus R cubed u j by 4 into minus 2 factor of 2 here, because I have x j by r square plus x i square x j by r power 4, x i square x j by r power 4 is just x j by r square. So, this just becomes equal to 2 x j by r square, here I have used the fact that x i square which is just x 1 square plus x 2 square plus x 3 square is just equal to r square. So, we get minus 2 x j by r power 4. So, I can take u j x j outside. So, i get 3 by 2 R U j x j by r square minus R cubed u j x j by r power 4.

And, now in this expression u j times x j is just equal to u times cos theta, u j times x j is just u times cos r I am sorry. U j x j is equal to U dot x, U is in this direction u is along the x 3 axis x vector is the position vector. So, U dot x is equal to magnitude of U times magnitude of x times cos theta. So, is equal to the magnitude of U which is capital U, magnitude of x which is r and then I have a cos theta. So, that can be used here to get the final expression 3 by 2 R u cost theta by r minus r cubed U cos theta by r cubed.

So, that is the final velocity expression for the velocity in the r direction. If you recall we had used this velocity in fundamentals of transport process I with some minor modifications, because when we did the high number transport, we had an expression for the fluid velocity field, around the sphere. In that case this sphere was stationary the fluid was moving with a velocity minus u, this is same as this except that I shift my reference frame from the center of the fluid, from the fluid at infinity to the center of the sphere.

So, you will get the exact same expression except it is modified, because I have to add the fluid velocity, because in this case the sphere is moving the fluid is stationary, in that case the fluid was moving far away and the sphere was stationary. So, this gives you a way of determining the velocity filed in a r theta phi coordinate system. We have calculated only the r component of the velocity, we have. Calculate only the r component of the velocity. We can also calculate the theta component of the velocity. The theta component of the velocity is perpendicular to the r component.

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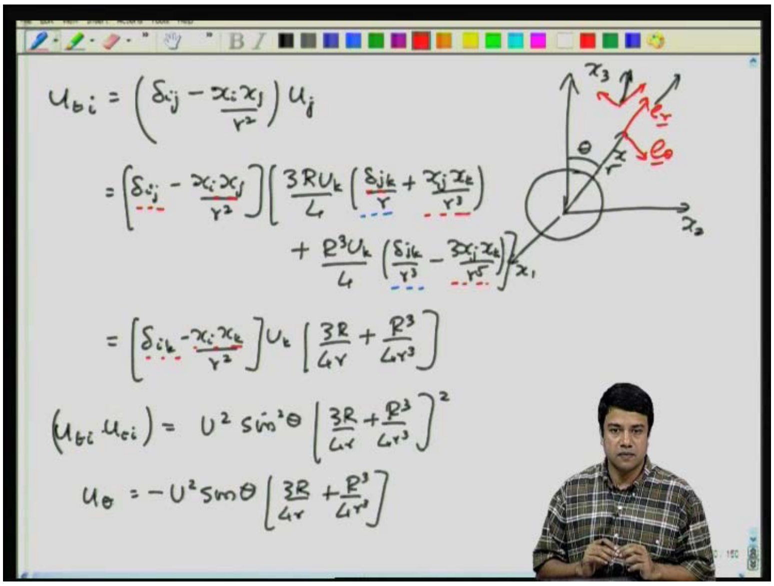

This is the radial direction and this is the theta direction. So, the theta component of the velocity is perpendicular to the r component of the velocity. If I take the theta angle is in this direction I should take it the other way. So, this is the angle theta e theta as I said has to be in the direction of increasing theta. So, e theta has to be in this direction. So, it is perpendicular to the r component of the velocity, it is perpendicular to x i by r it is perpendicular to that unit vector. The component of the velocity that is perpendicular to this vector you obtain by taking the transverse projection operator. So, if I have a particular vector e r here, x vector, so this position vector is the x vector and I want the component of the velocity perpendicular to x vector. Then the transverse component of the velocity u t i is equal to delta i j minus x i x j by r square into u j, as I said the unit vector in this case e r is equal to x i by r.

So, I get a component perpendicular to that by multiplying it that that by the traverse projection operator i minus e r e r, that is i minus x by r times x by r, that second order term, so I have multiplied by u j gives me the velocity that is perpendicular to this unit vector, that velocity has to be along the theta direction, because there is no flow along the phi direction just by symmetry.

So, I can multiply this to get the tangential component of the velocity minus x i x j by r square times. The fluid velocity field 3 R U j by 4 delta i j by r plus x i I am sorry you should be careful, when using indicial notations here because I have already used i and j for the transverse projection operator; that means, I need to use other indices here 3 R U k by 4 plus R cubed by r power 5.

So, I have to multiply this transverse projection operator by this velocity vector and of course, if you multiply it by the unit vector x itself, you will get 0, because this is perpendicular to x. And you can see that these two components of the velocity are along the x direction x j, x k, u k, x k dot u k is x dot u times x j which is the vector along the x direction.

So, these two components, when you multiply with the transverse projection operator you will get identically 0. So, the only terms that will be left are the terms which are along the u vector, that is this first terms here, delta j k times delta i j minus x i x j. So, basically what I will get here for this is equal to delta i k minus x i x k by r square into u k into 3 R by 4 r plus R cubed by 4 r cubed. When I multiply delta i j by delta j k I just get when I multiply delta i j by delta j k i get delta i k, and when I multiply x i x j by delta j k I get x i x k, the second term of course, is identically equal to 0.

And now in order to find out the magnitude of the velocity I have to take the square of this and then find out the magnitude, that is easily done. So, I multiply u t by itself and then take the square root, I would not go through the details I will just leave it as an exercise, this basically will give you u t i dotted with u t I, this is dotted with itself, I have to multiply this by itself again, and I will get U k square into delta.

Let me just give you the final expression this just becomes equal to u square into sin square theta into 3 R by 4 r plus R cubed by 4 r cubed the whole square. So, that is what you get for the final expression, because if i take u dot u I get U k delta i k times u I, that is u square minus U k times x I, x k. x k u k is equal to cos theta and x i u i is also equal to cos theta, you will get cos square theta 1 minus cos square theta will end up giving sin square theta.

So, I will leave that as an exercise for you. So, finally, I will get u theta is equal to u square sin theta into 3 R by 4 r plus R cubed by 4 r cubed of course, when I take the square root it could be plus or minus, if I take the square root it could be plus or minus but, from the direction of the vectors here, you can easily verify that the velocity flied has to actually have the minus sign. Velocity field has to have the minus sign simply because as the sphere is moving the velocity vector is in this direction, it is in the outward direction over here at a given position the velocity vector actually goes into this outward direction, and that is due to that has two components; one is along the plus e r, and the other is along the minus e theta direction.

So, this gives you a way of finding out the velocity fields, that we had obtained in the previous high (( )) number limit. We had used these velocity fields in order to calculate the transport rates in fundamentals of transport processes one, and here we have shown how you evaluate those velocity fields for the flow around a sphere. So, this completes our discussion I had also discussed in the previous lecture, how these velocity fields can now be used, in order to in the point particle approximation to get an equation for the disturbance to the velocity, in terms of the force exerted by the sphere.

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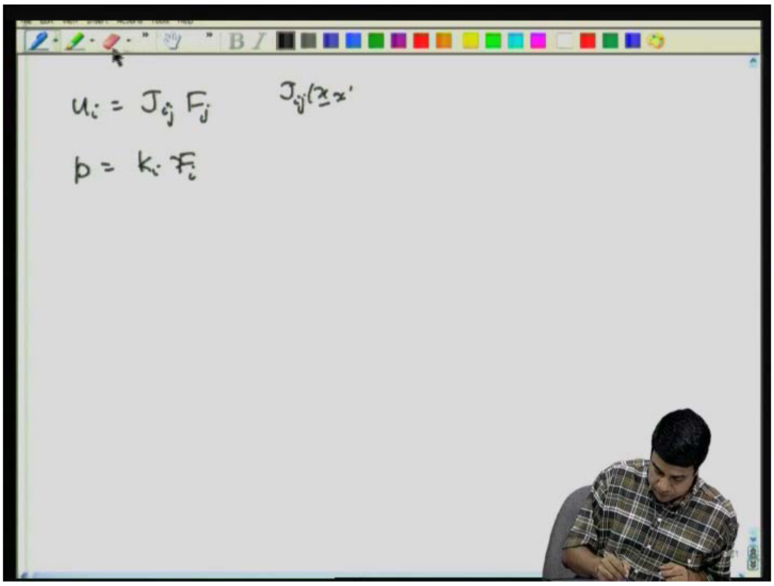

That expression was of the form u i is equal to J i j times F j and p is equal to k i F I, where J i j is the oseen tensor. Let me just write that in a longer form u i at a given location x vector is equal to J i j at x minus x prime F j at x prime, and p and x is equal to k i of x minus x prime F i of x, where J i j is a second order tensor and k i is a vector, these are called the oseen tensors.

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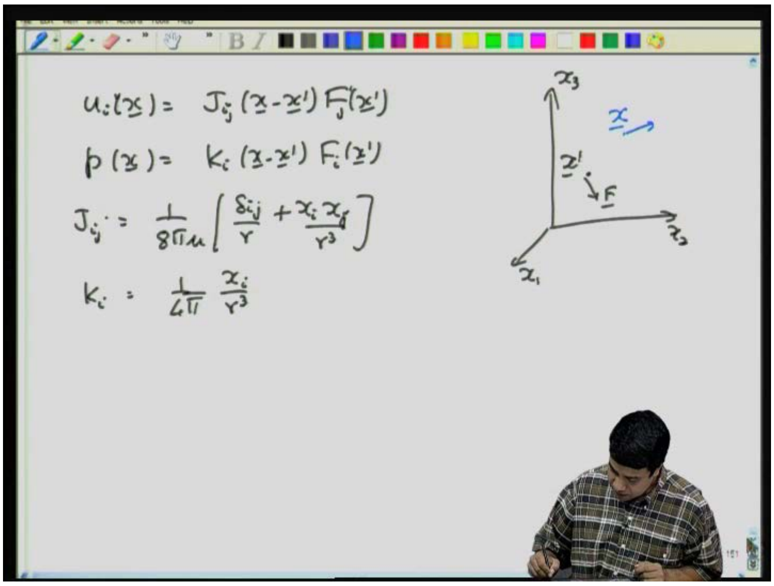

J i j is equal to 1 by 8 pi mu and k i is equal to by r cubed. So, the idea is this, if I have a 3 dimensional coordinate system x 1, x 2, x 3 and I have some object that is exerting a force f vector at the location x prime vector at the location x prime, and I want to know what is the fluid velocity field due to this force at the location x vector I want to know what is the fluid Velocity field due to this force at the location x vector.

That fluid velocity field is given by this expression; should note that this expression contains only a part of it this is only the slowest decaying term, as we saw in the case of the velocity field due to a sphere, there is one part that decays proportional to 1 over r, there is another part that decays proportional to 1 over r cubed; this contains only the part that decays proportional to 1 over r, this is the slowest decaying term.

In the limit as the particular goes to a point, what do mean by what do we mean by the point particle limit; that means, that the distance from the object to the location at which you are measuring the force the velocity is much smaller than the distance, between x minus x prime that is the particle radius r is much smaller than x minus x prime, in that case you can effectively set the particle radius r equal to 0, and take only the contribution to the velocity that is independent of the dimensions of the particle, it depends only on the force. As I told you if you rewrite the velocity field due to a sphere, in terms the force acting on the sphere, rather than the velocity of the sphere, it tends if you express in terms of the force. The term promotional to 1 over r becomes independent of the radius, and in that limit as the radius goes to 0, the force remains finite, you get to the point particle limit, in which the velocity disturbance decays as 1 over r, the distance from the object.

So, this was for a particular type of disturbance, where the sphere was translating with a constant velocity in the fluid. You could do the same thing for the case, where the sphere is rotating for example, to illustrate how the difference between the rotation case and the translation case.

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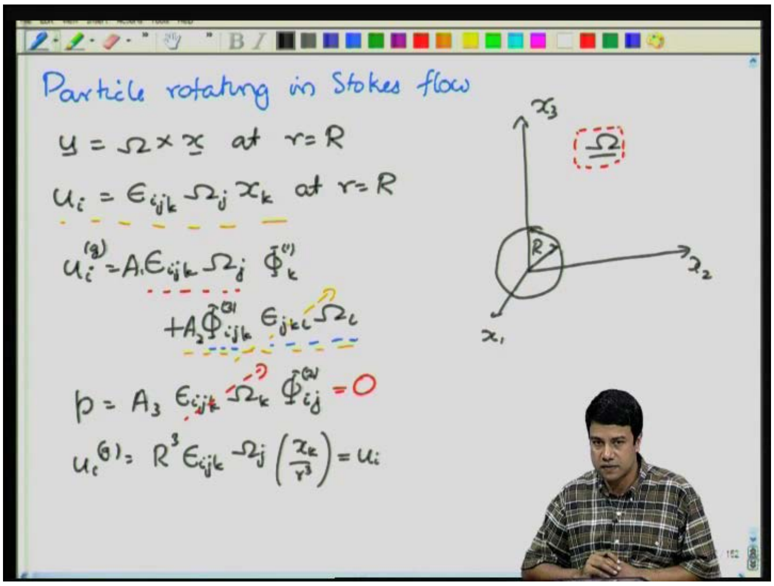

So, now let us take in stokes flow. So, we have a sphere once again of radius R, which is rotating under stokes flow condition. And the angular velocity of the sphere is the vector omega that is the angular velocity of the sphere. So, once again you have to solve the the stokes equation for the velocity and the pressure fields subject to the boundary conditions. In this particular case the boundary condition is that the velocity at the surface of the sphere, u is equal to omega cross r, the angular velocity cross product with the position vector. Let me just write the position vector as x vector at the surface of the sphere r is equal to capital R. So, velocity is equal to omega cross the position vector at the surface of the sphere r is equal to capital R.

So, that is the boundary condition and the equations are the stokes flow equations as usual, this boundary condition can be written in indicial notation, as follows u i is equal to epsilon i j k omega j times x k at r is equal to capital R. Now I have to find out the expressions for the general velocity, the expression for the pressure put them together to get the final velocity, check that what determined the constants in the expression from the condition that, the divergence of the velocity is equal to 0, and the boundary condition of the surface of the sphere is stratified.

If you recall, when we looked at the example of the of the sphere translating in a fluid, we said that the velocity field has to be linear in the velocity of the sphere. In this in that particular case, what was forcing the flow was the motion of the sphere at constant velocity, in this case what is forcing flow is the rotation of the sphere, therefore you would expect that the velocity field has to be linear, in this angular velocity omega vector.

So, if I were to increase the speed of rotation by a factor of two, you would accept the velocity at each point in the fluid to be increased by a factor of two. If you reversed the direction of rotation, the velocity at each point in the fluid would be reversed and therefore, the velocity field that I get should be linear in this angular velocity omega, and it should be linear in one of the spherical harmonics. This is once again an external flow. So, we need to take into account only the decaying spherical harmonics.

But, not just that one has to careful here, as i told you the angular velocity is a pseudo vector, it changes sign when you go from a right handed to left hand coordinate system. The fluid velocity on the other hand is a real vector. Since, the fluid velocity is a real vector, it does not change sign, when you go from a right handed to a left handed coordinate system, it is a real thing you can actually measure it in the excrements. Angular velocity the direction of that is depends upon the coordinate system that you use.

So, therefore, the velocity cannot be linear in just omega alone, it has to be linear in epsilon times omega, because epsilon is a pseudo vector the anti symmetric tensor is a pseudo tensor, it is direction changes when you go from a right to a left handed coordinate system, omega is also a pseudo vector. So, when you multiply the two, you end up with a real vector.

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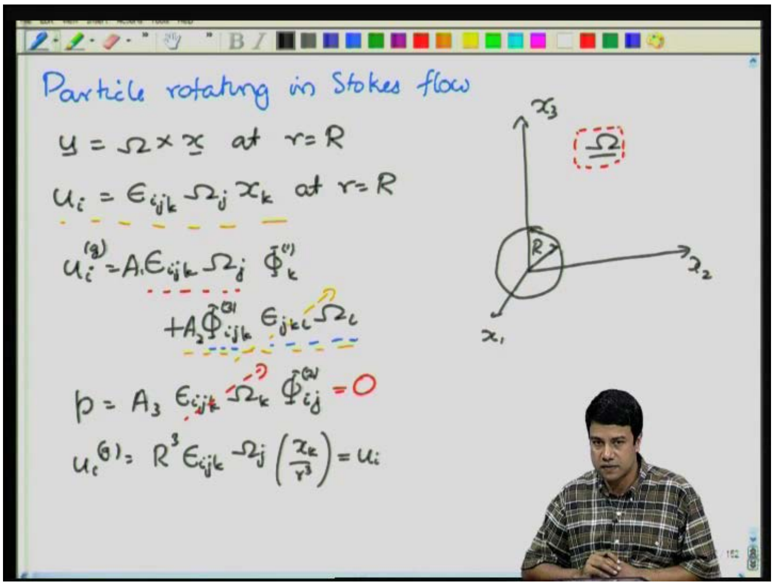

So, therefore the velocity u i has got to be equal to epsilon contacted with omega times a one of the spherical harmonics time's one of the spherical harmonics. So, epsilon i j k times omega k is a second order tensor, it is second order tensor because I have two unrepeated indices i and k. So, the second order tensor has to be multiplied, either you have to take the product of this with the first or a third order tensor in order to get a vector. So, therefore the only way that I can get a vector solution, which is linear in this second order tensor, as well as linear in one of the spherical harmonics is going to be to take the solution of this form.

So, I have to take the solution as a combination of two solutions; one is phi 1 k that is a vector. So, this second order tenser dotted with this vector would give me a vector, plus I can take 5 3 i j k epsilon j k l times omega l. So, this once again you can sees the vector, third order tenser dotted with a second order tenser will also give me a vector. So, I have two undetermined coefficients here, so I have one coefficient in terms of this one and other coefficient a two.

So, these as you recall only for the general part of the velocity field, it is only for the general part of the velocity field. So, I have forcing which is in the form of a second order tenser, now epsilon i j k times omega j. So, it has two frames i and k, out of that i would get a velocity field on the left hand side. So, I can either I can dot the second order tenser with a vector to get a velocity, or I can take two dots of this with third order tenser also to get a velocity, because if I have a third order tenser, and take two dot products of the second order tenser, I will get a vector, so there is the second party.

And now the pressure also has to be liner in this second order tenser epsilon i j k omega k. So, the pressure also has to be liner in this, the only way I get a scalar from the second order tenser, which is dotted two times with a second order tenser; that is the only way that I can get a scalar dotted two times with this second order tenser. So, that is the only way you get a scalar. The problem considerably simplifies for this particular case, the reason is as follows; if you recall phi 2 i j is symmetric tenser, you take two gradients of the fundamental solution is equal to delta i j by r cubed minus 3 x i x j by r power 5, in that tenser if I interchange i and j I get back the exact same result the symmetric tenser, obtained by taking two gradients on the fundamental solution 1 over r, epsilon i j k on the other hand is anti-symmetric.

It changes sign when i and j are interchanged I have a product of symmetric anti-symmetric tenser here, and therefore that product will be identically equal to 0, as because epsilon i j k omega k is symmetric in i is anti-symmetric in i and j, phi 2 is symmetric in i and j, and multiply in a anti-symmetric tenser I will get 0. So, the pressure is just equal to 0 by default, physically understandable if you rotate a particle, we do not except to generate a pressure gradient anywhere in the flow. Of course, if this is just the this pressure is determine only to a constant value, you can add any constant to this to get the pressure.

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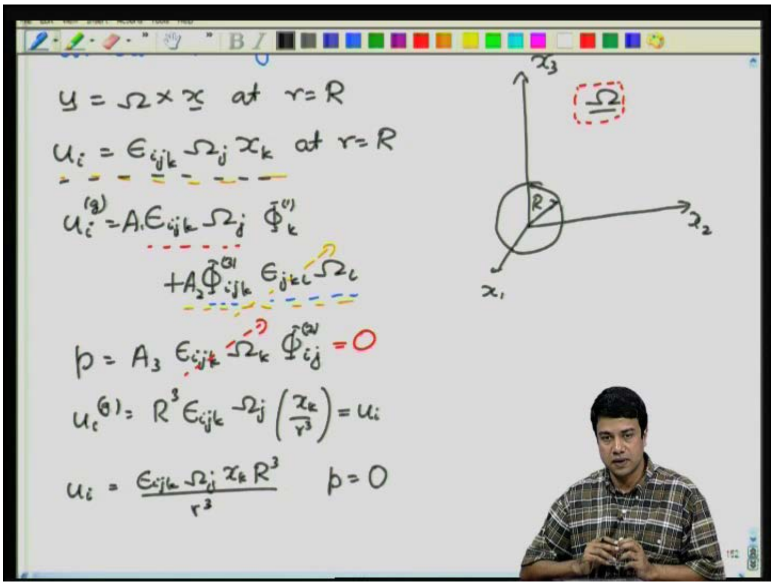

But, so this tells you, that there is no variation in the pressure, around the particle due to the rotation. This is understandable, because the rotation is actually related to the anti symmetric part of the rate of the definition tenser, where as pressure is the isotropic component of the stress tenser, and the rotating particle cannot generate net pressure. Similarly, in this expression for the velocity field phi 3 i j k is symmetric in the indexes j and k, because it obtain by successive gradient of that fundamental solution 1 over r epsilon j k l is anti-symmetric, and because of that this also has to be equal to 0, so both of these have to be identically equal to 0, and therefore the solution for the velocity field.

Now, the general solution for the velocity field is just equal to is equal to epsilon A 1 epsilon i j k omega j times phi 1 k is x k by r cubed, that is the final solution for the velocity field. So, if the pressure is equal to 0, this is not the general, but the entire velocity field, because the total velocity was equal to this general velocity plus 1 by 2 mu times p x I, pressure is identical equal to 0, therefore this is total velocity field.

The constant and if you take the divergence of this, I would not go through that I will leave it as an excesses for you, if take the divergence of this velocity field we get 0. So, this velocity I field identically satisfies this 0 divergence mass conservation condition. The constant A 1 is determine from the condition the boundary condition given here, the constant A 1 is to determine from this boundary condition, and you can easily verify that, the constant A 1which satisfies this boundary condition, A 1 has to be identically equal to R cubed, because if A 1 is equal to r cubed I get x k by r cubed on the surface of this sphere, when r is equal to capital R I will recover this boundary condition here.

So, there is the final velocity for a rotating sphere. Therefore, the velocity field is given by u i is equal to epsilon i j k omega j x k R cubed by r cubed. So, there is the velocity pressure is equal to 0. So, that is the velocity and the pressure field in the case. Now you would except that if the sphere is rotating in a fluid, there is no net force. So, there is no net force, because there is no reason for the force to be exerted in one way or the other, this sphere is only rotating; however, there will be a net torque acting on the sphere.

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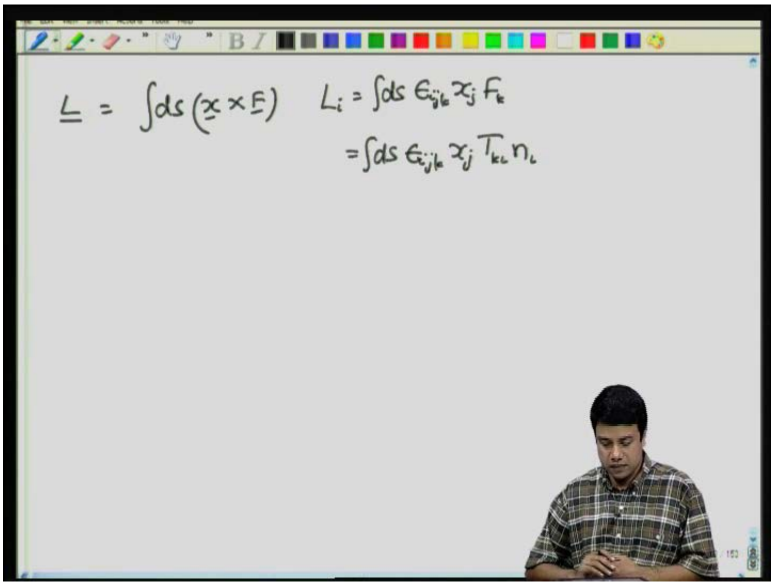

The net torque acting on the sphere, has to be the integral over the surface, I will call the torque as L i or let us go to write general expression, the torque vector is equal to into the surface of x cross, the force at each location of the surface. We except the net force to be 0, but there would in general be a non zero net toke, or if i write this in notation rotation a L i is equal to integral d s epsilon i j k x j times F k, where F k is the force per unit area acting on the surface. And the force can be written per unit area can be written in terms of the stress x j, stress I have to use T k L times unit vector n L.

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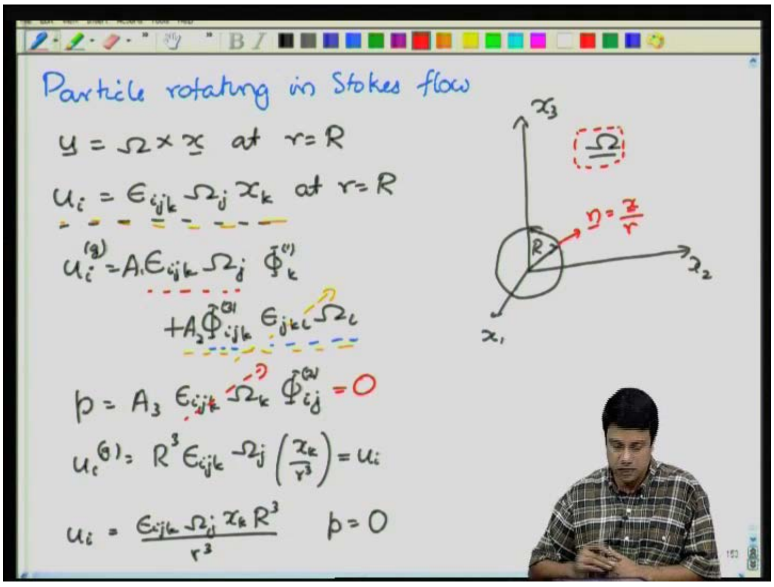

And the unit vector on the surface of this sphere, as you all know n vector is just equal to x by R, because on surface of the sphere, the unit radius vector is equal to just equal to x by the radius R. So, I have to multiply it by that in order to find out, what is the total torque.

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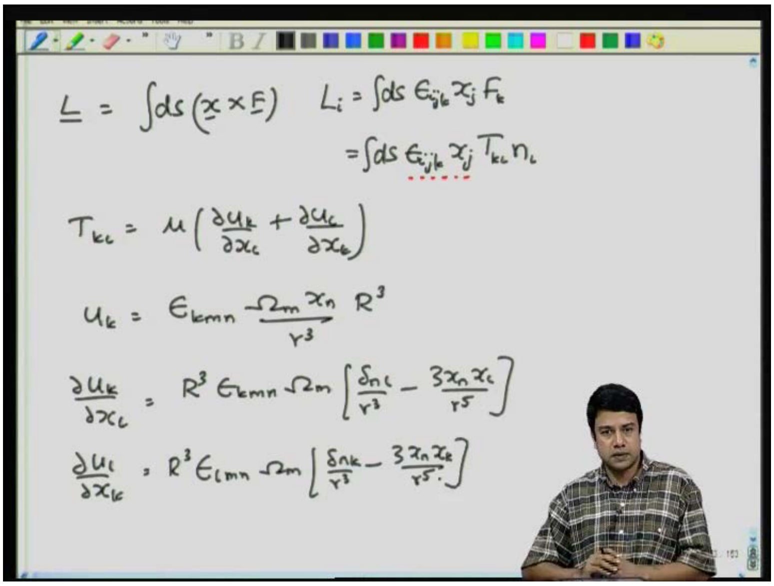

So, since the pressure is equal to 0, T k L is equal to partial u k by partial x L. We get the viscosity here, partial u k by partial x L plus partial u L by partial x k. Just to rewrite the velocity u k i had an expression for the velocity u i but, however, I since I have used i and j over here, I have to use a different index, now for the velocity, so the velocity u k is equal to epsilon k m n omega m x n by r cubed times. So, there is the expression for the velocity that I will use.

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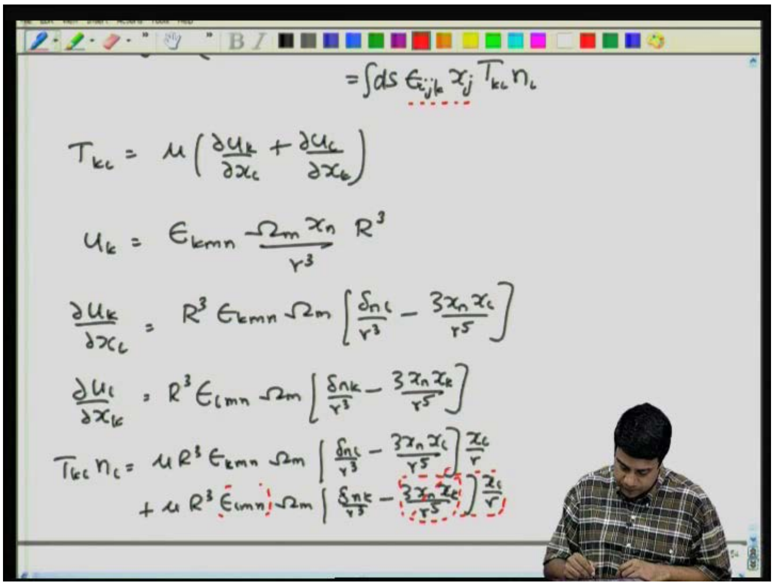

Now partial u k by partial x L is equal to R cubed epsilon k m n omega m into delta n L by r cubed minus 3 x n x L by r power 5. So, that is partial u k by partial x l, you can see the second order tenser with this is k n L unrepeated and m n L are both repeated. So, similarly, I can get the transpose of this is equal to R cubed I have to interchange k and l. So, I will get l m n omega m delta n k by r cubed minus 3 x n x k by r upon 5. So, those are the two components for the rate of deformation tenser, I have to multiply this by the viscosity to get T k L, this I will get mu R cubed epsilon k m n omega m into delta n l by r cubed minus 3 into x l by r x l by r. There are some simplifications that can be made quite easily over here, first thing first if you look at this term I have x n x k times x L times epsilon l m n. So, this is symmetric in this product here is symmetric in n and l, where as this is anti symmetric and therefore, this term goes identically equal to 0.

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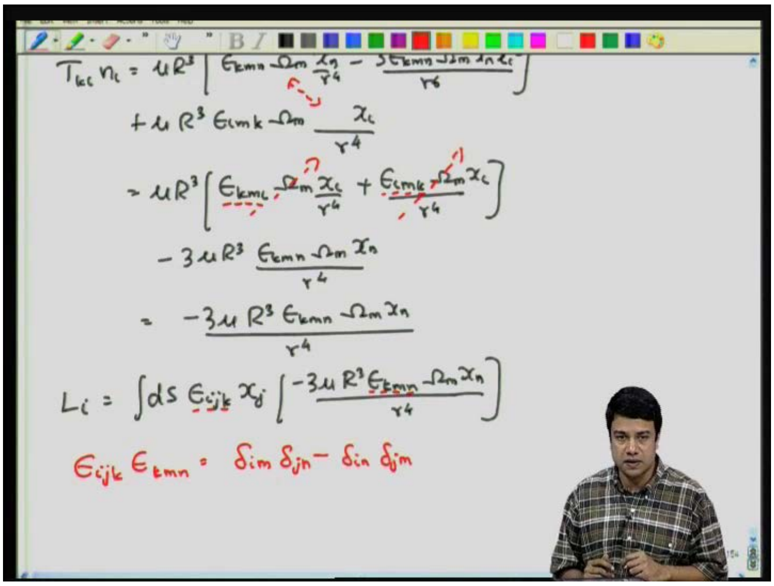

Similarly, if I expand out this term I will get T k L n L is equal to mu r cubed into epsilon k m n omega m x n by r minus 3 epsilon k m n omega m x n x l square, this will be there for hold by r power 6. So, there is a first term just expanding, it out the second terms becomes mu r cubed epsilon n m n omega m, then I have delta n k times x l, delta n k means that I can just replace n by k, delta n k x l by r upon 4 I can just replace n by k here. So, when I replace n by k in this expression, this becomes l m k and I can remove the delta here and I just get x l by r upon 4.

Now, you can see this epsilon k m n this term, here is epsilon k m n omega m x n and this is l m k omega m x l, and because of this these two are actually, the opposite of each other, only thing is if I have to rename. The repeated index from n to l, between these two terms here, if I just rename the repeated index n to l I get the exact same result but, it is the opposite. So, since our index is repeated I can just rename it; however, I want I can reach I can change l to k quite easily. So, if I do that, I will get mu R cubed into epsilon k m l omega m x l by r power 4 plus epsilon l m k omega m x l by r power 4 minus 3 mu R cubed epsilon k m n omega m x n, x l square just r square, so just x n by r power 4.

And you can easily verify that this is just the opposite of this one, because I get it by interchanging one index, between l and k and k and l, you can interchange one index. So, this is the opposite of this and therefore, these two are identically is equal to 0, and my solution becomes minus 3 mu R cubed epsilon k m n omega m x n by r power 4. So, there is the equation for the stress, in order to find out the angle of velocity.

The angle of velocity are L i is equal to integral d s epsilon i j k x j times T k l n l, which is minus 3 mu R cubed epsilon k m n omega m x n by r power 4. Now, here it becomes very useful to have a vector identity, because I have the product of two epsilons I have the product of two epsilons a product of two epsilons is n it is a tenser, which is independent of coordinate system, because both epsilons are either plus 1 minus 1 or 0 It is an event of coordinate system.

And it is real, epsilon itself was a pseudo vector a pseudo tenser but, if you take epsilon and multiply it by itself, we get a real tenser. So, let me just write out the result for you, you can easily verify that it is true, the result that we will use, here is epsilon i j k epsilon k m n is equal to delta i m delta j n minus delta i n delta j m, you can expand out the left and the right hand side for different values of m and n, and verify that this is a valid solution, you can expanded it out for different values and verify that this is a valid solution.

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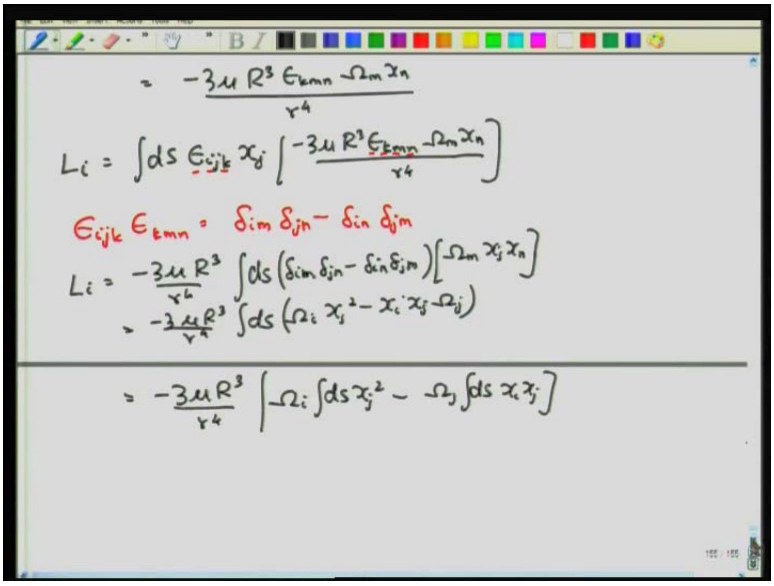

So, will use this to get the torque acting on the particle, so, first thing first I can take minus 3 mu R cube by r power 4 integral d s of delta i m delta j n minus delta i n delta j m into omega m x j x i. So, there is a final solution, so this is equal minus 3 mu R cubed by r power 4 integral d s omega i delta j into x g x n is just x j square minus delta i n. So, this is x i x j omega j, and this thing now be simplified. So, this is equal minus 3 mu R cubed by r power 4 into the first term omega i x j square omega i integral d s x j square, x j square is just equal to r square. So, this omega i times x j square is just equal to omega i times r square, integral over the surface minus omega j integral of the d s of x i times x j integral of x i x j over the surface, once again there is a little trick that we can use to get this.

Since, we are running short of time, kindly keep this in mind, we will continue this in the next lecture, in order to find out this integral of x i x j over the surface, there is a little trick that we will use. And once we get that, you will get, what is the torque acting on this spherical particle. So, we will continue this little bit in the next lecture, before we go on to another subject that is the effective, viscosity of a suspension of particles. We continue this in the next lecture, we will see then.