Fundamentals of Transport Process II
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Lecture - 18
Force on Moving Sphere

So, this is lecture number eighteen of our advanced course on fundamentals of transport processes. In the previous lecture, we were starting to solve, the Navier stokes equation, in the limit of lower Reynolds numbers, for a stokes flow. In the limit of lower Reynolds number, you neglect the inertial terms in the conservation equation, and you are left with just the pressure and the viscous terms. In a sense this is equivalent to the diffusion dominated regime that we had done earlier, in our fundamentals of transport processes one, where we basically solve the Laplace equation, for the concentration or the temperature fields, in the limit where convictive transport is negligible, except that this is more complicated, because I need to solve two equations; the mass conservation, and the momentum conservation equation.

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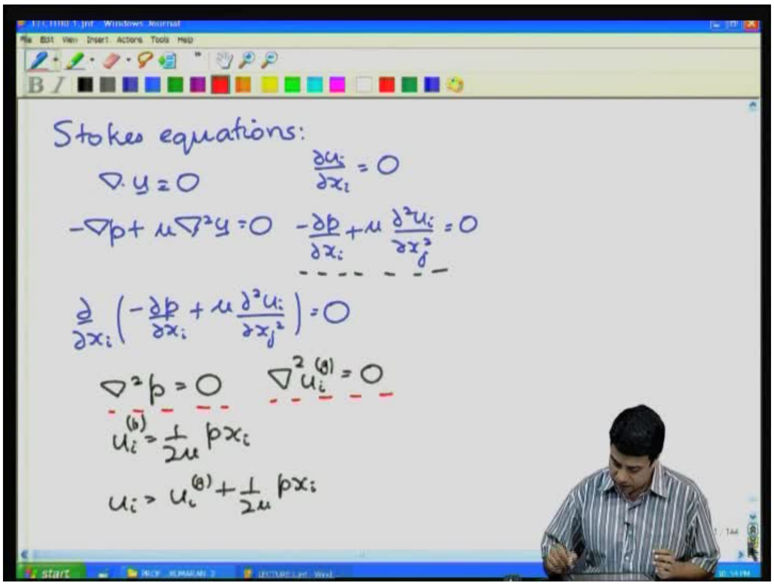

So, in the limit low Reynolds numbers, the equations had governed the flow or the stokes equations, the diversions of velocity is equal to zero; that is the mass conservation equation for an incompressible fluid, and then the momentum conservation equation minus grad p plus mu times the Laplacian of u is equal to 0.

We shall write this using indicial notations for most of the lecture; that is the diversions of the velocity is equal to zero and minus partial p by partial x i plus mu is equal to zero, so you will completely neglected all the inertial terms in the conservation equation. And I had briefly shown you how one can use, how can reduce these equations, to a set of Laplace equations. The way you do that, is by taking the diversions of the momentum conservation equation. So if I take the diversions of the momentum conservation equation. The second term is the diversions of the Laplacian of the velocity, since one can interchange the order of differentiation, is also equal to the Laplacian of the diversion of the velocity the diversions of velocity is zero. So this equation basically reduces to minus, this reduces to del square of the pressure is equal to zero, because partial by partial x i of partial by partial x i of the pressure is equal to zero. So the momentum, the mass conservation conditions, requires that the diversions of the Laplacian of the pressure, has to be equal to zero.

So using this one can evaluate the pressure, and now the pressure is an in homogenous term, in the equation for, in the momentum conservation equation for the velocity. So since it is an in homogenous term, the momentum conservation equation is an in homogenous, linear equation for the fluid velocity. This can be solved, by separating out the solution into two parts; one is a general solution, which satisfies the homogenous equation.

The homogenous equation, is the equation, without the pressure gradient in it, and therefore, the general solution, satisfies del square of u i general is equal to zero. So that is the general solution for the velocity. The particular solution is any one solution that satisfies the in homogenous equation. Note all the constants of integration, are all contained in the general solution. The particular solution, is any one’s solution, which does not have to contain any constants of integration. And we saw in the last lecture, that one particular solution, that one can get, is that u i is equal to 1 by 2 mu p times x i, this is the particular solution, which satisfies. It is one solution which satisfies the in homogenous equation. And therefore, the total solution u i is written as the sum of u i general plus the particular solution. So now the problem of finding a solution, reduces to the problem of finding the solutions of two Laplace equations; del square p is equal to zero and del square of u i general is equal to zero, and if you find the solutions of these two, you can put it into the equation, and then satisfy the boundary conditions. So we were trying to solve this in a general coordinate system, we were trying to obtain solutions in the form of scalar, vector, and tensor solutions.

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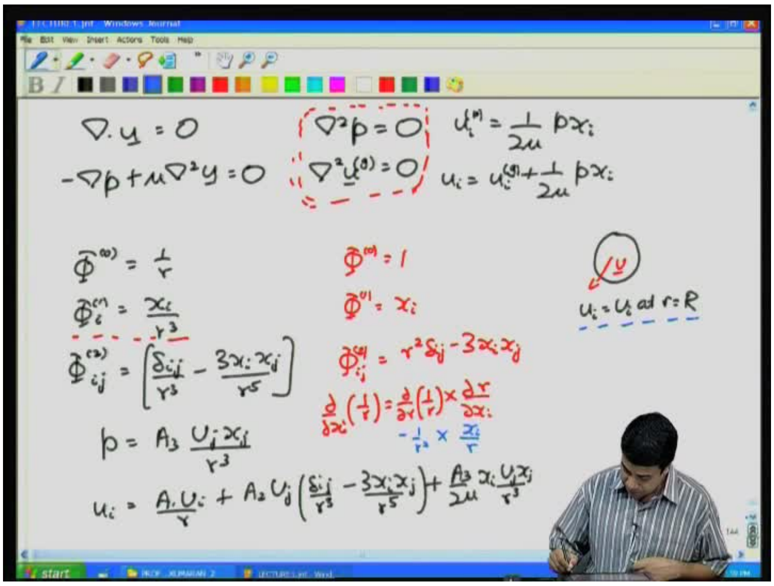

And the way that we obtained these solutions was, as follows. No, let us solve these two solutions, solve these two solutions and then obtain the solution for the total velocity field as sum of a general plus a particular solution. We know one solution of this; that is scalar solution; that is the solution for a point source, which we had solved in fundamentals of transport processes one. If we have a point source in an infinite fluid, the system is axis, it is spherically symmetric, and therefore, the temperature field is only a function of the distance, from the point source.

We can of course, place the origin of the coordinate system, at the point source, to the outlaws of generality. So since the solution is only a function of the distance from the point source, in a spherical coordinate system, there are conservation equation reduces to an ordinary differential equation, since there is no dependence on the theta and the pi coordinates, and that solution, if you solve in the Laplace equation in a spherical coordinate system, the solution is just equal to a constant divided by r, where r is the distance. So if there is a point source in the solutions are constant divided by r that is the flux, is independent of… Sorry the flux goes as one over r square, which means that the total heat or mass coming out of the volume, remains constant, independent of the distance from the origin. So that gave us the first solution, the scalar solution.

Now if this solution pi naught which is 1 over r, satisfies the Laplace equation, its gradient also satisfies the Laplace equation, because if I have an equation of the form del square pi naught is equal to zero. If I take a gradient of this; that is also equal to zero, because if a field, some function is equal to zero everywhere, its gradient is also equal to zero; that means at the gradient of pi naught, is also a solution of the Laplace equation. So from this, we get the solution pi 1, which is the gradient of pi naught and I showed you how to take the solution, here take partial by partial x i of 1 over r, and if you recall partial by partial x i of 1 over r, is equal to the derivative with respect to r of 1 over r into partial r by partial x i. And if you recall partial r by partial x i r is equal to square root of x 1 square plus x 2 square plus x 3 square. Therefore, partial r by partial x i is just equal to x i by r, and partial by partial r of 1 over r is minus 1 over r square. So to within an multiplicative constant, the solution becomes x i by r cubed. The constant setting in front does not have any physical significance, because these are just general solutions.

When I try to solve the equation up, subject to boundary conditions, I will be choosing an appropriate constant, in front of this function any way, that constant rule of course, chain sign, if if this if I use it negative sign in this function, but the final result will end up being the same. So minus x i by r cubed satisfies the Laplace the equation plus x i by r cubed also satisfies the Laplace equation. So this constant itself does not have any physical interpretation, further general solutions. Of course, I will put a constant in front and then solve it, when I do calculate the solution for actual problem. So in this case, by taking successive gradients, we get the first solution, which is a vector, the second solution which is a second order tensor, the third solution the third order tensor and so forth. And in the previous lecture, in fundamentals of transport processes one, we had solved these equations, using separation of variables in spherical coordinates. I showed you that, the solutions are obtained using the separation of variables are identical to these solutions. For example, one over r is the same as a solution obtained, using separation of variables for n is equal to zero.

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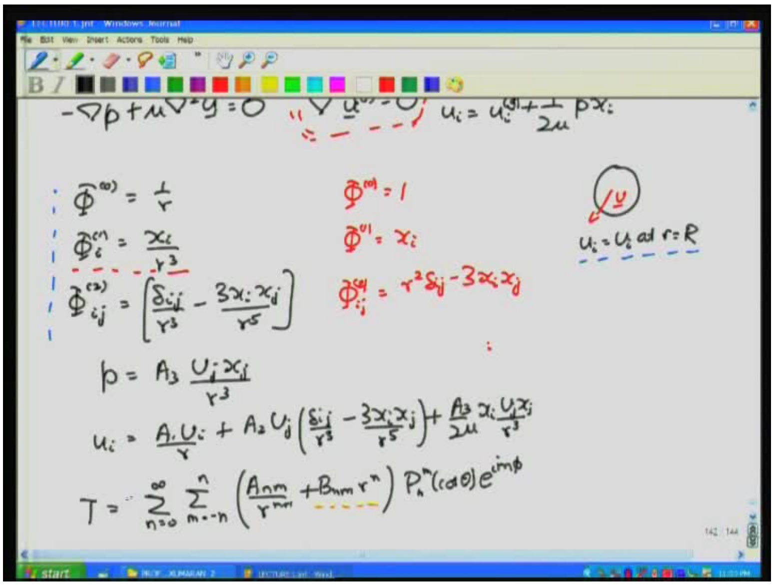

If you recall when we use separation variables, we had obtained the solution of the form, for the temperature field is equal to summation n is equal zero to infinity summation m is equal to minus n to plus n A n m by r power n plus 1 plus B n m r power n P n m or cos theta e power i m phi. So I showed you that this separation of variable solution for n is equal to 1, is exactly the scalar solution 1 over r. Separation of variable solution, for n is equal to 1. I am sorry for n is equal to zero, it is a scalar solution n is equal to 1 it is a vector solution, and in the separation of variables procedure you get three solutions; n is equal to minus 1 zero plus 1 for n is equal to 1. Similarly, for n is equal to 2 3 and so on. So these are exactly the same solutions that we got, they are linear combinations of these solutions. So corresponding to these solutions, these solutions are, what are called decaying harmonics.

The first solution pi naught goes as 1 over r, so as r increases or decreases as 1 over r. The next one decreases as 1 over r square, the next one decreases 1 over r cubed and so on. In a similar manner, one can also have what are called growing solutions, so the decaying solutions are only one part, the other part are the growing solutions. We got pi naught is equal to 1 pi 1 is equal to x i pi 2 is equal to r square delta i j minus 3 x i x j and so on and so forth. So corresponding to the decaying solutions, there also exists growing solutions, which correspond to this part, this part of the series solutions that we obtained by separation of variables. So using these two, we can construct any general solution. Since the solutions that we obtained in terms of legendary polynomials, are orthogonal to each other; that means the scalar vector solution, tensor solutions ,they are also orthogonal to each other. There are one to one correspondence between those solutions, and these one. Therefore, using a linear combination of these, we can construct the solution to any problem. In that linear combination, the constants will be chosen in such a way, as to satisfy boundary conditions. For external flows, for flows for example, outside a particle, so you have a an immersed object in a fluid; that means you are satisfying no slip, or no stress boundary conditions on the object, depending upon whether its solid, liquid or gas.

So for those kinds of flows, you also have to satisfy the condition, that the velocity field has to go to zero, far away from the object. So in that case, the solutions for the Laplace equations, cannot include the growing harmonics, because these growing harmonics go to infinity far away. So in that case, you would construct the solutions, using only the decaying harmonics alone, in the opposite case, for internal flows, where you have flows which are confined, within a finite domain. The origin of the coordinate system will be somewhere, within the domain, usually if the domain is symmetric at the centre of the domain. So in those kinds of internal flows, you have to make sure that the solutions that are used, are finite at the origin.

So clearly the decaying harmonics, are not suitable solutions, because they diverse as 1 over r power n, as you go to the centre of the origin. So in that case you have to use only the growing harmonic solutions. These finite at the origin, they do diverge as you go to infinity; however, for an internal flow infinity is not within the domain, so there is a finite volume, within which the flow takes place. So it is on this basis that one would chose, which harmonics to use. The last lecture we had derived the equations, for the flow around a sphere.

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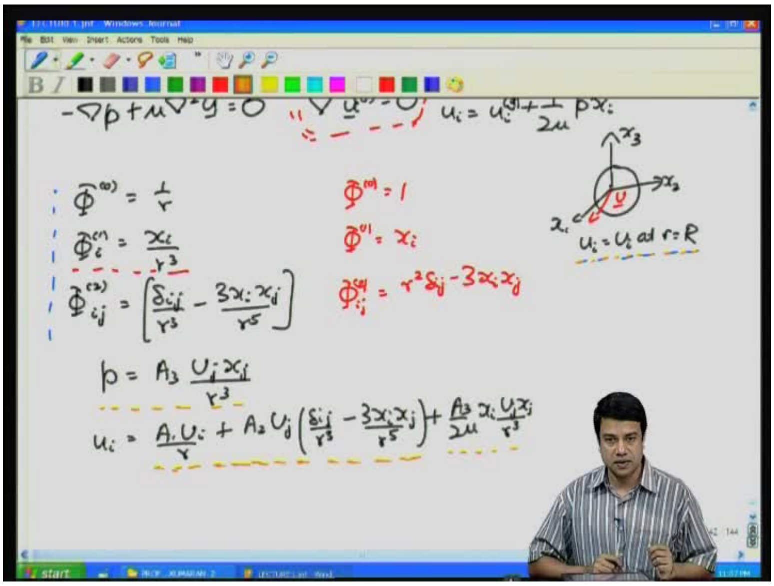

And in that case if you recall, we had used only the decaying harmonics. We had placed the centre of our coordinate system, the origin of the coordinate system, at the centre of the sphere, and then use this in order to evaluate the equations of the velocity field, due to this solution. Now the solutions firstly, since the general solution for the velocity, and the solution for the pressure, both satisfy the Laplace equation. They have to be a linear combination of the spherical harmonic solutions. In this particular case, there are linear combinations of only the decaying solutions, so there to be a linear combination of the spherical harmonic solutions. Also for stokes flow, we know that the solutions, have to be a linear function of the velocity. I shown you that since the equations, for the velocity and the pressure field are all linear, a change in the velocity at the bonding surface, is going to result in a change in a proportionate change, in the velocity and the pressure, everywhere within the domain.

The stress is linear in the velocity, and the pressure. The momentum conservation equation is linear in the velocity and the pressure, therefore if I have a solution for one particular velocity. In order to get the solution for any other velocity, I just need to multiply the velocity of the pressure, everywhere in the fluid, by that same factor, that still satisfies the equation, because of linearity. So because of that we have to get solutions, which are linear in the velocity, as well as in one of the spherical harmonics, for both the general part of the velocity, as well as for the pressure. And we had evaluated that in the last lecture, if you recall we got the pressure, to be of this form. The pressure end has to be a scalar, and it has to be linear in the velocity, as well as in the spherical harmonic. The only way you can get that, is by dotting the velocity of the sphere, with the vector spherical harmonic.

Similarly, the general part of the fluid velocity field, also had to be linear, one of the spherical harmonics, as well as in the velocity of the sphere. This is the velocity field everywhere in the fluid. It has to be linear in a spherical harmonics, as well as in the velocity of the sphere. You can get it two ways, either you multiply the sphere velocity by a scalar, or you dotted with a second order tensor, and these two are the solutions for the general part of the velocity profile, the general solution for the velocity profile, add to that the particular solution, to get the final solution for the fluid velocity, around the sphere. And these constants a 1 a 2 and a 3, if you recall in the last lecture, were derived from the condition; that firstly, the flow is incompressible; that is the diversions of the velocity has to be zero, and secondly, the velocity on the surface of the sphere, is equal to capital U; that is the boundary condition u i is equal to capital U i at r is equal to capital R. Using these two conditions we have derived both the pressure, as well as the velocity field.

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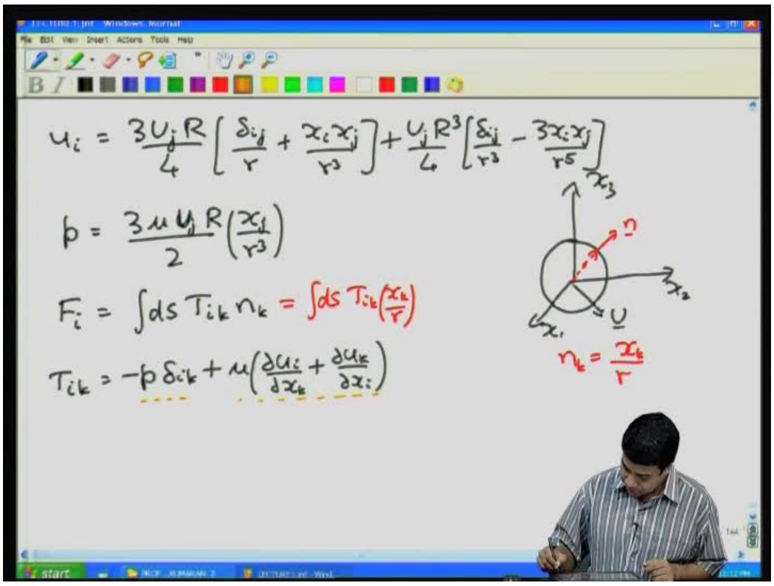

And the final result that we got for the pressure and for the velocity field were, u i is equal to 3 u j r by 4 into delta i j by r plus x i x j by r cubed plus u j r cubed by 4 into. So this was the velocity field, that satisfied all the boundary conditions, as well as incompressibility condition, and the pressure was given by 3 mu u j r by 2 into x j by r cubed, so those were the velocity and the pressure. So we had to go through some detailed algebra to get to the stage, but it is important, because I will be making some points with respect to this little later on, which will be important for general solutions of stokes flow for general objects, and I will come back to that a little later. So our work is not done yet, we still have to evaluate the force, acting on the spherical particle. The spherical particle is moving with some velocity u, and I need to calculate the force, acting on the spherical particle.

The force acting on the spherical particle, is of course, the integral, of the stress dotted with the unit normal, integrated over the surface of this particle. So force acting on the particle, is equal to integral over the surface, of the stress dotted with the unit normal. Now the unit normal at any position, the outward unit normal, at any position on the surface, is exactly along the position vector to that position on the surface. The outward unit normal, is exactly in the same direction, as the position vector. So it is parallel to the position vector, and it has unit length; therefore, this outward unit normal, is just equal to the position vector x k. Recall that when we use indicial notations x k represent, has summation as well as unit vector, so it is equal to x 1 plus x 2 plus x 3, I am sorry is equal to x 1 e 1 plus x 2 e 2 plus x 3 e 3, so that is the position vector at the surface.

Of course, the unit normal has to have unit length; that means I have to divide by the magnitude, of the position vector, to that point on the surface, note I am placing my origin at the centre of the sphere. Therefore, the length of the position vector is just equal to the radius itself. So n k is equal to just x k by r, where x k is the position vector to that point on the surface, and r is the radius, the distance from the origin. So this force becomes integral d s T i k x k by r, integrated over the entire surface of the sphere, so that is what is going to give me the force. Of course, the stress itself T i k is equal to minus p delta i k plus mu into partial u i by partial x k plus partial u k by partial x i, so that is T i k. So I have to put in two parts; one is due to the pressure, and the other is due to this symmetric part of the rate of definition tensor. So this is going to involve a little bit of algebra, but I will go through it, just to illustrate how this is done. It is an important exercise for us, to see how one can actually do this calculation, and get a result in the end.

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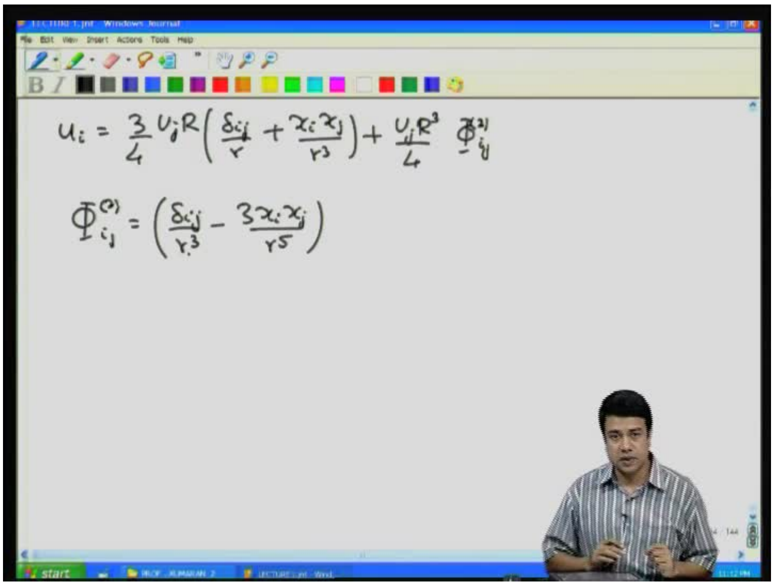

This is the only calculation we will go through in detail, for the rest I will just leave it, as an exercise for you to do it. So I will write this as u i is equal to 3 by 4 u j r into delta i j by r plus x i x j by r cubed plus r cubed by 4 into the second solution, for the spherical harmonics, where if you recall the second solution that we had, for the spherical harmonics was equal to delta i j by r cubed minus 3 x i x j by r power 5; that was the exact value of phi 2 i j. I will leave it in this form for the present, just so that it makes the algebra simple. So in order to calculate the stress tensor, i need mu times the gradient of the velocity. The first term that I have there, is mu times partial u i by partial x k.

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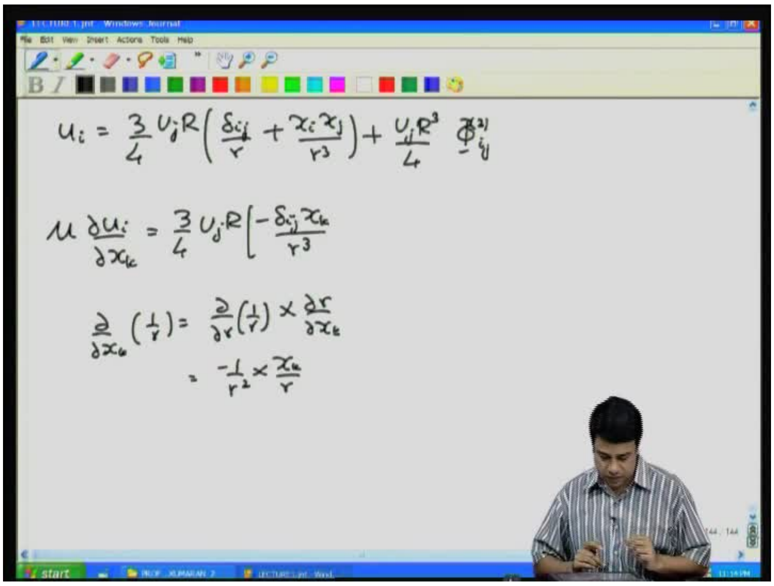

So I have to take the derivatives of each of those terms in the brackets, with respect to x k, as well as the derivative of the last term there on the right, so let us just go through that. This 3 by 4 u j r into when I take partial by partial x k of 1 over r a partial r by partial x k is just x k by r, and the derivative of 1 over r with respect to r is just minus 1 over r square. So this just becomes equal to minus delta i j x k by r cubed. Just to explain it once again, partial by partial x k of 1 over r is equal to partial by partial r of 1 over r times partial r by partial x k which is minus 1 over r square derivative of 1 over r with respect to minus one over r square partial r by partial x k we had shown, was equal to x k by r. So we will use this for all powers of r, which are there in the denominator in this expression.

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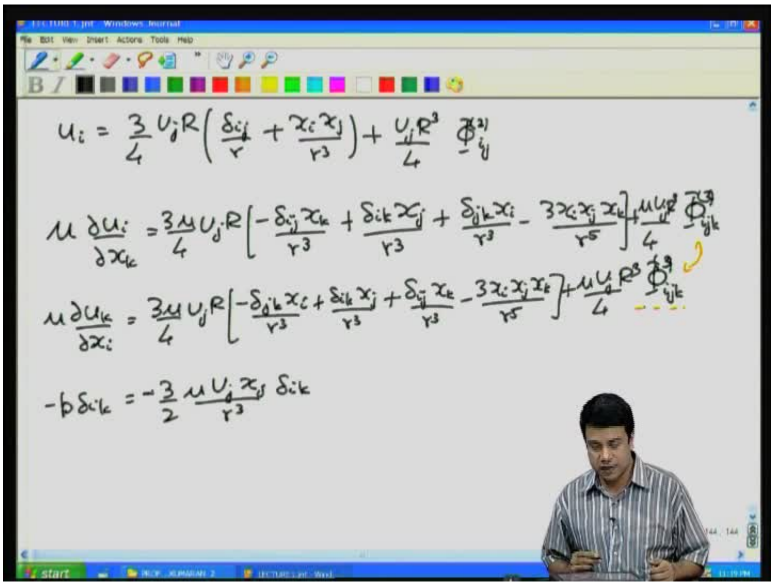

And then the second term, I have three derivatives; one of x i with respect to x k x j with respect to x k and 1 over r cubed with respect to x k. So when I take the derivative of x i with respect to x k partial x I by partial x k is 1 phi is equal to k it zero phi is not equal to k, so this just gives me delta i k 1 phi is equal to k and zero phi is not equal to k x j by r cubed plus delta j k x i by r cubed, and then I have to take the derivative of 1 over r cubed the denominator 1 over r cubed gives me minus 3 by r power 4 times x k by r. So if you work it out it will be minus 3 x i x j times x k by r power 5, because this minus 1 over r cubed times x k by r 1 minus 3 by r power 4 times x k by r plus the second term here which for the present I will just leave it as; u j r cubed by 4 times partial by partial x k of phi 2 i j is just the third solution, the third order tensor solution, which is just phi 3 i j k, the third tensor solution for the spherical harmonics. So that was the first part, the mu times partial u i by partial x k. If you recall the stress tensor also contained mu times partial u k by partial x i. I should put in a factor of mu everywhere, and then I have a factor of mu in the last term.

So in my expression for partial u k by partial x i I just interchange i and k in the expression for partial u i by partial x k; that is just equivalent of taking the transpose. So I interchange i and k in the first term, I will get minus delta j k x i by r cubed plus delta i k x j by r cubed plus delta i j x k by r cubed minus 3 x i x j x k by r power 5 when I take the gradient of phi j k with respect to i in this last term here. I get something that is identical to what I had got here, because the, both of these are obtained by taking three gradients, of the fundamental solution. Take the fundamental solution, partial with respect to k gives me phi 1 k, partial with respect to j gives me phi 2 j k, and then with respect to i gives me phi 3 i j k. The way you take those derivatives does not matter; therefore, you get exactly the same result for this last term, which we will evaluate a little later. And finally I have my first term here minus p delta i k, using the expression for the pressure that I had, I will just get 3 by 2 mu u j x j by r cubed times delta i k. So I just take the pressure, and multiplied by delta i k with the negative sign there.

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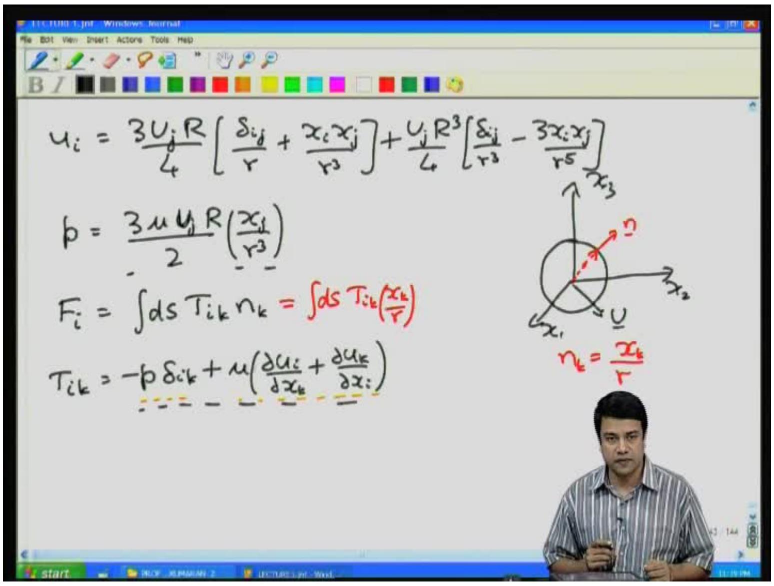

If you recall this was the expression for the pressure, and I just have to multiply with the negative sign, and multiply by delta i k. So this gives me all of these terms, in the equation for the stress, now I have to simplify this.

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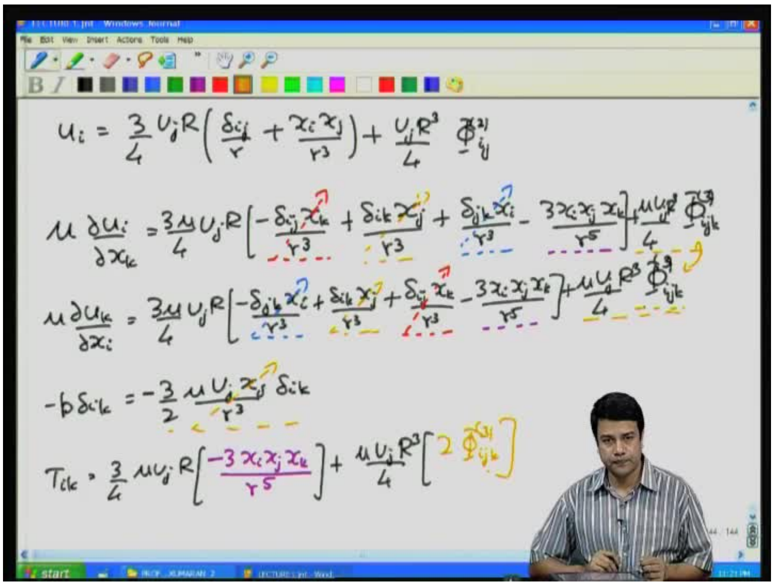

So you can easily see that, when I add these up there are certain terms, in the first set of brackets that will cancel out. This term and this term are the same, except that they appear with the negative sign. Therefore, this term, when I add it up, this term will be cancelled with this term. Similarly, this one and this one are the same, except that in the second expression they have a negative sign, so these two will cancel out, that leaves me with a term in the middle in all of these cases. those appear with the positive sign, in the expressions for the velocity gradient, but you can see that an identical term to these, is appearing with a negative sign, in the expression for the pressure.

So when I add up the pressure, to these two velocity gradients, these terms will all cancel out, and I will be left with an expression, for the stress tensor as T i k is equal to 3 by 4 mu u j r. I only have these terms remaining, they are both of the same sign, exactly equal in magnitude x i x j x k by r power 5, and I have minus 3 times that, so I get minus 3 x i x j x k by r power 5 plus u u j r cubed by 4 into these last two terms here, these last two terms here. As I said once again, they are both identical, once again they are both identical, so I just get two phi 3 i j k. So that is the final expression for my stress tensor.

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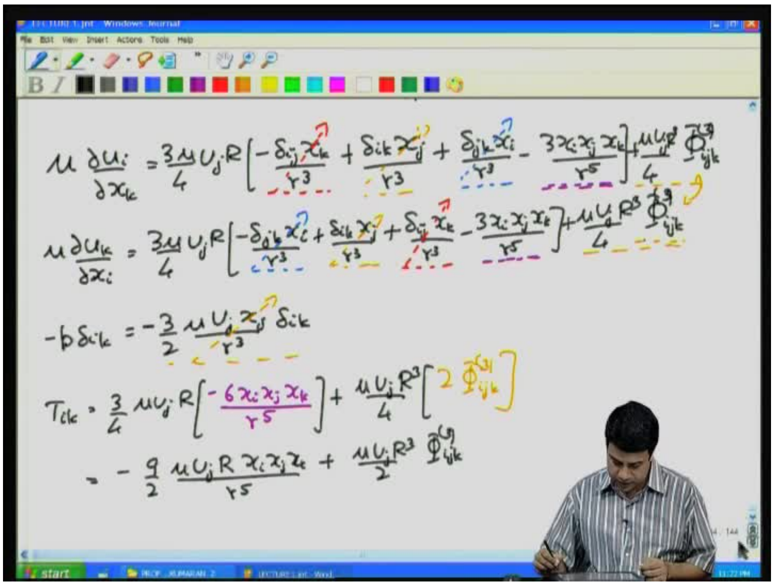

And this I can write it as minus. So I have two of these terms here, so these terms that are, there are two of them, so I get 3 into 2 here, so minus 3 minus 6 x i x j x k, because I have two of those terms there. So I get minus 9 by 2 mu u j r x i x j x k by r power 5 plus mu u j r cubed by 2 into phi 3 i j k. So that is the final expression for my stress tensor.

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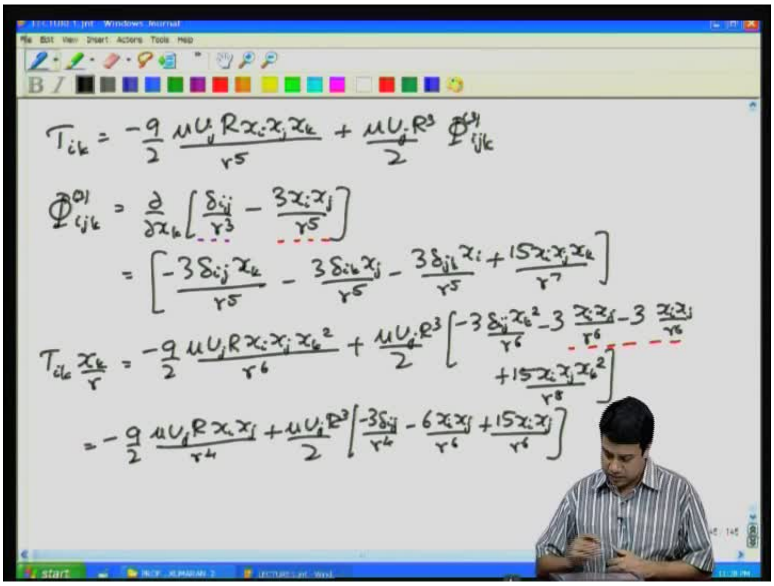

So let us proceed, and I will simplify the last term there T i k is equal to x k by r power 5 plus. And now we have to evaluate this term phi 3 i j k is equal to partial by partial x k of, and evaluated exactly the same way that we have been doing earlier. This is equal to minus delta i j x k; that is the first term here. And then at the derivative of the second term I have, the derivative of the second term I have a derivative with respect to both the numerator, as well as the denominator. So the derivative of the numerator will give me minus 3 delta i k x j by r power 5 minus 3 delta j k x i by r power 5 plus 15 x i x j x k by r power 7. So that is the expression for phi 3 i j k. Now, in the expression for this force, we ultimately had T i k dot n k T i k dotted with the unit normal. So T i k n k was equal to T i k times x k by r.

So in the expression for the stress T i k x k by r; that will be equal to minus 9 by 2 mu u j x i x j x k square by r power 6, put a r in here, so that is the first term. And the second term, it has plus minus 3 by 2 delta i j x k square by r power 6, let us put the three factor there; that is r cubed by 2 into k square by r power 6 minus 3 by 2 delta i k times x k just gives me x i x j by r power 6 minus 3 by 2 x i x j by r power 6 plus the last term there, which is plus 15. I should remove the two's here, because two is already outside, 15 x i x j x k square by r power 8. So this can be simplified, because x k square which is just x 1 square plus x 2 square plus x 3 square is equal to r square, because x k square is equal to r square I can simplify this, to get minus 9 by 2 mu u j r x i x j by r power 4 plus mu u j r cubed by 2 minus 3 delta i j by r power 4 minus. These two terms are identical, so I get minus 6 x i x j by r power 6 plus x i x j x k square by r power 8 this gives me 15 x i x j by r power 6.

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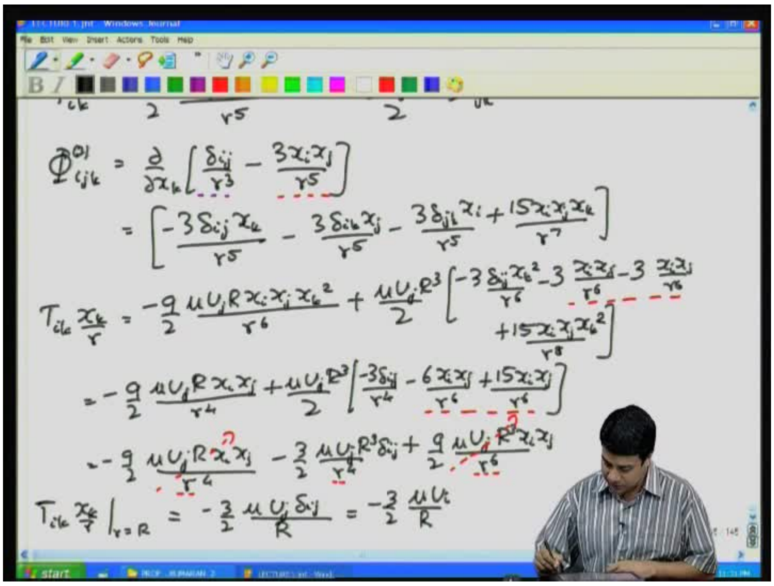

Now you can just add up these last two terms here, to give me minus 9 by 2 mu u j r x i x j by r power 4 minus 3 by 2 mu u j r cubed by r power 4 delta i j plus 9 by 2 mu u j r cubed x i x j by r power 6, so that is the final expression for the stress dotted with the unit normal. Now the simplification starts, because I need the stress at the surface of the sphere. The surface the sphere is at r is equal to capital R. So in this expression, I just have to insert capital R instead of small r. in this expression i have to insert capital r instead of small r So T i k x k by r at r is equal to capital R. I just have to insert capital r instead of small r in this case, in these two terms in the denominator, in these terms in the denominator and when I insert capital R instead of small r in the terms in the denominator, you can easily see that this one, will exactly cancel with this one, because they are both identical except of negative signs, and this gives me minus 3 by 2 mu u j delta i j divided by R, because that r cubed in the numerator cancels with r power 4 in the denominator to give me 1 over r and delta i j times u j is just equal to u i.

So this is the stress dotted with the unit normal on the surface of the sphere. Note that the result that we have got is a constant. It does not depend upon position on the surface, does not depend upon small r, and it does not depend upon x i and x j, so it is just a constant. So the stress on the entire surface, is just a constant independent of position for the stokes flow. How do I find the total force. I just integrate the stress dotted with the unit normal, over the entire surface; however, the stress dotted with unit normal is independent of position on the surface. So I get the total force, just by multiplying this by the area.

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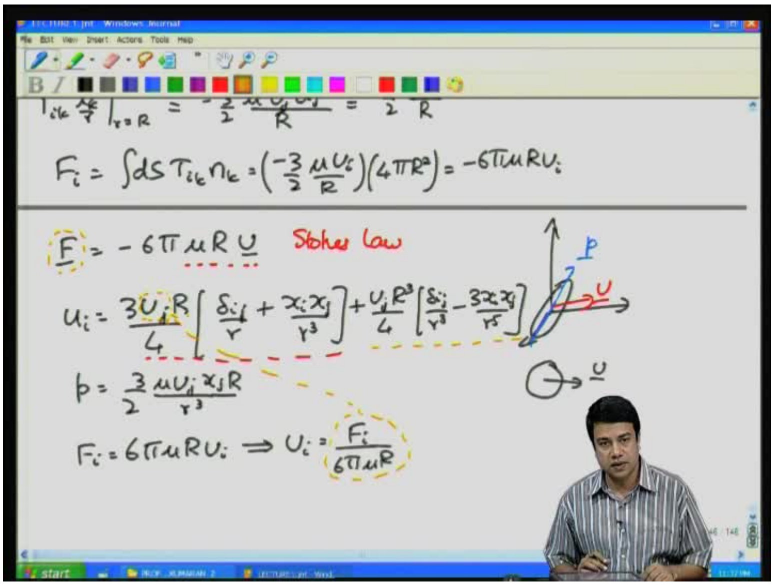

So therefore, the total force F i is equal to integral d s T i k and k is equal to minus 3 by 2 mu u i by r into the surface of the sphere surface of area of the sphere is 4 phi r square; that is the surface area of the sphere, is equal to minus six pi mu r times u i. So this has given me the force on the sphere f vector is equal to minus 6 by mu r times u vector. This is the famous stokes law, for the drag force exerted on the sphere in the limit of low Reynolds number. If you recall in the fundamentals of transport processes one course, we had actually derived this, almost up to a multiplicative constant, just using dimensional analysis. we have got the force, as some constant times mu u times r. Now by solving the stokes equation exactly, we have actually managed to find out what is the constant, in this particular case the constant is 6 times pi.

The negative sign just implies with the force, exerted is opposite in direction to the velocity of the sphere; that is the force in the fluid exerts on the sphere, is opposite in direction to the direction of velocity of the sphere. So this is an important result, we have managed to get the drag force in the sphere, by starting with the Navier stokes equations, and solving them, in order to find out the total in the net force. If you had an object of some other shape, if you had an object of some other shape; say that I had, in object that look something like this, I cannot use the same argument that I had used here. The reason is because if this object was moving with some velocity in some direction. I have two vectors one is the velocity itself, as well as the direction of the orientation of this object.

In my spherical harmonic expansion, I had managed to get a finite set of terms for the sphere, for the solutions of the velocity and pressure, because I knew that the result had to be linear in the velocity, as well as linear in one of the spherical harmonic expansions. So because of that I was able to get a finite set of terms in the equation for the velocity. If I have something like this, the velocity field can also depend upon the direction of, orientation of this direct of p. In general for an object you have to specify at least two coordinates in order to completely specified orientation. So there will be in general two unit vectors, if the object is axis symmetric, you can do it with just one unit vector. So even though the solution has to be linear in u as well as the spherical harmonics, it can in general be a complicated non-linear function of p, and because of that a combination of u p, and spherical harmonic, you can get many more terms in the expansion. And therefore, it is in general difficult to apply this exact same procedure, for other shapes of objects. However there are certain things in the solution, which will work even for, objects of other shapes.

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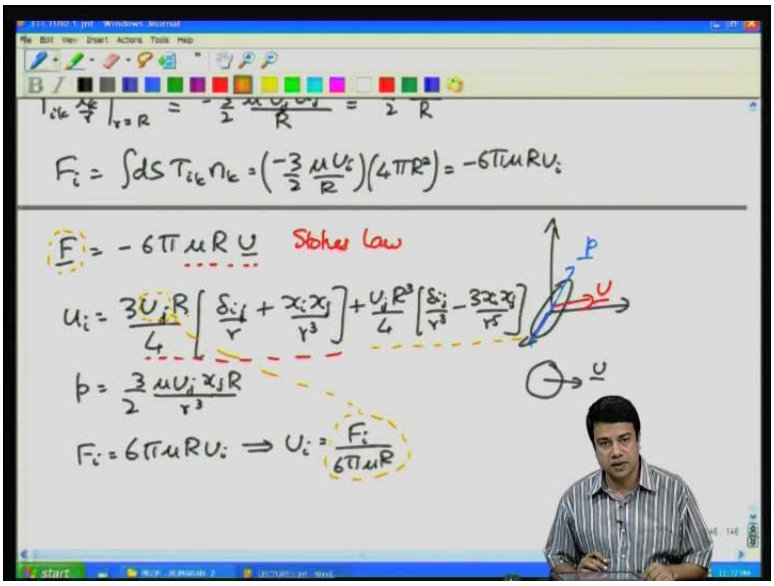

If you recall the velocity field that I had, is equal to 3 u j r by 4 into delta i j by r plus x i x j by r cubed plus u j r cubed by 4 into delta i j by r cubed minus 3 x i x j by r power 5 and p was equal to 3 by 2 mu u j x j by r cubed, I should have an r here. There are certain things in the solution, which will be valid even for objects of other shapes. You can see in this velocity field, these terms here decay as 1 over r. These terms here decay as 1 over r cubed; the limit as r goes to infinity. So if you are sufficiently far away from the object, the red terms will always be significantly larger than the orange terms. Therefore, you can neglect the orange terms, and retain just the red terms alone, and that will give me the velocity field sufficiently far away from the object. Now for the case of a sphere, rather than writing the velocity field in terms of the velocity, I could also write it in terms of the force, because I know, that F i is equal to 6 by mu r u i with a negative sign. So this is the force exerted by the fluid on the object.

The force exerted by the object on the fluid, is the negative of this, so the object is exerting a force in the plus u direction on the fluid. So if an object is moving in some direction u, the object exerts a force in the plus u direction on the fluid. The fluid exerts a force in the minus u direction on the object, what we calculated was a force exerted by the fluid on the object. If you calculate the force exerted by the object on the fluid, you will get exactly the negative of this, which is just plus 6 by mu R u, so this is the force exerted by the object on the fluid. Therefore, I can write down the velocity u i is equal to F i by 6 by mu R. Substitute this in the expression for the velocity here, substitute this in the expression for the velocity, to get the velocity not in terms of the velocity of the object, but rather in terms of the force, exerted by the object on the fluid.

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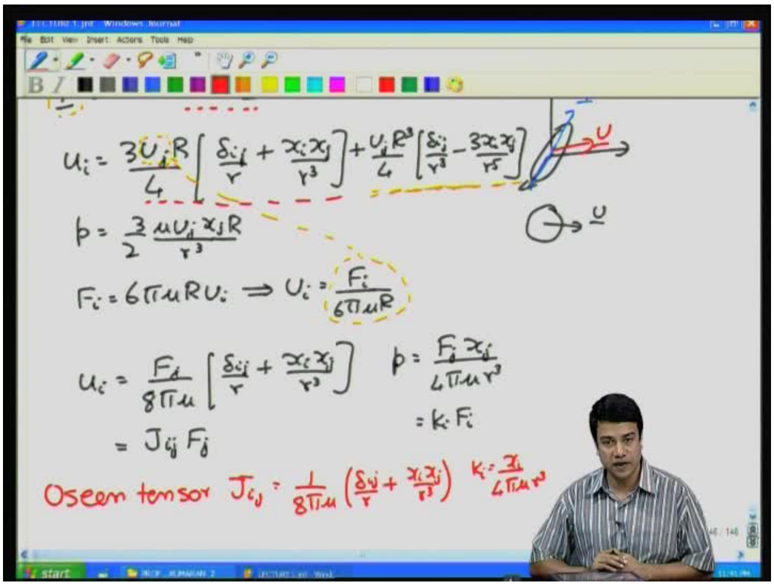

And if you substitute that you will get u i is equal to i have 3 by 4 times 1 over 6 pi I will get 1 by F i by 8 pi mu into r cubed, I am sorry should have F j there F j 6 by 8 by mu times this. And similarly, exert it in insert it in the expression for the pressure as well. I exert insert this into the expression for the pressure, I will get p is equal to F i F j F j x j by 4 pi mu times r cubed, so that is the expression for the pressure and the velocity. Expression for the velocity I have neglected the second term here, because I am assuming that we are sufficiently far away from the object. Therefore, I retained only the term proportion to 1 over r, I did not retain the term proportional to 1 over r cubed. Now the resulting expression for the velocity and the pressure, as you can see in both of these there is no dependence upon the radius of the object capital R. When I expressed in terms of the velocity of the object, there was a dependence on the radius of the object capital R. When I express it in terms the force, there is no dependence on the radius of the object capital R, that dependence would have come in, if I had retained the second term here, but since I retained only the slowest decaying term, that dependence did not come in. Therefore, the velocity disturbance due to a moving object, in a sufficiently far away, does not depends only upon the force, exerted by that object, not on the radius of the object, of the exact shape of the object. These expressions for the velocity can also be written in the form j i j F j and this is equal to k i f i, where j i j and k i are called the (( )) tensors, is equal to 1 by 8 pi mu, these are called the (( )) tensors.

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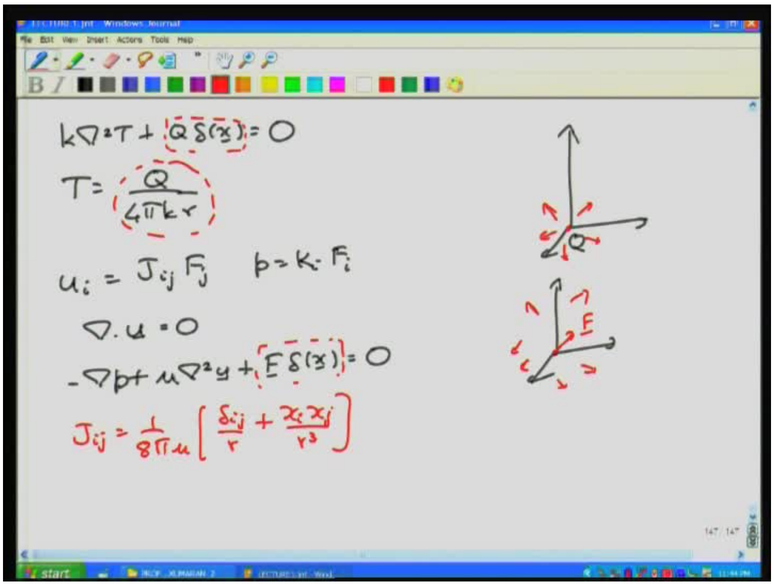

To place this in prospective, let us go back, to the solution, due to point source of heat. If you recall I had said, that if I had a point source of source strength Q at the origin, which is generating heat this point source is non zero only at this point, so it is a delta function, delta function point source. We had solved the equation k del square T plus Q delta of x is equal to zero. The in homogenous term is a point source of heat located at the origin. For that the solution was, the temperature is equal to Q by 4 pi k r, where r was the distance from the origin. The solution that I have got for the velocity field, I said this independent of the radius of the object, just as the solution for the temperature field in this cases independent of the radius of the object. It depends only upon the amount of heat coming out. So similarly, the solution for the velocity field, for point force, is independent of the radius of the object.

If I have a point force located at the origin; a point force, force of course, has some direction F, a point force located at the origin, it is going to cause some disturbance to the velocity field, at every point within the fluid. That disturbance to the velocity field at each point within the fluid, is we found out was equal to u i is equal to J i j times F j and p is equal to k i times F i the disturbance to the velocity and the pressure, where F j was the force exerted by the object, it is the entire force exerted by the object. So this is also a solution for a delta function force, located at the origin, for the stokes equation, for a viscous flow. So this is the solution for del dot u is equal to zero and minus grad p plus mu del square u plus f delta of x is equal to zero. So this is a delta function forcing, in the equation for the momentum conservation equation for the fluid, in a manner similar to the delta function force, in the heat equation in the case of heat conduction. So this is the solution for a delta function, for a point force. Force has direction, and therefore, you had the velocity field is equal to second order tensor times that force, the pressure field is equal to a vector times that force.

If you recall J i j is equal to 1 by 8 pi mu into delta i j by r plus x i x j by r cubed. So this also, this also is proportional to 1 over r, in a manner similar to this one over r solution, for the temperature field, for a point source of the origin. So this is a solution due to point force, the velocity field decays as 1 over r ,as you go away from the surface. The pressure field of course, goes as 1 over r square in this case. So these are the equivalents of the solutions for a point force, this is what you would see, if you are sufficiently far away from the object. Therefore, even if had an object that was not spherical, so if I took this particular object, if it is exerting a force, a net force on the fluid, the velocity disturbance if you go sufficiently far away from this, is going to still be J i j times F j, where F is the total force that is exerted; that is when you retain only the terms proportional to 1 over r.

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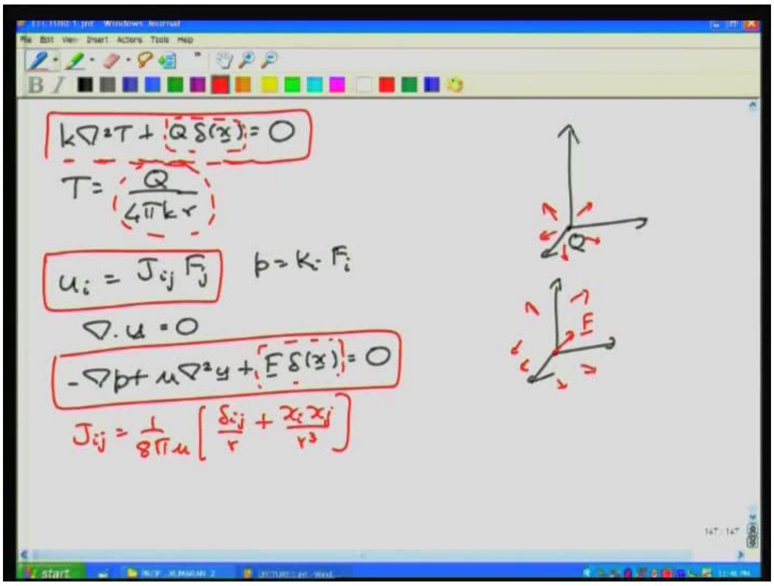

Of course, there will be higher order terms, but those will decay faster, as we go sufficiently far away. So this basically gives us the response to a point force, an object that axis of source of point force, within a fluid, delta function source. If you recall the in a in our discussion on a fundamentals of transport processes one, we got the solution for a delta function, for temperature field, a delta function heat source. Here many such heat sources, is just add up, the temperature due to each one. Similarly, I have many point forces in the fluid, I will just add up the velocity fields due to each one, stokes equations linear. Therefore, the velocity field due to a sum of many point sources, it is just the sum of the velocity fields, due to each individual point source. If you recall we had also discussed, the greens function solutions in the last fundamentals of transport processes one, as the solution of this equation.

So the greens function solution in this case, solution of this equation, this is the equivalent greens function equation, and the solutions of that, is this one. So this is your point source. In the next lecture, we will proceed, and we will look at some other more complicated solutions, solutions due to point dipoles. If you recall we had obtained for the heat transfer case, the solution for the temperature field due to a point dipole, source and sync of equal strength, separated by a small distance, such that the source strength times the distance remained finite, even as the distance goes to zero. Similar things can be done in the stokes law case as well, and you will discuss that in the context of the effective viscosity of a suspension, in the next lecture. So kindly go through the solutions that we have done over here, a little bit of the algebra is complicated, but I thought I should go through this, just to show you how you can actually get an exact solution, in the stokes law, for the simple case of a sphere settling, and how that can be extended to other cases, if we just retain only the slowest decaying term in the conservation equation. So, kindly go through this, and we will continue to look at dipoles in the next lecture, we will see you them.