Fundamentals of Transport Processes II
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Lecture - 15
Viscous Flows - Part I

This is lecture number 15 of our course on Fundamentals of Transport Processes. In the last lecture, we had discussed the fundamental equations and the boundary conditions for the viscous flow, I am sorry for the Navier stokes equations for a Newtonian fluid. Newtonian fluid, the stress is linear in the rate of deformation transfer and the proportionality there is the coefficient of viscosity. Incompressible, the density is a constant and therefore there is no radial part of the rate of deformation transfer, the divergence of the velocity is equal to 0.

In the previous lecture, we had discussed unidirectional flows, many of the results that we had derived in the previous course on fundamentals of transport processes one using shell balances, where we explicitly selected a shell within the flow and wrote down the conservation condition for momentum. The rate of change of momentum and the sum of the applied forces, those same results we had derived in the last lecture using the Navier stokes equations, which were derived using vector notations.

In that particular case for unidirectional flows, I showed you how these same equations, the Navier stokes equations, reduce to the far simpler equations, that we have for the unidirectional flows in the previous lecture. So, in this lecture, we will start looking at special cases, where we solve the Navier stokes equations in certain limiting situations with diffusion dominated regime that is when the Reynolds number is small. The convection or the inertia dominated regime when the Reynolds number is large, in those limiting situations, we try to solve the equations using some physical intuition.

As I said the Navier stokes equations, the mass and momentum conservation equations, they are partial differential equations in the velocity field, and they have two other important characteristics. First of all, they are explicitly time dependent, so in general at a particular instant and time depends upon the history, the previous velocity fields at earlier instance and time. And they are also non-linear, because of the inertial term in the conservation equation, which goes as u dot grad u.

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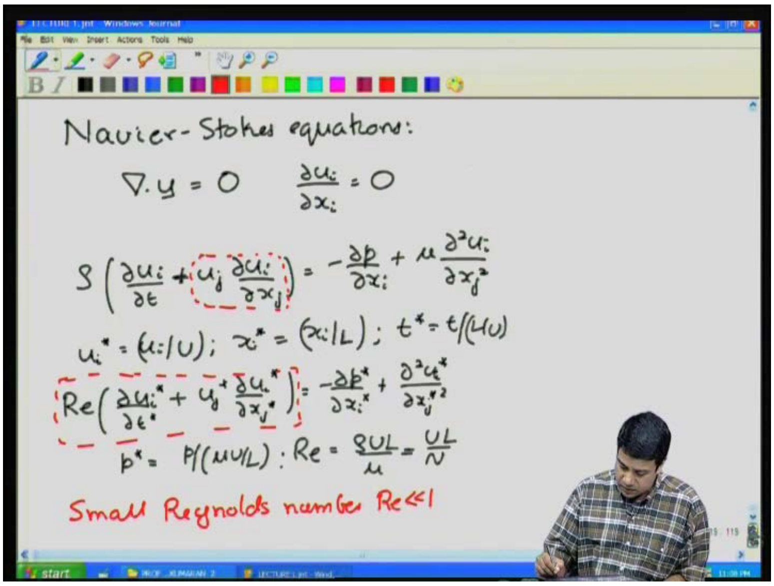

So, if you recall the Navier stokes equations as we had written in the last class, as we had written in the last lecture, we have two characteristics, two equations. One for the mass conservation del dot u is equal to 0 or partial u i by partial x i is equal to 0, this is for an incompressible flow. And in addition, you have the momentum conservation equation rho times partial u i by partial t plus is equal to minus partial p by partial x i plus mu. Here, the complete equations, they are time dependent and they are non-linear, because of this u dot grad u term.

As of that, when you multiply the velocity by a factor of two, not all terms multiply by that same factor and therefore, they are non-linear. We can consider this in different limits as I showed you in the last lecture, if we scale this equation, we scale the velocity by characteristics scale, the distance by characteristic scale. And due to this, the time gets defined as t by L by u, if we scale it in this fashion then we get the scaled equations as R e partial u i by partial t. So, in this exercise, I have scaled the pressure by the viscous terms in the conservation equation.

That is, I define the scale pressure as p by mu u by L and if you scale it in this way, the Reynolds number is the ratio of inertia and viscosity. It can also be written as the ratio of convection and diffusion where, mu is the convectic viscosity, the ratio of the viscosity and the density, it is also the momentum diffusion coefficient. So, if I scale it in this fashion, the Reynolds number multiples the inertial terms of the conservation equation and it gives me the ratio of inertia and viscosity.

So, the first limiting case, that we will consider is the case where the Reynolds number is small so small Reynolds number, R e small compare to 1. So, in that case, what you would do is, you neglect the entire inertial term in the mass conservation equation and the reduced equations in the limit of small Reynolds number is called the stokes equations.

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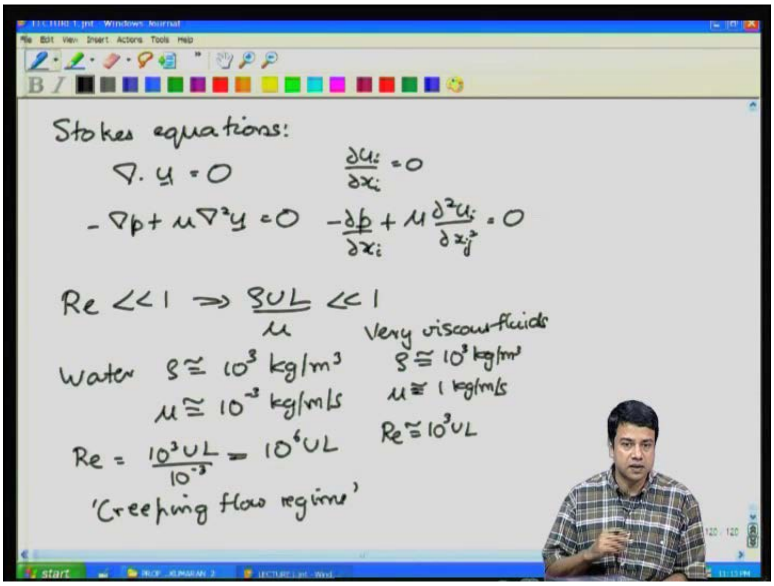

These reduced equations are del dot u is equal to 0 and minus grad p plus mu del square u is equal to 0 or in indicial notations, partial u i by partial x i is equal to 0. So, those are the stokes equations. So, first let us look at some idea of, where one encounters this low Reynolds number regime, R e is less than 1 which implies that rho u L by mu is small compared to 1. For for usual fluids, let us say take a typical fluid as water, for that the density is of the order of 10 power 3 in SI units kg per meter cube.

Whereas, the viscosity is actually 1 Centipoise that is 10 power minus 2 poise, which is actually 10 power minus 3 in kg meter in the SI unit system, 10 power minus 3 kg per meter per second. So, in this case, the Reynolds number is equal to 10 power 3 u L by 10 power minus 3 is equal to 10 power 6 times u times L where, u is in meters per sec and L is in meters. So, you can clearly see, that for common applications, the length is of the order of a centimetre or so the velocity is of the order of a centimetre per second to a meter per second.

For those kinds of applications, the Reynolds number is clearly large because I have put a centimetre in length and a centimetre per second in u, I will end up with a Reynolds number that is, 100. So, in common applications, that we are used to or common every day experience, the Reynolds number is actually large. The Reynolds number is going to be small only if, the velocity is very small or the length is very small or both.

So, for example, if you have a very slow flow for example, in in water treatment plants in the flow through or in ground water, the flow or fluid through the ground for example, the velocity is very small less than a millimetre a second. And the the length scale in that case will be for example, the pores between the solid particles there, the sand particles that is a very small length. And therefore, you will have a small Reynolds number, it is called the creeping flow regime, a very small length or a very small velocity the the the motion of for example, micro-organisms in fluids, these microorganisms are very small length scale, they are typical about 100 microns or less.

In that case, the velocity is also small therefore, the Reynolds number is small and because of that, many of the things that we are consider that we are used to intuitively in our everyday experience, do not work at those small length scales. The other system for which the Reynolds number can be small, is for very viscous fluids. For example, glycerines, silicon oil, they have a viscosity that is about 1000 times that of what or higher. In that case, though the density is still of the order of 10 power 3 kg per meter cubed, the viscosity is actually of the order of 1 kg per meter per sec.

And therefore, the Reynolds number becomes about 10 power 3 u L where, u and L once again in meters and in meters per second. So, in that case, as as well the Reynolds number can be small so there are very specify applications, in which the Reynolds number is small and one can neglect the inertial terms in the conservation equation. And in those applications or intuition that is built up, looking at flows of of macroscopic scale for example, flows of water and streams and so on, that kind of intuition does not apply at these very small scales and we will discuss the reasons, why?

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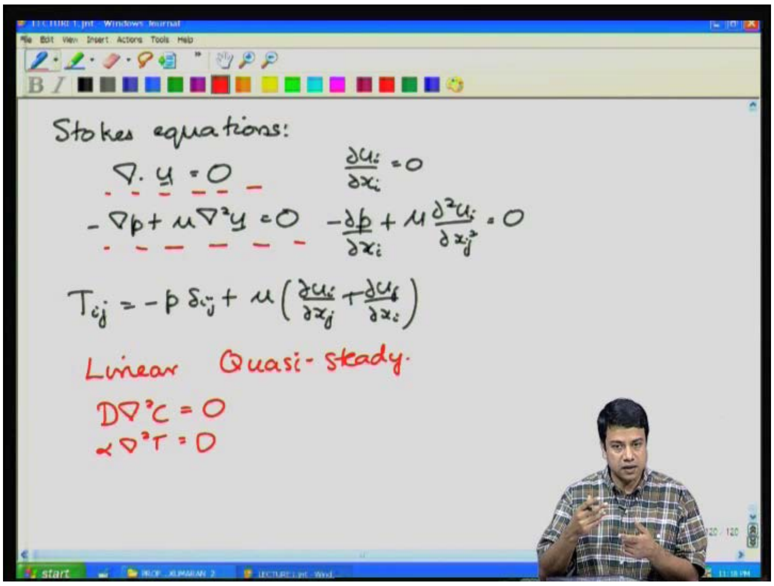

These are the stokes equations minus divergence of velocity is equal to 0 and minus grad p plus mu del square u is equal to 0 and these have to be augmented by the equation for the stress. So, the stress in this case, tau i j is equal to minus p delta i j plus mu into, this I should write as the total stress includes the pressure as well this should be the total stress and includes the pressure as well this is the stress, the force per unit area acting on the surface.

So, these definitions of stress, pressure, the momentum equation and the mass equation, completely specify the equations of motion for the system. The first thing you notice about this is that, the equations now are linear in the velocity and the pressure, the stress is linear in pressure and velocity. The momentum conservation equation is linear in pressure and velocity and the mass conservation equation is linear in velocity, contains no pressure.

So, these now constitute a linear set of equations and because of this, one is guaranteed that there exists a solution and that that solution is unique subject to well post boundary conditions of course. So, for these equations, if we have well post boundary conditions, one is guaranteed that a solution exists for these equations and that solution is unique. The other property of this is, what is called Quasi steady recall, in the original momentum conservation equations, I had the partial derivative of velocity with respect to time, that was in the inertial term.

We neglected the inertial term in the limit of lower Reynolds number and therefore, these equations do not contain an explicit time derivative. What that means is that, the velocity at an instant of time can be determined purely on the basis of the velocity that is, the fluid velocity at an instant of time can be determined purely on the basis of the velocity boundary conditions, in the stress boundary conditions at that same instance of time.

So, if I have a particular configuration given to me at an instant in time with specified velocity or stress boundary conditions on the surface, the fluid velocity at that instant in time can be solved can be obtained with that information alone. You do not need any information about the time history of the fluid flow, what the fluid flow was at earlier instance in time. So, in that sense, for a given instant in time, if you are given well specified boundary conditions, you know what the velocity is everywhere within the fluid.

So, that is the consequence of it being Quasi steady, you are not solving a partial differential equation in time. You recall that, when we look at diffusion dominated transport in mass and heat transfer as well, the similar case was was obtained. In that case, the equations were of the form D del square C is equal to 0 or alpha del square T is equal t 0.

So, these once again do not contain any time derivatives and therefore, given configuration and a set of boundary conditions at an instant in time, you can solve for that equation for for the temperature consideration fields at that instant in time without needing to know anything about the history of, how the system got there. And physically this is a consequence of the fact that, when you assume that diffusion dominates it implies that, the diffusion takes place everywhere instantaneously.

That is, you are assuming that, whenever a temperature perturbation or a concentration perturbation is placed at some location of the fluid, the entire field response instantaneously to that perturbation because there is no time derivative. The assumption is that, the response time of the entire field over the entire domain is fast enough that, the information that there has been a perturbation at one particular location reaches everywhere instantaneously.

Of course, that is not strictly true, it is true only up to a certain limit and towards the end of the course, we will try to analysis what will I am sorry towards end of this section, we will try to analyze, what that limit will be. But, further moment, we will assume that the Reynolds number is low and therefore, the equations are completely independent of time themselves. Boundary conditions could depend upon time, you could have a moving surface within the fluid but the motion of that surface is instantaneously communicated everywhere within the fluid.

Because, diffusion is a extremely fast process so that is the assumption here so that is the implication of a Quasi steady set of equations. The other the linearity also has it is own implications, one of the implications is that, one can use linear superposition in order to find solutions to the equations. That is, when an equations are linear, let us say that I have sphere that is falling a fluid, I have a boundary condition on the surface of that sphere.

The velocity of the fluid at the surface is equal to the velocity of the sphere itself or any other object, does not have to be spherical in general. Now, if I want to change the velocity of the sphere itself by a factor of two, since the equations are linear, the velocity at each point in the fluid also has to change by that same factor. So, let us try to explain that in a little detail.

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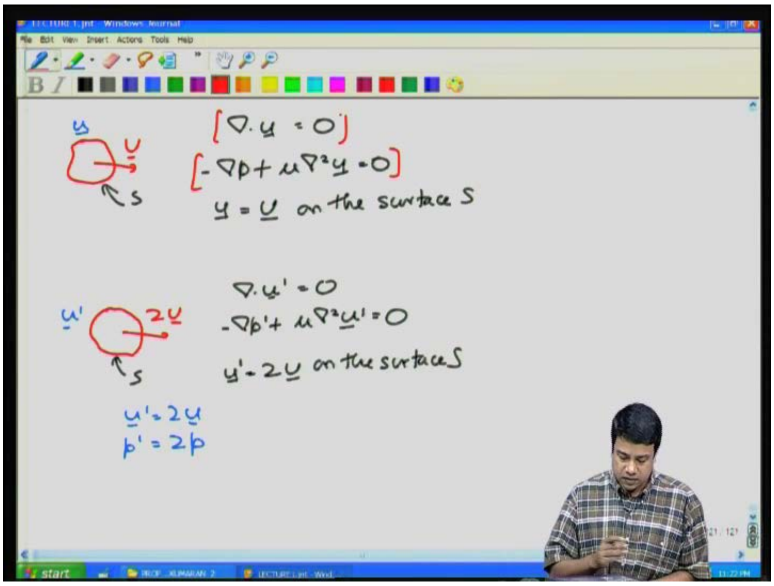

Let us say, I had some general object that was moving with some velocity u, in some direction it need not be spherical. What I solve is the equations, del dot u is equal to 0 minus grad p plus mu del square u is equal to 0, with the boundary conditions u is equal to capital U or on the surface so let us call this the surface S, that is the surface of the sphere S. Now, let us say that I change the velocity of the sphere, the exact same object is not spherical in general but the exact same object. If I change it is velocity to 2 times U on the same surface, on the same surface S, I change the velocity to 2 times U.

So, I have u is equal 2 times U on the surface S so as this is a since this is a problem in which the velocity has become 2 times, I will use the velocity u prime for this, u prime is the solution for the velocity field with the velocity being 2 times u on the surfaces. And I am solving the exact same equations, del dot u prime is equal to 0 minus grad p prime plus mu del square u prime is equal to 0. One can easily verify that, if I use the velocity field if I use the velocity field, u prime is equal to 2 times U, this exactly satisfies the equation this exactly satisfies the equation.

That is, I have to use a velocity field, in which the velocity u prime for this new configuration, velocity is u prime here and velocity is u here, u prime is equal to 2 times U and the pressure in this is equal to 2 times p. With this new velocity and pressure, the equations once again are identically satisfied because if I because putting in u prime is equal to 2 times U in the mass and momentum conservation equation, is equivalent to multiplying the original equations that I had here, by factor of two that is it.

And since if something is equal to 0, you multiply it by 2, it is still equal to 0 so therefore, you can get a solution for this by just multiplying the velocity at each point in the fluid by that exact same factor. You multiply the velocity at each point by that same factor, the pressure as well multiplied by that same factor and what that means is that, the stress is also multiplied by that same factor. The pressure is linear and the velocity, stress is linear and pressure and velocity.

So, if I have a solution for this particular configuration, for this particular velocity 2 times velocity u, I can get the solution for the velocity with boundary velocity 2 times U by just multiplying by that same factor, by a factor of two. So, if the velocity of the surface is doubled, the velocity of the fluid at each and every point within the fluid, the velocity and pressure are both increased by that exact same factor of two, does not have to be in the same direction.

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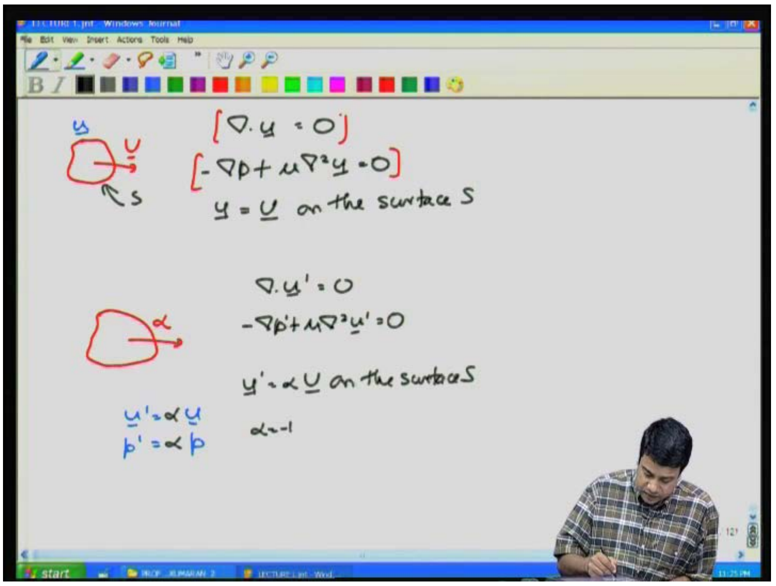

So, for example, if I take a configuration where, instead of just taking the velocity and multiplying it by a factor of two, I take that same object and multiply it by any factor. You can multiply it by any factor, multiply it by some factor alpha. So that, the velocity u is equal to alpha times U on the surface S and the equations are once again the same, del dot u prime is equal to 0 minus grad p plus mu del square mu prime is equal to 0 on the surface within the fluid.

And you can easily see that, the solution is one where, the velocity and pressure are both multiplied by the same factor alpha at each point within the fluid. So, if the boundary velocities are multiplied by some factor, the velocity of the fluid everywhere is multiplied by that same factor, that factor could be positive, it could be negative as well. So, if I use alpha is equal to minus one that means, that the velocity is reversed that means, that the velocity on the surface is exactly reversed.

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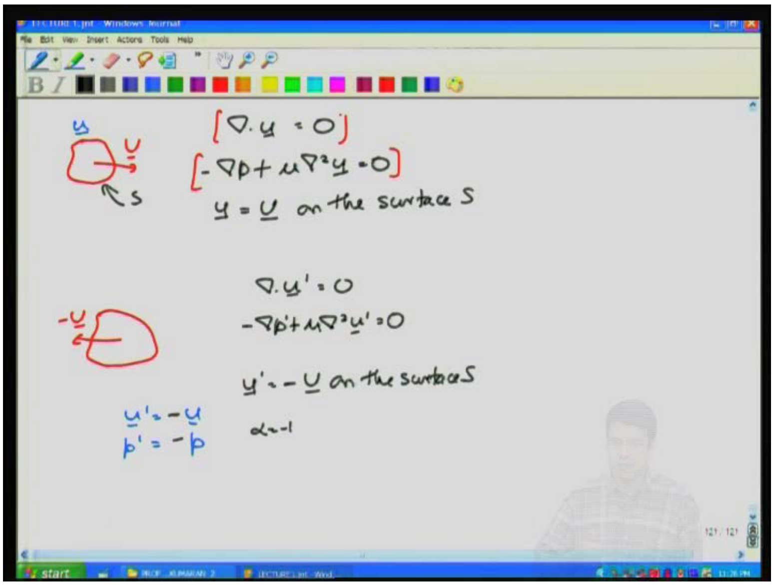

So, the velocity rather than being from left to right, is actually from right to left, the velocity is minus u on the surface therefore, I have u prime is equal to minus U and the solution the equations are identically satisfied provided, both the velocity and the pressure are exactly reversed at all locations. So, in this case, both the velocity and pressure are both reversed at all locations, so this is a varied solution of the equations. If I reverse the direction of the object of the velocity on the surface, the velocity at each point reverses.

The same thing holds for stress boundary conditions because stress is linear in both the pressure and the velocity. So, rather than imposing a velocity boundary condition on the surface, I could as well impose a stress boundary condition on the surface. So, if I have an object, which is exacting a certain stress on the surface for example, if it is settling under gravity, that force per unit area is equal to the stress the, that gravitational force per unit area exerted in the downward direction is equal to the stress.

So, if the stress is doubled that is, you increase the mass 2 times while keeping the shape of the object the same then the velocity as well as the stress and the pressure at each point is doubled. The direction of the stress is reversed on the object then the velocity, the pressure and the stress at each point are exactly reversed. So, that is the consequence of linearity and this leads to things that we would not intuitively expect when based upon our experience with microscopic flows.

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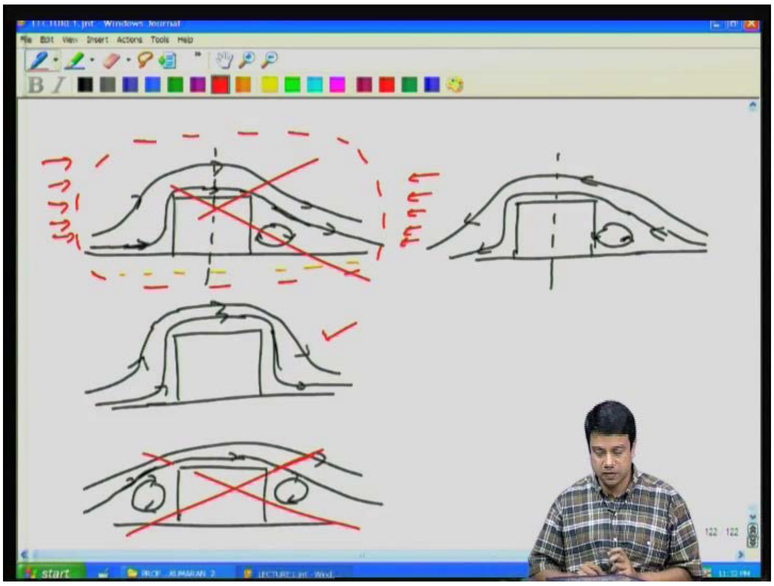

For example, let us say we had the flow pass, some kind of a rectangular object like this, let us say we had the flow pass some kind of a rectangular object like this flow is coming in from the left with some constant velocity. And based upon our everyday experience, we would expect that the stream line coming in would go along the surface and then it will come out like this. Because, this is a bluff body, it expect the velocity to go something like this with some circulation zone at the downstream edge, that is what we would normally expect based upon our everyday experience with macroscopic objects.

This is not what happens at low Reynolds number when we neglect inertia, the reason is that this is inconsistent with the idea of reversibility and linearity. I told you that, when the direction of the velocity is reversed, the direction of the velocity at the boundaries is reversed, the direction of the velocity at each point within the fluid is also reversed. And for this then this is not a reversible flow, how do I reverse the direction of the velocity on this at the boundaries.

That means, that rather than going from right to left, I would have the velocity at the boundaries going from right to left, instead of going from left to right. And this is my object, the direction of velocity at boundaries is reversed, the direction of velocity at each point is revered. Stream lines are the same, only the direction of arrows on those stream line change, so linearity and reversibility were telling you is that, if this is the solution, the stream lines that I would get for the fluid going from left to right.

Then, for the fluid going from right to left, the steam line that I should is as follows, the stream should remain exactly the same, only the direction of the arrows should change. That means, the stream lines remain exactly the same and only the direction of the arrows change, I should get a flow that goes like this, with a circulation which is now going in the opposite direction. This is a direction of the circulation opposite to what I have here, these two are clearly incompatible because this object has left right is left right symmetry, this object is clearly symmetric about this axis.

Therefore, when I did reverse the direction of the flow, the direction of the velocity at each point reverses. And if the object has to retain it is symmetry and the flow has to retain its left right symmetry, you cannot have this kind of a velocity profile as a solution of your equations. Because, when I reverse it, I do not (( )) the same thing which I would expect based upon symmetry and we know very well that, since the equations are linear, there exists only one solution. So, in the limit of very low Reynolds numbers, the only kind of solution that one can have is actually one where, either you have the flow going along the object and coming back down along the object on the other side. If this were the flow then if I reverse the direction of the flow, get back the same thing.

So, it is consistent with the requirements of symmetry so in that sense, it is not what we expect at the microscopic scale. Of course, you could also have a flow, where you had circulation on both sides, you could for example, have a flow that went something like this, you could of course, have a flow which went something like this, this is also reversible. But however, in actual actually find that, this second flow is not what you get, what you actually get is the first one. So, the requirement of linearity itself imposes some strict conditions on the kinds of flow that you can get. This is clearly not a flow that can happen at low Reynolds number because it does not satisfy the condition of reversibility and linearity whereas, this is a possible flow profile at low Reynolds number.

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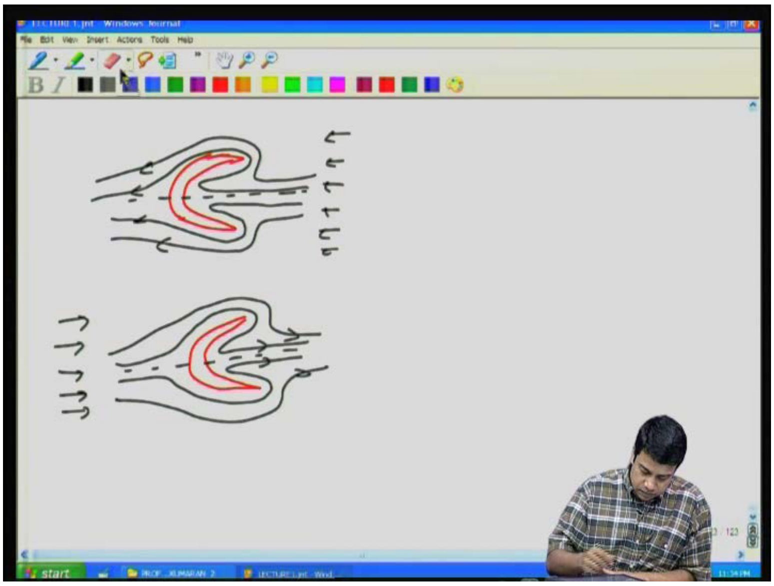

Similar things apply for other objects as well, if you have for example, an object that looks like this and when the flow is going from left to right, you get I am sorry you should go in the other way. If the flow is going from right to left, you get stream lines that look something like this then for that same object, if the flow goes from left to right for that same object when the flow goes from left to right, you will actually get stream lines that go all the way back and down to the centre of the object. You will get flow that look something like this, you will not get separation from this subject and there is sort of circulation at the back in the limit of low Reynolds number, this is the only flow that satisfies the conditions of linearity as well as reversibility.

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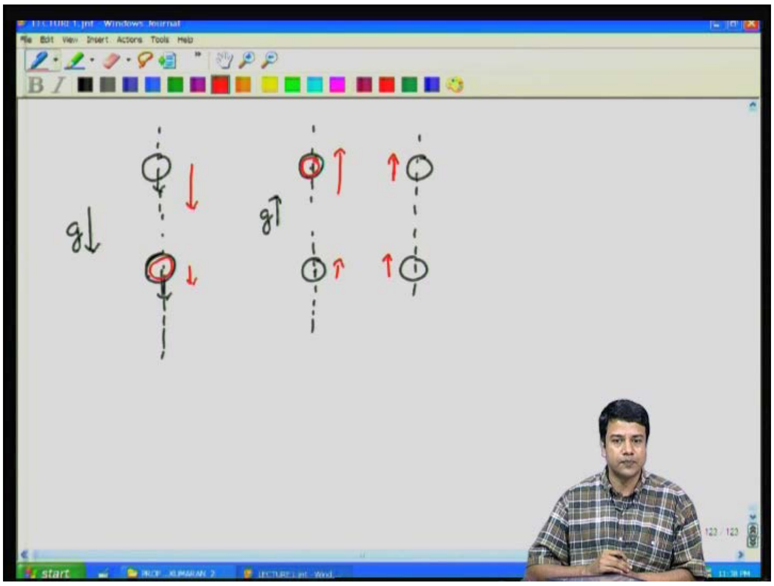

You can also answer other simple questions for these kinds of flows for example, let us say, that I had two identical spherical objects two identical spherical objects settling under gravity. So, the gravitational acceleration was in this direction, if you had one object, you would it would of course, settle at it is terminal velocity. The question is, what happens when there are two, let us assume that they are both aligned vertically with respect to each other in the same vertical line.

We will assume that, they have the same vertical line, they are aligned vertically with respect to each other. In that case, one can ask the question, to these two particles, does the one that is below settle faster, does the one that is above settle faster or do the both do both of these settle at equal speed. In other words, that is the distance between the two objects, reduce or increase as these are settling. Different people will give different answers for this question, most people will say based upon their experience that, the one that is above settles faster.

Because, it is following in the wake of the object that is below and because it is in the wake, the resistance to flow is lower and therefore, the one that is above will settle faster and the distance between the two will decrease. So, this question can also be answered simply on the basis of linearity and reversibility, both these objects are identical. So, let us for the moment assume, that the one that is behind settles faster and the one in front settles slower.

What happens when you use reversibility, when you reserve the direction of the force, the direction of velocity at all point within the fluids is exactly reversed. So, let us reverse the direction of the force in this new configuration, I have two objects and I am reversing the direction of gravity. The direction of the velocity everywhere in the fluid is exactly reversed that means, the direction of the velocity is exactly reversed. This configuration, the one on the right is exactly the same on the one as the one on the left except that, the direction of gravity has changed.

So, in this case what happens is, this object this object in front this object in front is the one that is down stream that is, further down along the direction of flow whereas, in the previous example, this this object was the one that was downstream, further down along the direction of flow. So, on the left hand side, I am getting the solution as saying, I had assumed, I had I had postulated for that for the configuration on the left hand side with gravity acting downwards, that the downstream object travels slower and the upstream object travels faster.

So, that the distance between the two reduces, when I reverse the direction of gravity so the gravity is upwards, the direction of velocity at each point in the fluid reverses. And this tells me that, the downstream object travels faster, the upstream object travels slower therefore, the distance between these two actually increases. These are clearly not compatible, these two results are clearly not compatible with each other. The only result that would be compatible when I reverse the directional of gravity was to say, that both of these objects actually travel with the exact same velocity.

If both of these travel with the exact same velocity, then whether I reverse the direct of the gravity or not, in this case the distance between the objects remains exactly the same, as they are settling. Even when I reverse the direction of gravity, their distance between the same between the two remain exactly the same, as they are travelling. So, the velocities have both got to be the same. So, the requirements of linearity and reversibility is telling you that, when two identical objects are settling, they cannot travel at different speeds, they have to travel at the same speed.

If you postulate that the downstream one travels slower so that, the distance decreases, when you reverse the direction, you get the opposite result, they are not compatible. If you postulate that the downstream one travels faster so that, the distance increases, when you reverse the direction of force, you get the exact opposite. The only result that is compatible with the requirements of linearity and reversibility is that, both of these identical objects have to travel at exactly the same speed in a viscous flow. So, those are kinds of simple results that you can immediately infer, based upon the requirements of linearity and and reversibility.

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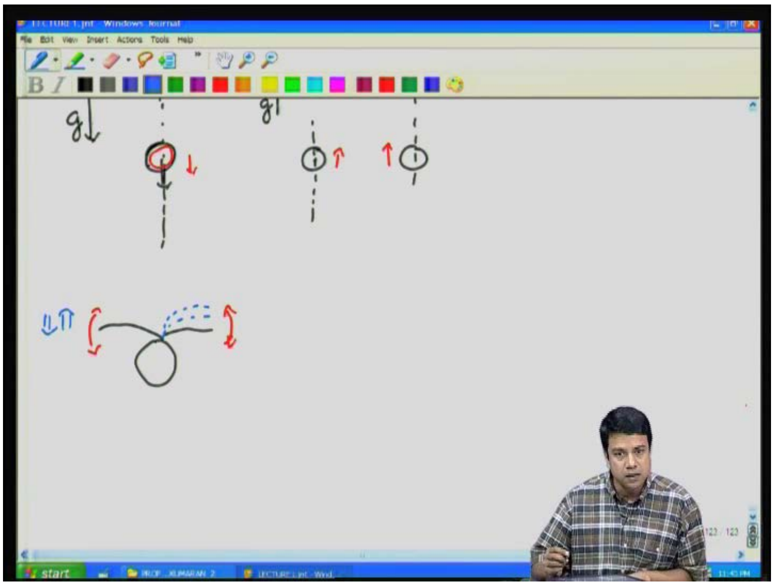

Another interesting thing, swimming at the micro scale is actually very much different from swimming at the microscopic scale. Small objects which swim have to use, they cannot use periodic motion to swim for example, if I had a small micro organism, which was using some flagella to swim. In macroscopic applications, we are used to just you know a periodic motion of these resulting in a thrust in one particular direction. In microscopic applications, where are used to this situation where, this periodic motion can lead to a net force.

However, in the microscopic for microscopic objects, which typically swim at very low Reynolds numbers, this is not true. Because, let us say that, you take a cyclical, if this under goes a cyclical motion then if if the object if the object on on the up stroke if the object moves in one particular direction. On the down stroke, the velocity at each point is reversed to that in the up stroke if it moves in a periodic manner if this flagella moves in a periodic manner. So that, along the up stroke it moves in one particular direction, along the down stroke the velocity at each point along that flagella is exactly reversed.

So that means, if the velocity in the fluid at each point is reversed note that, the velocity at a instant in time only depends upon the velocity on boundaries at that instant in time, in this particular case, on this moving object at that particular time. So, the velocity at each point in the boundary is reversed between the up stroke and the down stroke. That means, the velocity within the fluid at each point is also reversed, on the up stroke and the down stroke.

Therefore, if the, if this microorganism moves a certain distance along the up stroke, if the down stroke is exactly the reverse of the up stroke, it has to move in equivalent opposite distance on the down stroke. Therefore, you cannot use periodic motion in order to, for microorganisms to move at in the limit of very low Reynolds numbers. We have to use something that is non-periodic, a cyclical motion for example, in order to move in one particular direction. This is very different from what, you have used to at high Reynolds numbers. At high Reynolds numbers because the non-linear term, there is no reversibility and because of that periodic motion, there is no reversibility and it is not Quasi static either. And because of that, a periodic motion could generate a thrust in one particular direction.

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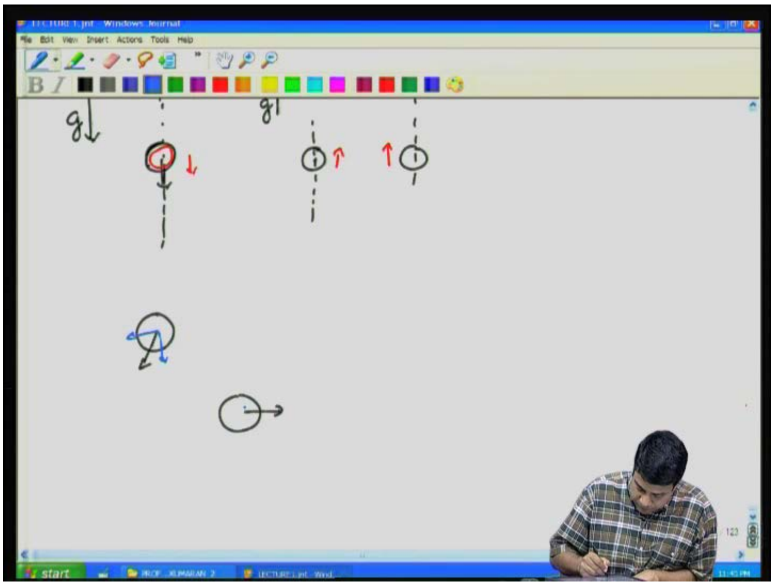

Super position can also be used to advantage in solving problems for example, if I wanted to solve for the velocity field of a fluid, which contains two object, which were moving in different directions. I could in general split this, the fluid velocity, into two. I am sorry, the velocity of the object into two components.

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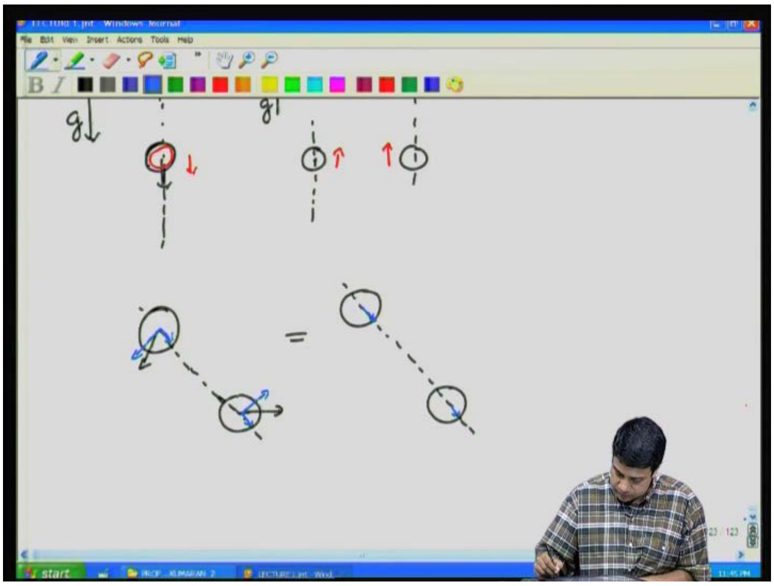

So, I have two objects, which are moving in different directions, which have a particular alignment with respect to each other. Two objects which are moving in different directions with a particular alignment with respect to each other, the velocity of each object, I can separate into two parts, one is the velocity along the line joining the centres of the two particles and the other is the velocity perpendicular to the line joining the centres of the two particles. And I could use this to solve two separate problems, one is this one where, I include only the components of the velocity along the line joining centres plus a second problem.

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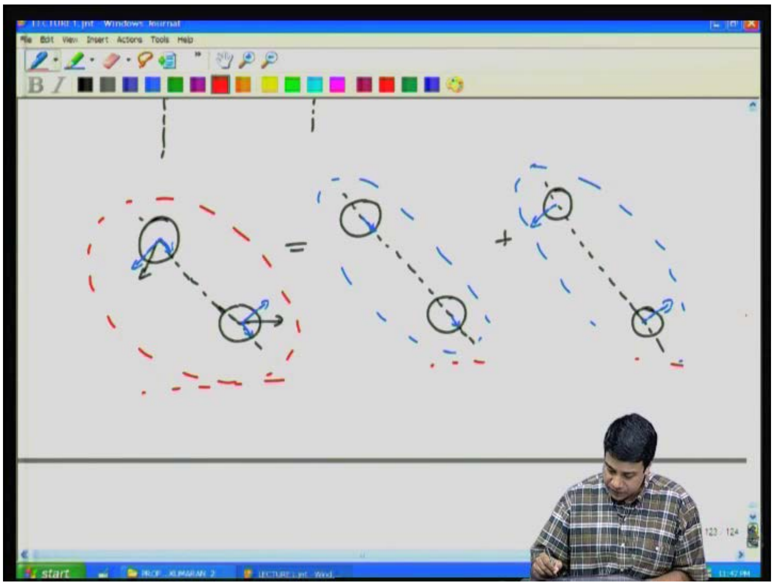

Plus a second problem, where I include only the components of the velocity in the direction perpendicular to the line joining centres. So, first I solve this problem, find out the velocity field at each point in the fluid then I solve this problem, find out the velocity field at each point in the fluid. Add up the velocities due to each of these problems and I will get the solution for this problem. The velocity field for this one that is, at each point in the fluid, I can add up the velocity fields of these two problems.

The result will be equal to the velocity field of the problem that I originally wanted to solve that is because for these two, the sub problems that I have for these two, the velocity on the boundaries, when I add these two up is exactly equal to the velocity for the first problem. And I know that, if if the velocity of the boundaries is added up, the equations are linear. So, if I add up the velocities for these two, the resulting velocity field also satisfies the equations of motion.

That is, the mass and momentum conservation equation, if they are satisfied individually by the velocity fields in these two cases, the sum of those two velocity fields also satisfies those equations. The sum of the velocity fields satisfies the equations, the sum of the velocity fields satisfies the boundary conditions therefore, the sum of the velocity fields is a solution of this original problem. So, there is the advantage, you can use linear super position in order to solve the problem. An important point when you use linear super position, one has to be careful that the boundaries are all the same in all of those sub problems, that you have divided the main problem into, let me illustrate that.

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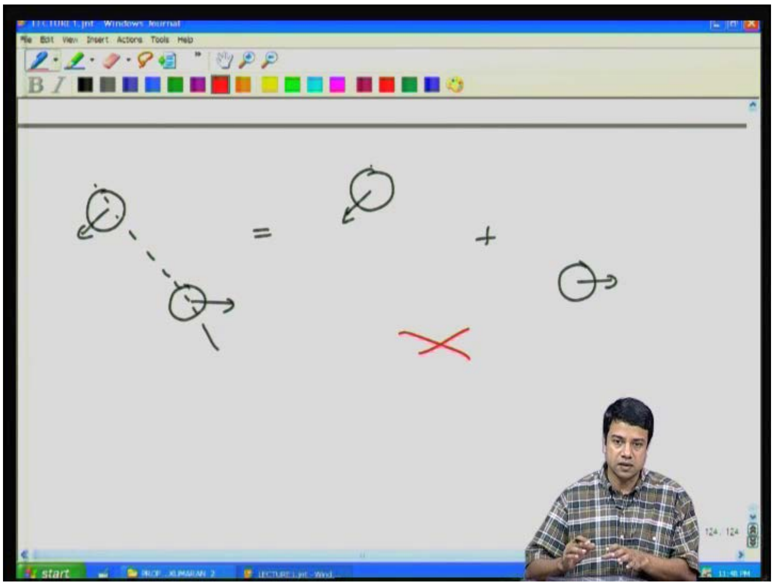

If I had a problem in which I had to objects, I just showed you that, you can divide into two sub problems where, you separate out the velocities into two different components. So, what I cannot do is to separate out into two problems where, I retain only one sphere in one case and retain only the second sphere in the other case, this is clearly wrong. That is because the boundaries have not been kept a constant, while I was dividing the problem into sub parts.

I need to divide the problem into smaller bits in such way, that the boundaries, all of the solid boundaries are the are the the interfaces, in each of the sub problems is identical to what was there in the original problem. You can separate out the velocity on boundaries into different parts, you can separate out the stress into different parts. For example, if you were prescribing a force on the boundaries, you could separate out that force into different parts, you could separate out the velocities into different parts.

But, you cannot just separate out the objects or the boundaries into separate parts. You have to make sure that, in each of those sub problems, the boundaries that are chosen are exactly the same as those in the original problem, though the velocity condition in the stress conditions can be separated out into sub problems. So, if you can do that, that gives you an important advantage, you can use this linearity and super position in order to separate out a problem into multiple components and solve for each of those components separately.

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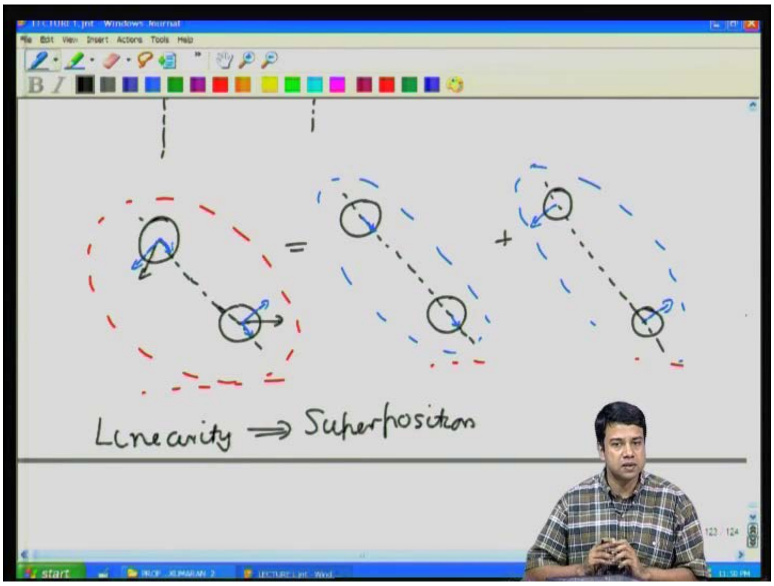

So, that is the advantage of having linearity of the equations gives rise to the possibility of super position. Linearity of the equations means, that you can use super position to separate out a problem into small orbits then solve each of those separately. So, we have discussed some qualitative features, important features, Quasi steady, linear, reversible and use of the super position principle to solve the equations and you could get some simple results based upon this intuition. We did the particular case of two objects settling in gravity to show that, the distance between the two has to remain exactly the same.

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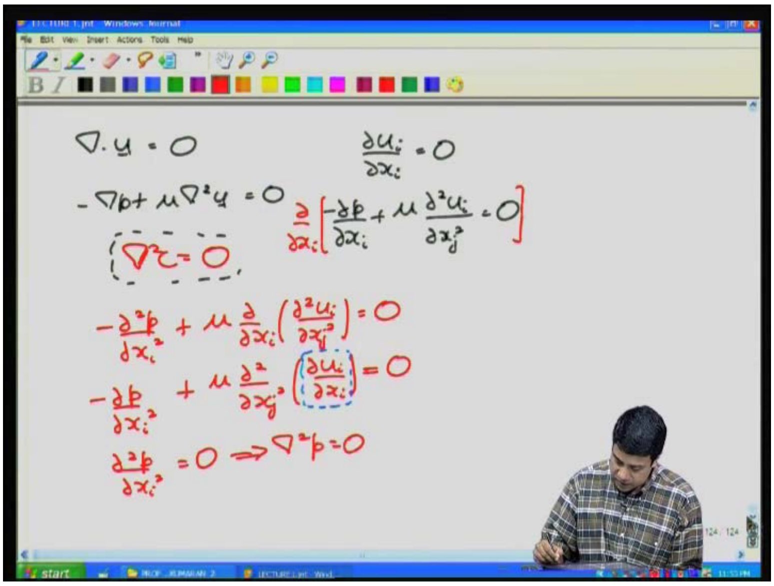

So, let us proceed to try and solve this equations, the equations of the form, divergence of velocity is equal to 0 and minus grad p plus mu del square u is equal to 0. And just to recall, the equivalent of this in the case of mass and energy conservation were for example, D del square C is equal to 0 of the equivalent point for temperature conservation for the energy conservation equation. In this case of course, the diffusion coefficient does not matter, you solving for the Laplacian of the concentration field is equal to 0.

So, in notation, I would write this equation as minus partial p by partial x i so how do we solve this, as you would expect we would we we would try to reduce the equations to some kind of a Laplacian equation. And then we had discussed various methods and the fundamentals of transport equation I, to solve this Laplacian of something is equal to 0 in order to find the concentration field or the temperature field. Transport, we can do a similar thing here and that is by taking the divergence of this entire equation, you just take the divergence of this entire equation.

I guess, divergence of this entire equation but I get is minus partial square p by partial x i square equal to 0. In this second term here, I can interchange the order differentiation, because the x's are all in timed variables. So, I can write this equivalently as this is equal to 0 then I have the first term here, which is the Laplacian of the pressure. Now, this term is clearly 0 from the mass conversation condition because it is the divergence of the velocity.

So, it is equal to 0 because the mass conservation condition therefore, I take the diversion of the momentum conversation equation. I find that, partial square p by partial x i square is equal to 0 which implies that, del square p is equal to 0. So, basically you get the Laplacian of the pressure is equal to 0 by just taking the divergence of the mass conservation of the momentum conservation equation. So, this del square p is equal to 0 can be used to solve the momentum conservation equation, subject to appropriate boundary conditions. Then how do you solve for the velocity field, the way the velocity field is solved is to go back to the original equation.

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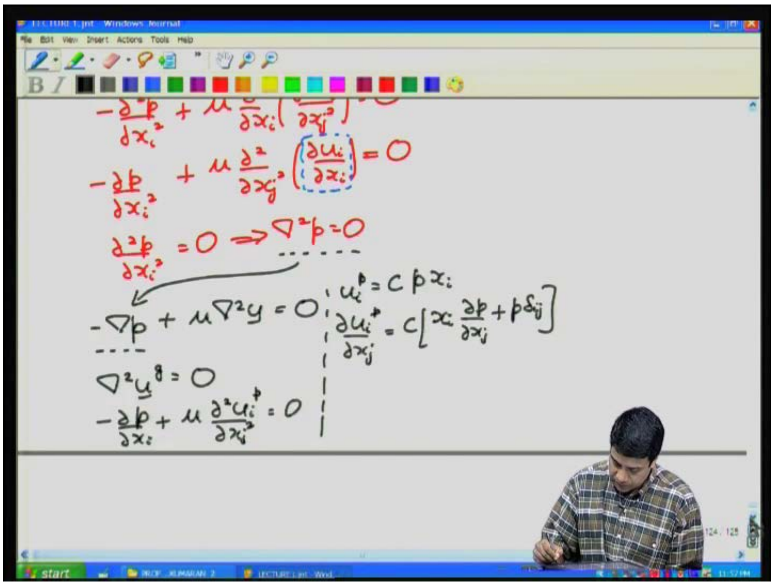

So, this you can consider as an inhomogenous equations for the velocity field u, you can consider this as an homogenous equation for the velocity u because I have already got the pressure from solving this equation. And I can put that into this equation in the creation of the pressure here, so I just use this solution to substitute for the pressure here. So, that gives me an inhomogeneous term in this equation. So, I have a linear inhomogeneous equation and obviously, the solution can be separated out into two parts the general solution as well as the particular integral.

The general solution is the solution of the equation without the inhomogenous part, the gradient of the pressure here is the inhomogenous part. So, the general solution is the solution of del square u general is equal to 0 note that, the general solution is the one with which, you enforce all the boundary conditions and this the one, that contains the constants of integration. Second order equation, you would expect two boundary conditions at each surface and this contains all the constant of integration.

The particular solution is any one solution that satisfies the equation, it is anyone solution, you can choose any solution, you do not have to have any, there are no integration constant in that, it just anyone solution that identically satisfies the complete equation. So, the particular solution is one that satisfies minus partial p by partial x i plus mu partial square u i particular by partial x j square is equal to 0. Any one solution that satisfies this equation and it turns out that, one can get a particular solution quiet simply in this case.

Since the gradient of the pressure balances the Laplacian of the velocity field, one would guess the particular solution to be of the form u i particular is equal to some constant p times distance. Because, the gradient of the pressure is balancing the Laplacian, there are one derivative on the pressure, two on the velocity. So, you would expect the particular solution to be of the form, some constant times p times x i. Now, how do I determine the constant, I put this into the conservation equation. If I put this in if I take one gradient of the velocity field, partial u i partial by partial x j is equal to some constant into x i times partial p by partial x j plus p times delta i j. Take another derivative because I have the Laplacian of the velocity field here for the particular integral.

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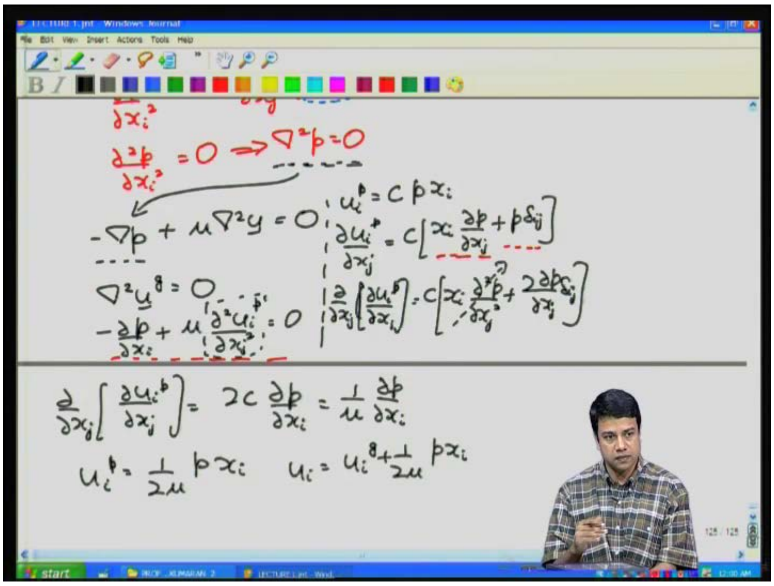

So, I take another derivative and I will find that, partial u i is equal to C into, I will get x i partial square p by partial x j square plus two partial p by partial x jdelta i j. First I take the derive this term first I take the derive this term, which gives me x i partial square p by partial x j square plus partial p by partial x j times del i j. And second with this term, which gives me another partial p by partial x j delta x j therefore, and note that, the pressure already satisfies the Laplace equation, del square p is already equal 0 here.

Therefore, this term which is for partial Laplace of the pressure is identically equal to 0 therefore, partial by partial x j of partial u i particular by partial x j is equal to 2 C. Partial p by partial x j times delta i j is just the gradient of the pressure p by partial x i partial p by partial x i. And I know that, partial minus partial p by partial x i plus mu times del square u has to be equal to 0 that means, that this has to be equal to 1 by mu partial p by partial x i for the equation for the particular integral to be satisfied, which means that the particular integral for the velocity is equal to 1 by 2 mu p times x i.

So, the strategy is clear, I divide the velocity into two parts, the general solution and the particular integral. General solution satisfies the equation without the inhomogeneous term, so that reduces to Laplace equation. Particular integral is any one solution that satisfies the equation, in this particular case, I have found one solution that satisfies the equation. Therefore, the total solution for velocity is equal to the general solution plus 1 by 2 mu pressure times x i.

So, I find the pressure by solving the Laplace equation, the general solution by solving the Laplace equation then I put it together to get the final solution. So, this is the strategy, we will use for solving the stokes equations in the limit of low Reynolds number. As I expected, we have managed to reduce it to two equations, each of which is Laplace equation, is a laplace equation for the pressure. There is a Laplace equation for the general part of the velocity and the the total velocity is this, particular plus the general part. Kindly go through the solution procedure that we had done for the Laplace equation, in the previous diffusion dominated transport lectures. And we will look at a different ways of getting the same thing in this particular lecture. So, kindly revise that and come for the next lecture and we will continue this in the next lecture, we will see then.