Fundamentals of Transport Processes II
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Lecture - 12
Boundary Conditions

So, welcome to this lecture number 12 of our course on Fundamentals of Transport Processes 2, where we were just getting into the constitutive relation for this stress in a fluid.

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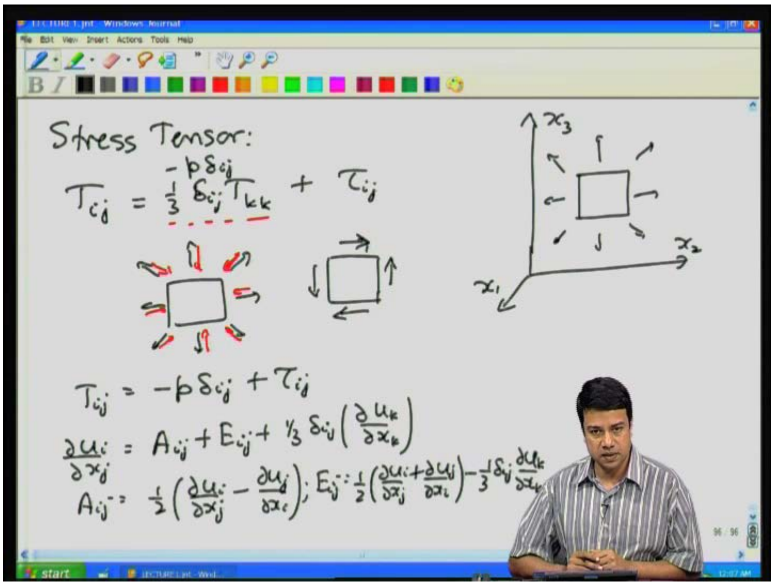

Stress Tensor:

$$T_{ij} = \underset{- p\delta_{ij}}{\frac{1}{3}\delta_{ij}T_{kk}} + \tau_{ij}$$

$$T_{ij} = -p\delta_{ij} + \tau_{ij}$$

$$\frac{\partial u_i}{\partial x_j} = A_{ij} + E_{ij} + \frac{1}{3}\delta_{ij}\left(\frac{\partial u_k}{\partial x_k}\right)$$

$$A_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right);\ E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{1}{3}\delta_{ij}\frac{\partial u_k}{\partial x_k}$$

So, as it is we briefly discussed in the last lecture, the stress tensor T i j, force per unit area in the i direction acting at a surface whose unit normal is in the j direction. So, T 1 1 if we have a coordinate system here, x 1 x 2 x 3, T 1 1 is the force per unit area in the x 1 direction acting at a surface whose unit normal is in the x 1 direction. T 1 2 is the force per unit area in the x 1 direction acting at a surface, whose unit normal is in the x 2 direction, and so on.

However it is far more profitable to think in terms of fundamental components of the stress tensor itself. From the angular momentum balance equation, I had shown you that the stress tensor has to be symmetric. Otherwise the angular momentum will not be balanced on individual volume elements. In most cases where there is no microscopic torque acting on the system, the stress tensor has to be a symmetric tensor. So, we have this 3 by 3 in three dimensions stress tensor a symmetric tensor, as with the rate of deformation tensor this can also be decomposed into fundamental components, that is one is the isotropic and the symmetric case less compound.

So, if I have a differential volume, there is one component which is isotropic, isotropic means it acts gradually inward or outward with this, so the isotropic component acting on this will act gradually inward or outward. It is outward if the force acting outward of the volume, inward if is acting inward of the volume, so that is one one part of the rate of definition tensor, the isotropic part one third delta i j times T k k. The T k k is the trace of this stress tensor T 1 1 plus T 2 2 plus T 3 3 on a differential volume, this basically acts outward at all points.

And the magnitude is the same in all directions for an isotropic you have either isotropic outward pressure acting from out to in or in to out, and there is a second component which is symmetric traceless. Symmetric traceless and their context of deformations, we had said there is a deformation in which there is no net rotation no net change in volume. And in this particular case symmetric traceless means at on any differential volume, I have the stress that is acting tangential to the surface.

There is no net rotation because so that the torque due to the clockwise rotation on top and below is exactly balanced by that top, due to the anti-clockwise rotation on the right and left. There is no net force because the magnitudes are in are all equal, so these are the 2 fundamental components of the stress. The symmetric traceless part is 0, if there is no fluid flow because for the symmetric traceless part to exist, that means that there has been momentum transport that is perpendicular to the direction of flow.

Momentum transport that is direct perpendicular to the direction of the unit normal, that can happen only due to momentum diffusion, momentum diffusion will require a gradient of the velocity. So, the symmetric traceless part is non-zero only when there is no flow, the isotropic part on the other hand could be non-zero even when there is no flow. In a static fluid you do have a hydro static pressure, the pressure acts inward on any differential volume at all points and it is along the inward unit normal.

So, the pressure in a static fluid basically, acts inward the pressure is acting inwards at all points within the flow. That means that this first component one third delta i j T k k can also be written as minus p times delta i j the pressure times the isotropic tensor. So, this is the pressure component which is non-zero even in a static fluid, the second is a symmetric traceless part the shear stress which is non-zero only when there is flow and there is of course, no anti symmetric component.

Now, this shear stress has to be related to the rate of deformation tensor rate of deformation tensor had 3 components, once again partially u i by partial x j is equal to the anti-symmetric part plus the symmetric part plus one third delta i j times the divergence of the velocity, one third delta i j times the divergence of the velocity. And the symmetric traceless part in the last class I had defined it as E vector where E i j A i j is equal to half partial u i by partial x j minus it is transpose half of the tensor minus it is transpose. And the symmetric traceless part is equal to half the vector plus it is transpose, but I have to remove the trace I have to remove the trace, because this is symmetric and traceless. So, I remove the trace so I remove the trace one third identity tensor times the divergence of the velocity, so we have the anti-symmetric and a symmetric traceless parts.

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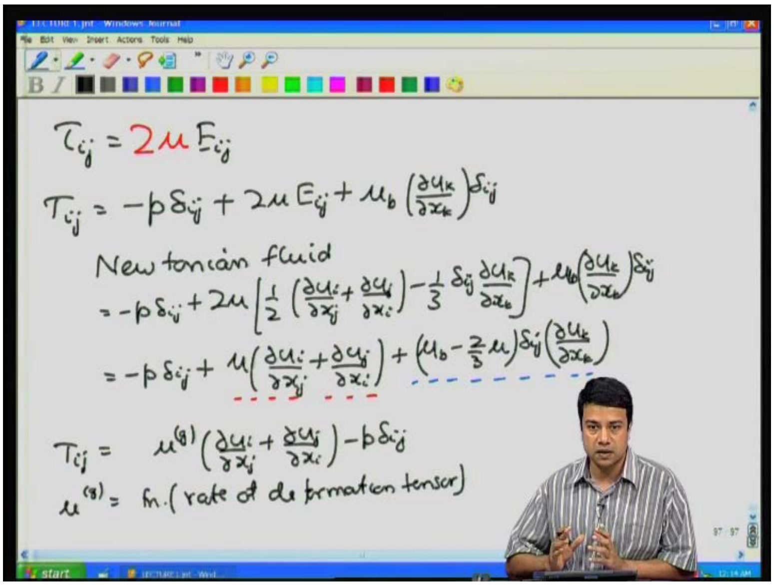

$$\tau_{ij} = 2\mu E_{ij}$$

$$T_{ij} = -p\delta_{ij} + 2\mu E_{ij} + \mu_b \left(\frac{\partial u_k}{\partial x_k}\right)\delta_{ij}$$

Newtonian fluid

$$= -p\delta_{ij} + 2\mu\left[\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{1}{3}\delta_{ij}\frac{\partial u_k}{\partial x_k}\right] + \mu_b\left(\frac{\partial u_k}{\partial x_k}\right)\delta_{ij}$$

$$= -p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \left(\mu_b - \frac{2}{3}\mu\right)\delta_{ij}\left(\frac{\partial u_k}{\partial x_k}\right)$$

$$T_{ij} = \mu^{(g)}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - p\delta_{ij}$$

$\mu^{(g)} =$ fn.(rate of deformation tensor)

For a Newtonian fluid the relation between the stress and the rate of deformation is a linear one, for a Newtonian fluid the relationship between stress and rate of deformation is linear so the symmetric traceless part of the stress has to be a linear function of the symmetric part of the deformation tensor. If it is linear the only way that you can have a relationship between these 2 is if tau i j is equal to some constant times E i j, because the left hand side is a symmetric traceless; that means, the right hand side also has to be symmetric traceless it is a linear relationship.

So, it has to be proportional to the first power of E and if it is a linear relationship I cannot have a dependence on the isotropic part, because you have dependence in the isotropic part and it is a linear dependence. This stress will become will not be traceless any more, and this coefficient here this is actually 2 mu times E where mu is the coefficient of viscosity. And for a linear stress rate of deformation relationship the general expression for this stress tensor is equal to minus p times delta i j, the part that is static that is that is non zero even in a static fluid plus 2 mu times E i j. And in general you could also have a contribution to the stress due to radial expansion or compression that would be the bulk viscosity times partial mu k by partial x k. So, this is the most I am sorry that times delta i j in other words the isotropic part could be the pressure plus a contribution due to the deformation tensor rate of deformation tensor that isotropic contribution to the rate of deformation tensor.

If it has to be linear in the rate of deformation tensor it should be proportional only to the isotropic part of the rate of deformation tensor, so this is the complete expression for the the stress tensor for a Newtonian fluid. And if I can expand this out this will be minus p delta i j plus 2 mu into half minus one third, that is the complete symmetric traceless part of a rate of deformation tensor. Symmetric I take the matrix transpose and divide by 2 then from that take off the trace, plus mu B delta i j and of course, I can expand this out and simplify it to get minus p delta i j plus mu plus mu B minus 2 by 3, so that is complete expression for a stress tensor for a Newtonian fluid. Not all fluids are Newtonian you could have for example, polymeric solutions suspensions and so on, which in general do not satisfy the linearity relationship between the stress and the rate of deformation tensor.

One simple way to extend the the relationship between stress and the rate of deformation is to just make the viscosity itself, depend upon the rate of deformation for non-Newtonian fluids. These fluids are usually incompressible fluids, the velocities are usually very small compared to the speed of sound, so for incompressible fluids the divergence with a velocity as I said has to be equal to 0. So, usually they have only this first part the rate flow of this stress tensor, the second part for these non-Newtonian fluids polymer melts polymer solutions, suspensions of particle sum suspensions of surfactants and so on. This first part is usually the second part in blue is usually 0, because the divergence of the velocity is 0, they are going at sufficiently small speeds that the fluid is effectively incompressible. So, in that case if one to want to write an expression for this stress tensor in terms of the rate of deformation tensor.

The simplest way to expand it is to write T i j is equal to 2 mu of the rate of deformation tensor, a function of the rate of deformation tensor let us just call it as 2. I will call it as a generalized viscosity 2 should not be there, this viscosity the generalized viscosity is in general of a function of the rate of deformation tensor, this is generalized viscosity which we use in a non-Newtonian relationship between the stress and the rate of deformation will in general be a function of the rate of deformation tensor.

What kind of a function should it be? Usually, it cannot depend upon the rotational part of the rate of deformation tensor, because a solid body rotation does not change the distance between nearby points, it can depend upon the symmetric traceless part. In general the isotropic part will be 0, because the fluid is incompressible, so I said if the fluid is incompressible you do not have radial expansion or compression therefore, the isotropic part of the rate of deformation tensor will in general be 0.

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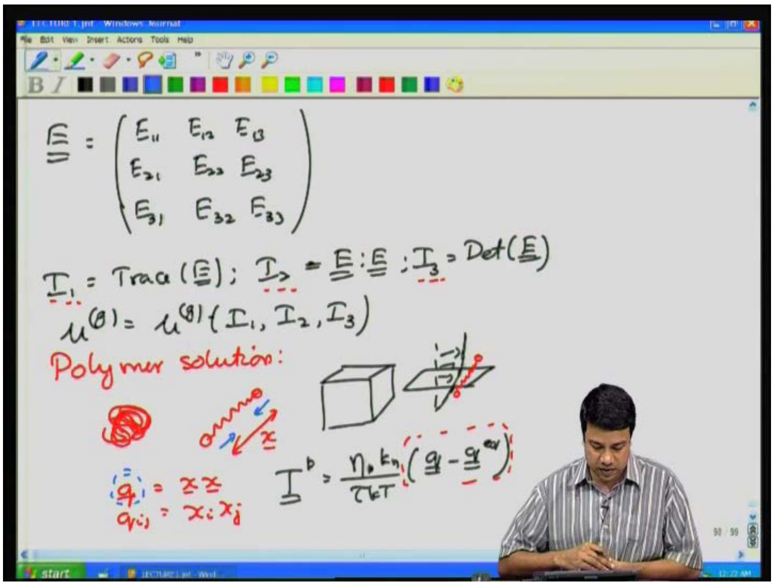

$$\underline{\underline{E}} = \begin{pmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33} \end{pmatrix}$$

$$I_1 = \text{Trace}(\underline{\underline{E}});\ I_2 = \underline{\underline{E}}:\underline{\underline{E}};\ I_3 = \text{Det}(\underline{\underline{E}})$$

$$\mu^{(g)} = \mu^{(g)}(I_1, I_2, I_3)$$

Polymer solution:

$$\underline{\underline{q}} = \underline{x}\,\underline{x}$$

$$q_{ij} = x_i x_j$$

$$\underline{\underline{T}}^p = \frac{\eta_p k_n}{\tau k T}\left(\underline{\underline{q}} - \underline{\underline{q}}^{eq}\right)$$

So, in general this can depend upon the symmetric traceless part of the rate of deformation tensor, how can it depend upon the symmetric up on the symmetric traceless part, because that is the second order tensor. That deforms that that transforms in certain ways upon rotation of coordinate systems, whereas mu itself is a viscosity so there are some general measures of this symmetric traceless tensor which do not depend up on the underlying coordinate system.

For example, if I had this tensor E which has E 1 1 E 1 2 E 1 3 E 2 1 E 2 2 E 2 3, for this tensor there are certain certain quantities, certain properties of this tensor that are invariant under coordinate transformation. And therefore, if you want to write a constitutive relation which is coordinate frame invariant that constitutive relation can depend only upon the invariant measures of this tensor and for this a 3 by 3 there are 3 frame invariant scalars there are 3 frame invariant scalars.

The first is just the trace of E the second is equal to the double dot product and the third is the determinant is the determinant of this tensor, so these 3 are frame invariant. And therefore, any viscosity that I write which is dependent upon these, this mu this mu g has to be a function of only these 3, it has to be a function only of these 3 frame invariant quantities. It cannot in general depend upon let us say the 1 2 component of this rate of deformation tensor, because if I rotate coordinate systems the value of that particular component is going to change.

However, the values on these invariant a scalars scalar measures of this tensor do not change, when I change coordinate systems therefore, in general the viscosity has to be written in terms of these three. You could have more complicated cases for example, if I have polymeric solutions, if have liquid crystalline media the stress in these cases may depend upon things like confirmation. For example, in the case of a polymer solution in the case of a polymer solution you know that the polymer is basically a long chain molecules, consisting of monomers which are covariantly bonded with each other.

So, it is a long linear molecule, however if you put into a solution, it is going to be in an elongated fashion, because the the entropy of stretching that is too much. So, in a solution the polymer is going to be in a highly coiled state, the solution will be in a highly coiled state, a useful representation for this highly coiled polymer is a bead spring model, where I imagine 2 beads and a spring between these two, a linear spring. A linear spring between which is between these two beads the beads themselves represents the ends of the polymer molecule, it is in a highly coiled state the distance between ends is relatively small and therefore, the force.

So, therefore, the polymer prefers to be in that configuration, if I stretch out the ends there is are storing force which tends to draw back the ends towards each other, and that restoring force is acting this this this modeled as acting on these beads. Now, the distance between ends will roughly scale as square root of the number of monomers that are there in a highly coiled state if the polymer actually is is in a perfectly coiled state in which it is a random walk.

The distance it is square root of the total number of monomers in the polymer, in this case it is not sufficient to just have a fluid viscosity, you also have a stress which is exerted due to the polymer. So, for example, if I have a differential volume there is of course, if I take a one particular surface of this volume take one particular surface of this volume there is of course, a transfer of momentum due to the mean fluid velocity across the surface there is of course, a transfer point into the velocity gradient across this surface fluid momentum diffusion.

However, I could also have a transfer of momentum, because one end of this bead is below the surface the other end of the bead is above the surface, so the end of the bead above the surface exerts a force and the end of the bead below the surface. And that is a net force that is exerted on the volume below the surface, so the presence of these polymers is also in an in an additional force that is exerted. There is a stress tensor due to this force and the stress tensor due to the force is basically proportional to what is called the end to end vector distance, so if I have a polymer and the distance between these two ends is x vector. I can create a vector which is related to I can create a tensor by taking the direct product of these two ends, I will create a q tensor which is equal to the product of x vector with itself of q i j is equal to x i x j second order tensor.

And there will be an additional stress, which is proportional to this second order tensor, in other words if x i and if if the polymer stretches x i and x j are greater therefore, there is a greater force that is exerted throughout the surface. And the force the stress tensor is usually written in this case for the polymers as proportional to x i x j there are some constants here times x i x j minus it is equilibrium value. Let us say it as where q is the actual stress within the fluid and q equilibrium is the stress in the absence of fluid, k n is a spring constant which basically tells you how much force there is in the polymers when you stretch them, tau was a relaxation time for the polymer and k T, is the Boltzmann constant time for temperature. So, in addition to writing equations for the fluid velocity one would also have to write equations for this polymer confirmation tensor, and couple these two because the stress flow in the fluid does depend upon the stress that is transmitted by the polymer molecules themselves.

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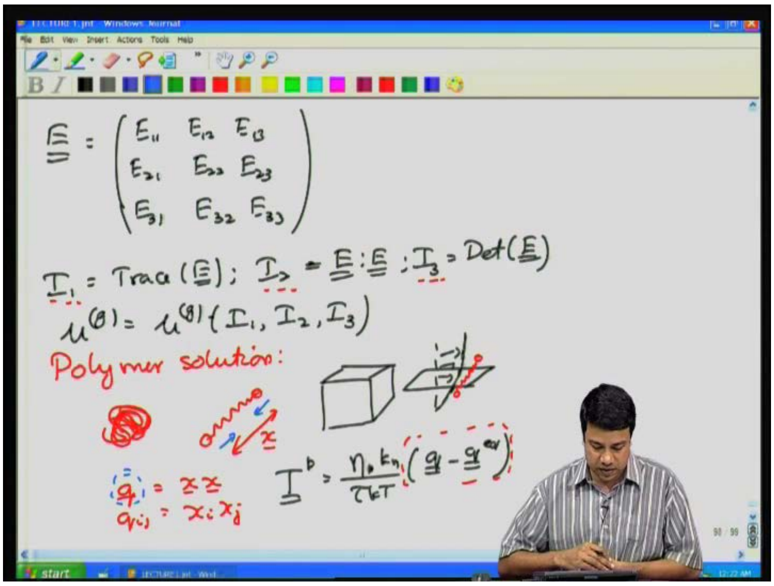

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$$I_1 = \text{Trace}(\underline{\underline{E}});\ I_2 = \underline{\underline{E}}:\underline{\underline{E}};\ I_3 = \text{Det}(\underline{\underline{E}})$$

$$\mu^{(B)} = \mu^{(B)}(I_1, I_2, I_3)$$

Polymer solution:

$$\underline{\underline{q}} = \underline{x}\,\underline{x}$$

$$q_{ij} = x_i x_j$$

$$\underline{\underline{\tau}}^{p} = \frac{\eta_p k_n}{\tau k T}(\underline{\underline{q}} - \underline{\underline{q}}^{eq})$$

So, in those cases it is necessary to write an additional conservation equation for this polymer stress tensor itself, in there are various ways that these constitutive relations are written so for the polymer stress tensor for the confirmation tensor. For this confirmation tensor q, one writes a constitutive relation, in order to determine this confirmation tensor in terms of all the other quantities. So, but in this course you will not deal with all of these additional complications relating to the presence of polymers within the fluid and so on, our attention will be restricted simply to Newtonian fluids, where we have a clearly defined relationship between stress and relative transformation tensor. Which, is just a brief introduction to how you would extend the formalism, in the case of non-Newtonian fluids we either have no complicated relationships within the stress and the rate of confirmation.

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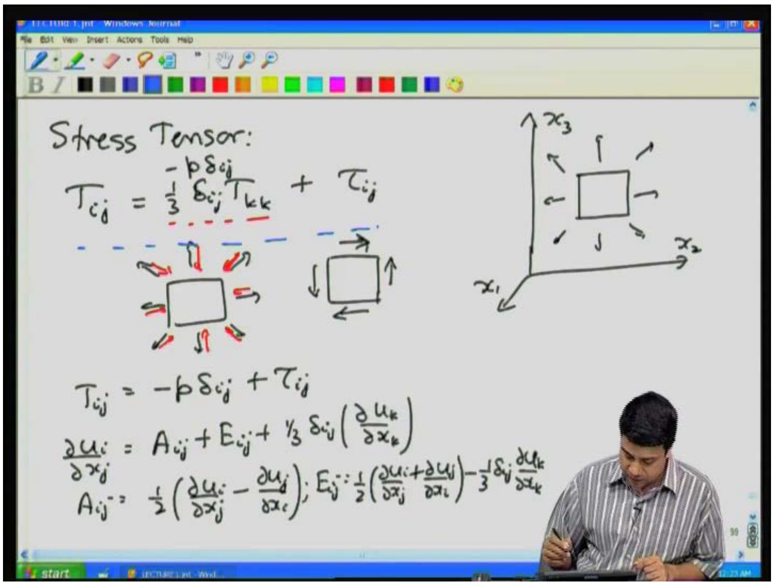

Stress Tensor:

$$T_{ij} = \overset{-p\delta_{ij}}{\underline{\tfrac{1}{3}\delta_{ij}T_{kk}}} + \tau_{ij}$$

$$T_{ij} = -p\delta_{ij} + \tau_{ij}$$

$$\frac{\partial u_i}{\partial x_j} = A_{ij} + E_{ij} + \tfrac{1}{3}\delta_{ij}\left(\frac{\partial u_k}{\partial x_k}\right)$$

$$A_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right);\ E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{1}{3}\delta_{ij}\frac{\partial u_k}{\partial x_k}$$

So, now putting all these things together, I have an equation for the stress tensor here a relationship between the stress tensor and the rate of deformation tensor here, and that has to be put into the momentum conservation equation.

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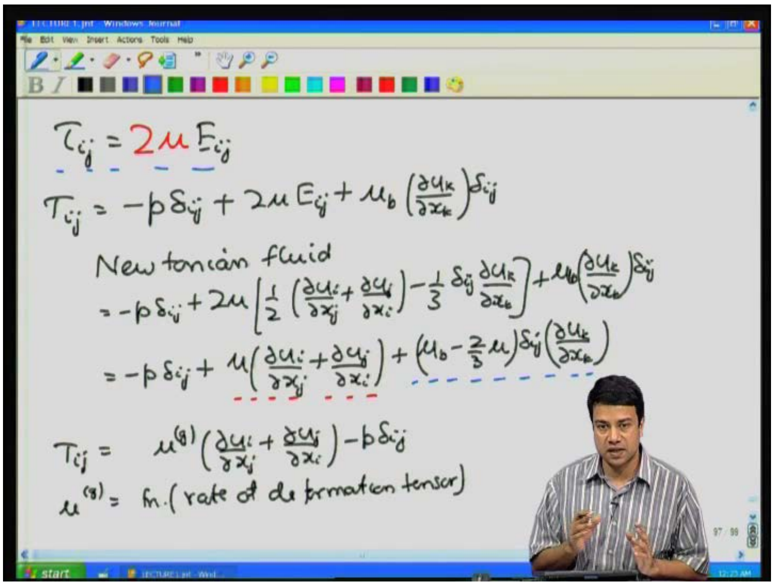

$$\underline{\tau_{ij} = 2\mu E_{ij}}$$

$$T_{ij} = -p\delta_{ij} + 2\mu E_{ij} + \mu_b\left(\frac{\partial u_k}{\partial x_k}\right)\delta_{ij}$$

Newtonian fluid

$$= -p\delta_{ij} + 2\mu\left[\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{1}{3}\delta_{ij}\frac{\partial u_k}{\partial x_k}\right] + \mu_b\left(\frac{\partial u_k}{\partial x_k}\right)\delta_{ij}$$

$$= -p\delta_{ij} + \underline{\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)} + \underline{\left(\mu_b - \frac{2}{3}\mu\right)\delta_{ij}\left(\frac{\partial u_k}{\partial x_k}\right)}$$

$$T_{ij} = \mu^{(g)}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - p\delta_{ij}$$

$\mu^{(g)}$ = fn.(rate of deformation tensor)

In order to finally, get a closed set of equations for the density the velocity the momentum and in the equation for the momentum I know now have a pressure, which is the force exerted in the statistic state.

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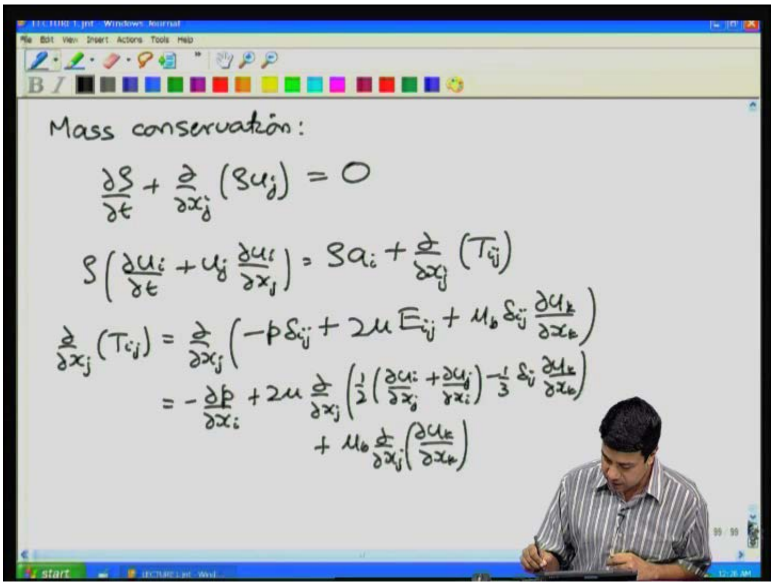

Mass conservation:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j) = 0$$

$$\rho\left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}\right) = \rho a_i + \frac{\partial}{\partial x_j}(T_{ij})$$

$$\frac{\partial}{\partial x_j}(T_{ij}) = \frac{\partial}{\partial x_j}\left(-p\delta_{ij} + 2\mu E_{ij} + \mu_b \delta_{ij} \frac{\partial u_k}{\partial x_k}\right)$$

$$= -\frac{\partial p}{\partial x_i} + 2\mu \frac{\partial}{\partial x_j}\left(\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{1}{3}\delta_{ij}\frac{\partial u_k}{\partial x_k}\right)$$

$$+ \mu_b \frac{\partial}{\partial x_j}\left(\frac{\partial u_k}{\partial x_k}\right)$$

So, let us look at what happens when we put all of these together, first the mass conservation equation partial rho by partial T plus the divergence of rho u is equal to 0. Momentum conservation equation can be written in one of two ways, the simplest way to write it is rho times partial u i by partial T plus u j partial u i by partial x j is equal to minus rho a i plus partial by partial x j of T i j. What is the divergence of this stress tensor partial by partial x j of T i j is equal to partial by partial x j of minus p delta i j plus 2 mu E i j plus the bulk viscosity.

So, that is the divergence of the stress tensor, partial by partial x j of p times delta i j is just partial p by partial x i, partial by partial x j of minus p times delta i j is just minus partial p by partial x I because delta i j is not 0, only when i is equal to j. If the viscosity is independent of position if the viscosity is independent of position then the second terms becomes 2 mu partial by partial x j of the symmetric traceless part of the rate of deformation tensor symmetric traceless part of the rate of deformation tensor. And of course, I have this other term here which is mu b partial by partial x j of partial u k by partial x k, so that is the complete expansion of the divergence of the stress tensor let us simplify this a little bit the first term is minus partial p by partial x k x i.

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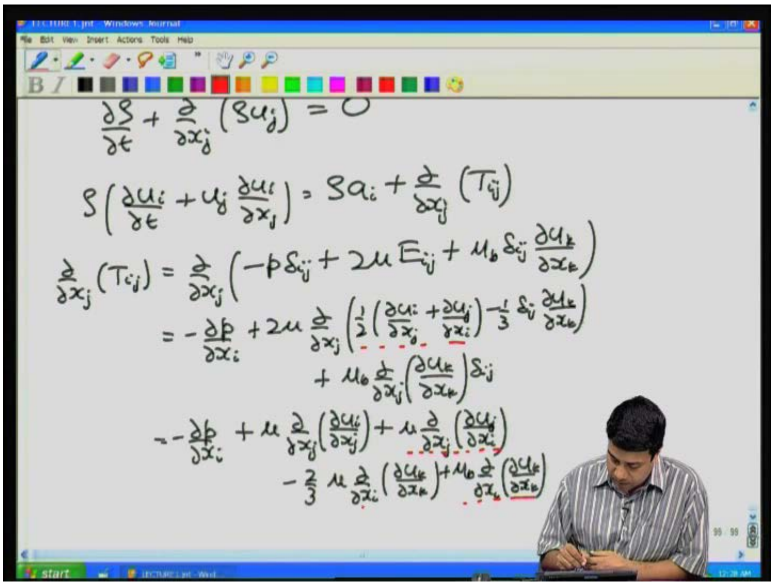

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j) = 0$$

$$\rho\left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}\right) = \rho a_i + \frac{\partial}{\partial x_j}(T_{ij})$$

$$\frac{\partial}{\partial x_j}(T_{ij}) = \frac{\partial}{\partial x_j}\left(-p\delta_{ij} + 2\mu E_{ij} + \mu_b \delta_{ij}\frac{\partial u_k}{\partial x_k}\right)$$

$$= -\frac{\partial p}{\partial x_i} + 2\mu \frac{\partial}{\partial x_j}\left(\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right) - \frac{1}{3}\delta_{ij}\frac{\partial u_k}{\partial x_k}\right) + \mu_b \frac{\partial}{\partial x_j}\left(\frac{\partial u_k}{\partial x_k}\right)\delta_{ij}$$

$$= -\frac{\partial p}{\partial x_i} + \mu\frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j}\right) + \mu \frac{\partial}{\partial x_j}\left(\frac{\partial u_j}{\partial x_i}\right) - \frac{2}{3}\mu\frac{\partial}{\partial x_i}\left(\frac{\partial u_k}{\partial x_k}\right) + \mu_b\frac{\partial}{\partial x_i}\left(\frac{\partial u_k}{\partial x_k}\right)$$

Second term is mu partial by partial x j of partial u i by partial x j plus viscosity into of partial u j by partial x i, so as the first and second terms in this these first 2 terms in this expansion. Then I have the isotropic part once I have separated out the isotropic part from to get a traceless tensor, so this will give me minus 2 by 3 mu delta i j times partial by partial x j is just partial by partial x i of partial u k by partial x k plus mu b, there should be delta j. Now, in this term you can interchange the order of differentiation you have partial by partial x j of partial u j by partial x i, since x i and x j are independent coordinates i can interchange the order of differentiation i can write this as partial by partial x i of partial u j by partial x j. However, partial u j by partial x j is identical to partial u k by partial x k, because it is repeated index both are equal to partial u 1 by partial x 1 plus u 2 by partial x 2 plus partial u 3 by partial x 3.

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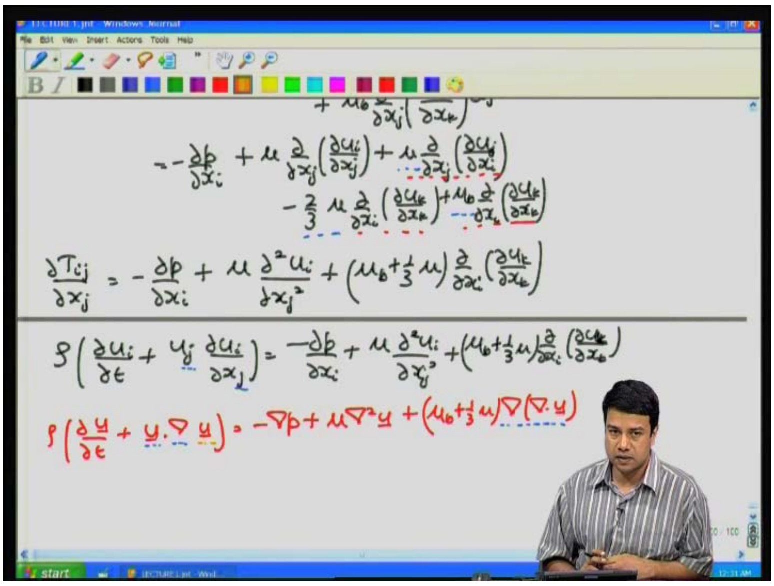

$$= -\frac{\partial p}{\partial x_i} + \mu \frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j}\right) + \mu \frac{\partial}{\partial x_j}\left(\frac{\partial u_j}{\partial x_i}\right) - \frac{2}{3}\mu \frac{\partial}{\partial x_i}\left(\frac{\partial u_k}{\partial x_k}\right) + \mu_b \frac{\partial}{\partial x_i}\left(\frac{\partial u_k}{\partial x_k}\right)$$

$$\frac{\partial T_{ij}}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j^2} + \left(\mu_b + \frac{1}{3}\mu\right)\frac{\partial}{\partial x_i}\left(\frac{\partial u_k}{\partial x_k}\right)$$

$$\rho\left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}\right) = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j^2} + \left(\mu_b + \frac{1}{3}\mu\right)\frac{\partial}{\partial x_i}\left(\frac{\partial u_k}{\partial x_k}\right)$$

$$\rho\left(\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u}\right) = -\nabla p + \mu\nabla^2\underline{u} + \left(\mu_b + \frac{1}{3}\mu\right)\nabla(\nabla\cdot\underline{u})$$

So, that is a common factor between these 3 terms here, there is a common factor between all of these 3 terms here, and I can add all of those up to get partial T i j by partial x j is equal to minus partial p by partial x i plus mu. This is partial by partial x j of partial u i by partial x j del dot del u, that is partial square u i by partial x j square, the second term I have plus mu minus 2 by 3 mu. So, there is a plus mu here minus 2 by 3 by mu plus mu b, so all put together this becomes equal to mu b plus one third mu partial by partial x i of partial u k by partial x k, so that is the divergence of a stress tensor.

So, putting all these into the momentum conservation equation I get rho into partial u i by partial T plus, so that is the final expression often written in short form as follows. In vector notation this is often written as rho times partial u vector by partial T plus u dot grad u minus gradient of p plus, this is mu del square u vector the Laplace an acting on u vector plus mu b plus one third mu, the gradient of the divergence of u vector.

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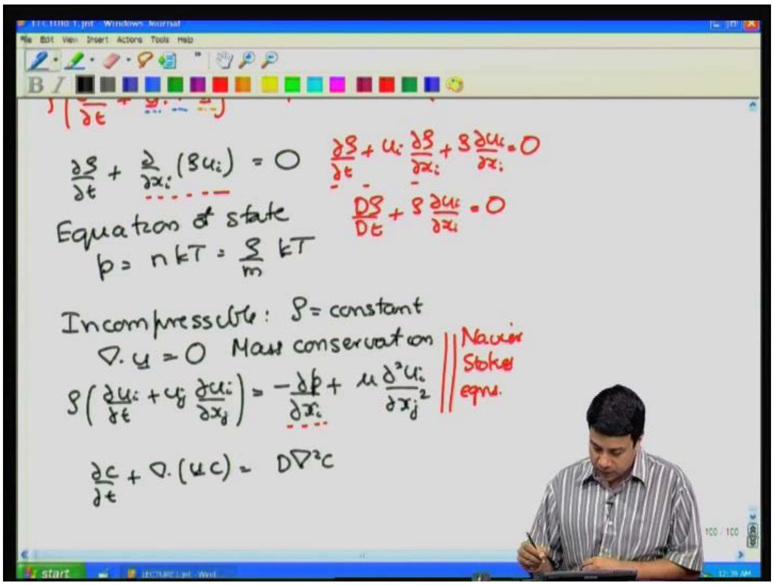

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i) = 0 \qquad \frac{\partial \rho}{\partial t} + u_i \frac{\partial \rho}{\partial x_i} + \rho \frac{\partial u_i}{\partial x_i} = 0$$

$$\frac{D\rho}{Dt} + \rho \frac{\partial u_i}{\partial x_i} = 0$$

Equation of state

$$p = nkT = \frac{\rho}{m} kT$$

Incompressible: ρ = constant

$\nabla \cdot \underline{u} = 0$ Mass conservation

$$\rho\left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}\right) = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j^2}$$

Navier Stokes eqns.

$$\frac{\partial c}{\partial t} + \nabla \cdot (uc) = D\nabla^2 c$$

Note the difference between these 2 terms here, here I have divergence of u which becomes a scalar and then I have taken the gradient of that one, in this case there is a dot product between the j here and j the here, therefore, I get u dot grad. Three index belong to this vector i the 3 index belongs to this vector this this i index is the 3 index, so this is the momentum conservation equation written in vector notation, this coupled with the mass conservation equation gives me the complete set of equations.

The mass conservation equation partial rho by partial T plus partial by partial x i of rho u i is equal to 0, note that the mass conservation equation is a scalar equation, because a rho is a scalar the density is a scalar, momentum conservation equation is a vector equation. One equation for each value of phi i going from 1 to 3 therefore, there are 3 equations in the 3 scalar equations contained within this one vector equation, put together I get a total of 4 equations, one mass conservation equation and 3 momentum conservation equations.

Now, what do I have to solve for here let us write that clearly for you, now what do I have to solve for here, I have to solve for the density rho 3 components of the velocity u one u 2 u 3 as well as for the pressure. So, one density one pressure 2 plus 3 components of the velocity 5, so I have total 5 scalar quantities to solve for I have 4 equations here, I need one more equation that equation will be a relationship between the pressure and the density.

That is an equation of state for the pressure density relationship for the ideal gas for example, the equation of state is basically p is equal to n k T is equal to rho by m times k T, where rho was the mass density and I was the molecular mass k is the Boltzmann constant. So, I need one more relationship between pressure and density, and we need a constraint here, because even though I have got this relationship between pressure and density, I have to specify, whether the flow is isothermal or adiabatic.

For those two situations from knowing that relationship between temperature and pressure, I am going to get one relationship between pressure and density. So, that equation of state along with one mass conservation equation and one vector or 3 scalar momentum conservation equations gives me 5 equations, which can be solved to determine the density, the pressure and the 3 components of the velocity. So, in general you need to know equation of state, for a ideal gases it is just this ideal gas law, for liquids you need to know how the density varies with temperature and pressure for the liquid state itself. So, that relationship is required in order to be able to completely specify the equations. A considerable simplification can be made for fluids that are called incompressible, for incompressible fluids the density is invariant, the density is a constant, incompressible fluids the density is equal to a constant.

Now, if the density is equal to a constant as I said I can also write the mass conservation equation in terms of the substantial derivative of the density by using chain rule for differentiating this term. So, if i use the chain rule for differentiation this becomes partial rho by partial T plus u i partial rho by partial x i plus rho partial u i by partial x i equal to 0, and these first 2 terms put together is d rho by d t. If rho is a constant then d rho by d T is equal to 0 and what; that means, is that partial u i by partial x i is equal to 0, so in that case the divergence of the velocity is equal to 0.

So, this is the mass conservation equation velocity is divergence free if the fluid is incompressible divergence free; that means, if I take any differential volume since the density is a constant it has no source of mass within that volume. Since, density is a constant there can be no net flow of fluid outside, because from inside to outside because the density is a constant. So, this divergence of u is equal to 0, is itself a mass conservation equation this divergence of u is equal to 0 is itself a mass conservation equation, and if the divergence of u is equal to 0, you can see that in this momentum conservation equation, there is this piece here related to the bulk viscosity which is proportional to the divergence of u. As this piece here which is the divergence of a velocity and that is equal to 0 therefore, that little bit which is proportional to the divergence the velocity disappears.

And finally, what I get is an equation which goes as rho times partial u i by partial T plus u j partial u i by partial x j minus partial p by partial x i plus mu. So, that is the momentum conservation equation for an incompressible fluid, and these two equations put together are what are called the navios stokes equations, is the navios stokes equations mass and momentum conservation equation for an incompressible fluid.

Now, these equations as you can see there is one scalar mass conservation equation, there are 3 scalar or one vector momentum conservation equation, and what are the unknowns here. That is pressure, there is a pressure gradient in this terms, so there is pressure and there are 3 components of the velocity, now I have 4 amounts density itself now is invariant, density is a function of the fluid. There is the density and both the density and the viscosity are not fluid properties, which are specified it is independent of pressure it is invariant and it is specified fluid property.

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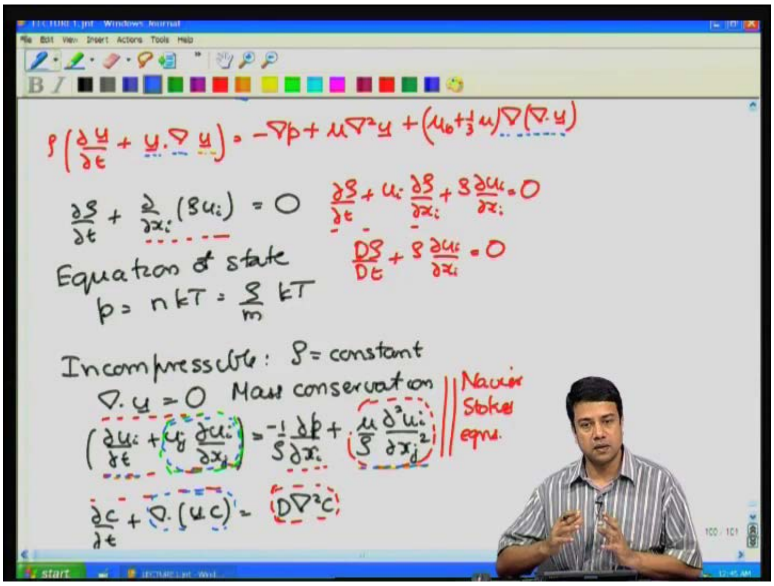

$$\rho\left(\frac{\partial \underline{u}}{\partial t} + \underline{u}.\nabla \underline{u}\right) = -\nabla p + \mu \nabla^2 \underline{u} + \left(\mu_b + \frac{1}{3}\mu\right)\nabla(\nabla.\underline{u})$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i) = 0 \qquad \frac{\partial \rho}{\partial t} + u_i\frac{\partial \rho}{\partial x_i} + \rho\frac{\partial u_i}{\partial x_i} = 0$$

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Navier Stokes eqns.

$$\frac{\partial c}{\partial t} + \nabla.(uc) = D\nabla^2 c$$

Therefore, I have one mass conservation equation scalar and one vector momentum conservation equation, which contains 3 momentum components and from these I have to determine now, 4 variables one is the pressure and the other 3 are the 3 components of the velocity. So, in that sense this is completely well specified system of equations, which can be solved in order to determine the velocity and the pressure field, note that this has a form that is very similar to the concentration and the temperature equations which we had derived in part one. So, if you recall my conservation equation for the concentration field was partial c by partial t plus del dot u c is equal to T del square c.

So, here if I divide throughout by the density if I divide throughout by the density in my momentum conservation equation, I will get here minus 1 over rho and this is mu over rho mu by rho is the kinematic viscosity, momentum diffusivity. So, it has exactly the same form, so this has exactly the same form as the diffusion term here, and the left hand side actually looks similar there is small difference, but it looks very similar to the left hand side the convictive terms and the time derivative in the concentration equation.

A big difference here two big differences, one is that we have a pressure in the momentum conservation equation, for which there is no analog in the mass and energy conservation equations. The diffusion term has an analog it is kinematic viscosity times the Laplacian of the velocity is analog that is to the diffusion coefficient times del square c, but there is no analog to the pressure. And the pressure in this case is required to ensure that the divergence of velocity is equal to 0 or that the incompressibility condition is identically satisfied. So, that is why we have a pressure term because I have to satisfy an additional equation, so an even additional variable.

So, the pressure at every point in the fluid is no longer something is determined from the equation of state, as it was originally for a compressible fluid but rather it takes whatever value locally is required or rather the pressure gradient takes, whatever value is required locally to ensure that the divergence to the velocity is always equal to 0. So, this pressure ensures incompressibility at every point within the fluid, and the pressure and the velocity 3 components of the velocity, 4 unknowns are determined from these 4 equations.

So, these are the navios stokes mass and momentum conservation equations, the other big difference is that you have a term here, which is a non-linear term u dot grad u it is non-linear in the velocity, you have no equivalent term in the mass and energy conservation equations, because this term here is linear in the concentration field, it is linear in the concentration field. So, the big difference is that the convictive term in the concentration equation is linear in the concentration field, where as the convective term in the momentum conservation equation is non-linear. If you have a linear equation subject to well posed boundary conditions, you have guaranteed that a solution exists and that the solution is unique for a non-linear equation it is no longer show.

And because of this the navios stoke equations mass and momentum conservation are considerably more difficult to solve than a concentration equation. In the concentration equation if I if you have a certain concentration field, which solves this equation which satisfies the mass conservation equation. If I multiply the concentration by a factor of 2 at every point, it stills satisfies the concentration equation, because each term in this equation gets multiplied by 2 and the result is still satisfying the mass conservation equation.

On the other hand if I have a velocity field and if I multiply the velocity vector by 2 at each and every point the linear terms, all get multiplied by a factor of 2 whereas, there is one non-linear term, which is not multiplied by a factor of 2. If I multiply the velocity everywhere by a factor of 2 the non-linear term gets multiplied by a factor of 4, so because of this the momentum equation is no longer linear in the velocity field. And when an equation is not linear in the velocity field then there is no guarantee that a solution exists, even if a solution does exist there is no guarantee that it is unique. So, that is why the momentum the the navios stokes equations are considerably more difficult to solve, and as you will seen in this course there are specified there are specified there are the, the ways that we use to solve these equations are based upon a physical understanding of the problem that we had considered.

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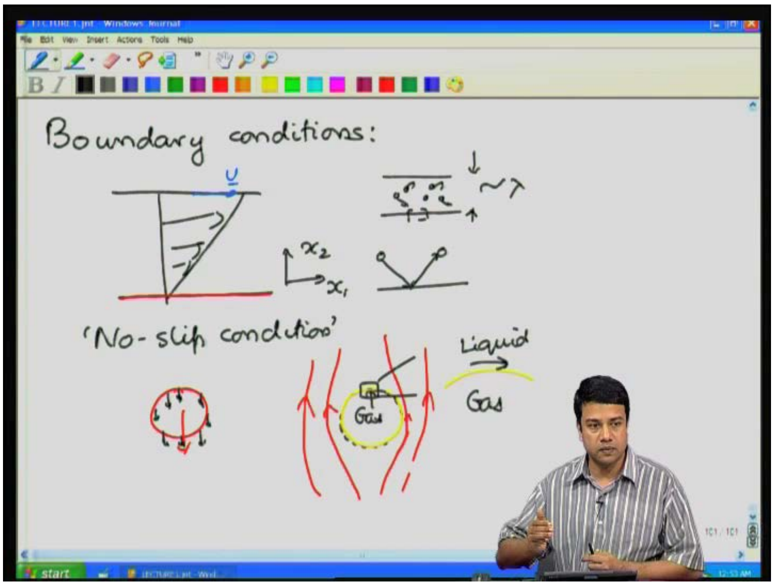

But, first these are the equations one also has to specify boundary conditions, these equations are second order in space and first order in time I have one time derivative here one time derivative here. And a second order derivative in the spatial coordinates for the velocity, the diffuse of term as always has the highest derivative, highest spatial derivative. That means that I need to specify two boundary conditions in each direction for the velocity field, two boundary conditions in each direction for each component of the velocity field.

So, in order to solve these equations in order to get a well posed set up equations, which can be solved any two boundary conditions in each direction for each component of the velocity. So, let us briefly discuss that boundary conditions in physical systems for example, if I have the flow between two flat plates or the flow in a pipe or something. For the flow between two plates not only the boundary condition at solid surfaces that you specify, is that the fluid velocity at the surface is exactly the same as the velocity of the surface itself, it is called the no-slip condition.

So, in this case if the bottom surface is stationary the fluid velocity is also 0 on that bottom surface. The top surface is moving with a velocity some velocity u, the fluid at that surface is also moving with exactly the same velocity, these conditions are general applicable to most cases unless, you have what is called slip at the boundaries, when the distances are sufficiently small that they become comparable to the mean free path.

In that case what happens if my system is sufficiently small in in micro-scale and nano-scale devices, of course, I have particles that are whose mean free path is comparable to the distance. And in that case you should look carefully at the wall itself, if you look carefully at the wall itself, you would expect that the molecule that is reflected off the wall has the same velocity, as the wall itself because when it is collided into the wall it is equilibrated with the wall. So, once I thus equilibrated with the wall it is going to come out with a velocity which is the same as the velocity of the wall itself.

However, the molecule that is incoming before collision, it need not have the same mean velocity as the velocity of the wall itself, and because of that one could end up having a net slip at the surface, but this happens only in micro-scale devices and for gases, where the length scale is comparable to the mean free path. For most macroscopic applications one can safely use the no-slip condition that is that the velocity of the fluid at the wall is equal to the velocity of the wall itself.

Important to note when you say the velocities are equal it means that each component of the velocity is equal, the x 1 component of the velocity of the wall of the fluid is equally the x 1 component of the wall velocity, x 2 component is also equal to the x 2 component of the wall velocity. In this particular case the top wall is moving only in the x 1 direction; that means, that the x 2 component of the wall velocity is equal to 0. There is no motion of the wall perpendicular to itself, that means that the x 2 component of the fluid velocity is also equal to 0, similarly for the x 3 component which is pointing into the plane of the wall that is also equal to 0. Similarly, if I have a fluid if I have a spherical particle for example, settling within a fluid that means that every point on the surface of the particle is moving with an equal velocity downwards.

The particle itself is a rigid object therefore, every point on that particle is moving with equal velocity downwards, that means that every point in the fluid on the particle surface is also moving with that exact same velocity downwards. So, in this case you enforce velocity boundary conditions you do not enforce stress boundary conditions, particle is sufficiently rigid that it can balance any stress, any fluid stress, that is exerted on it without deforming. So, you enforce velocity boundary conditions, you do not force stress boundary conditions at solid surfaces.

Similar, to in the case of for example, heat transfer you would enforce temperature boundary conditions at the surface, when you enforce the the temperature at the surface is equal to the temperature of the fluid. There is now no further necessity to say anything about the flux, we will just require the temperature at the boundaries has to be equal, so similarly, here if we are dealing with solid surfaces you require that the velocity vector at the surface is equal to the velocity vector of the surface itself.

The other kind of boundary condition that one can enforce is the stress boundary condition, analogous to the flux boundary condition in heat and mass transfer problems, these are the appropriate. For example, if I had rather than a particle, if I had a bubble had a bubble that is rising through the fluid so that there is gas inside which was rising through the fluid. And because the bubble is rising through the fluid I have some fluid flow around this bubble, I have some fluid flow that is around this bubble, now what are the boundary conditions at the surface of the bubble itself, inside the bubble there is gas.

So, if I look at the bubble surface, if I expand out one of the surfaces with a bubble, I have the surface of the bubble here, on this side is liquid and the other side is gas. Liquids typically have a viscosity that is about a thousand times more than gases, and at the surface of the bubble itself you require that the stresses have to be balanced that is the normal stress. In the liquid is equal to the normal stress in the gas the shear stress, in the liquid is equal to the shear stress in the gas, here you cannot enforce continuity of velocity boundary conditions because the gas itself can move.

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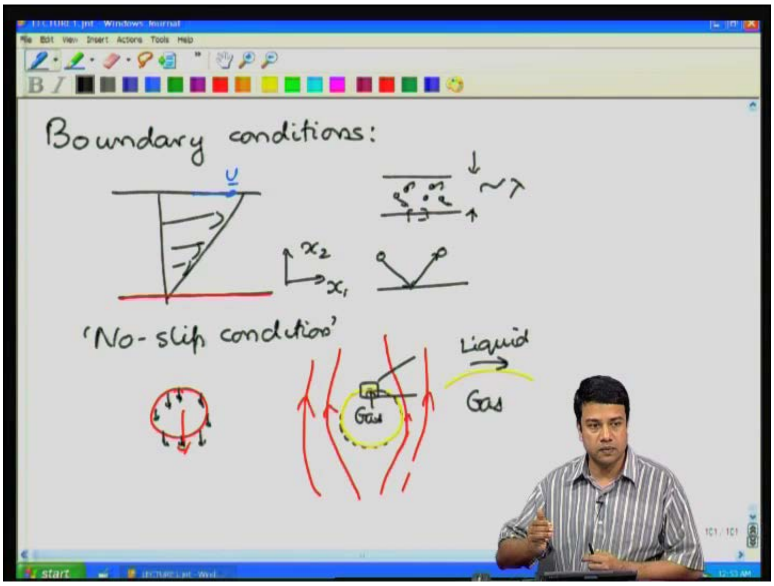

So, one can have a motion in the liquid motion in the gas such that at the boundaries the velocity is equal, so we do not straight away know the velocity in the gas of the of the bubble as it is moving. For a solid object of course, the velocity of the surface has to be the same velocity as the object for a gas it does not have to be same, so in this case you enforce zero stress boundary conditions. If you if there is a shear stress that is exerted by the gas on the liquid, if there is a share stress exerted by the gas on the liquid there is going to be a deformation within the gas itself. So, any shear stress that is exerted by the liquid on the gas is going to result in a deformation rate of deformation within the gas, rate of strength within the gas. However, the viscosity of the gas is significantly smaller than the viscosity of a liquid, so I need liquid stress that is exerted will result in an in an enormously large rate of deformation in the gas.

However, one cannot generate a rate of deformation that is significantly larger, because you require the velocities to be equal therefore, at the liquid gas interface it is the good approximation to state that the shear stresses, with unusual stress exerted by the liquid on the gas has to approach 0. As, you go from the liquid to the gas that is because if the stress were non zero the stress in the liquid is proportional to the viscosity of the liquid times the rate of deformation stress, in the gas is equal to viscosity of the gas times the rate of deformation. If there is some deformation are comparable the liquid viscosity is much larger than the gas viscosity the only way you can maintain a balance, in the limit as the gas viscosity goes to 0, compared to the liquid viscosity is if the shear stress in the gas the tangential stress acting at the surface if the gas goes to 0.

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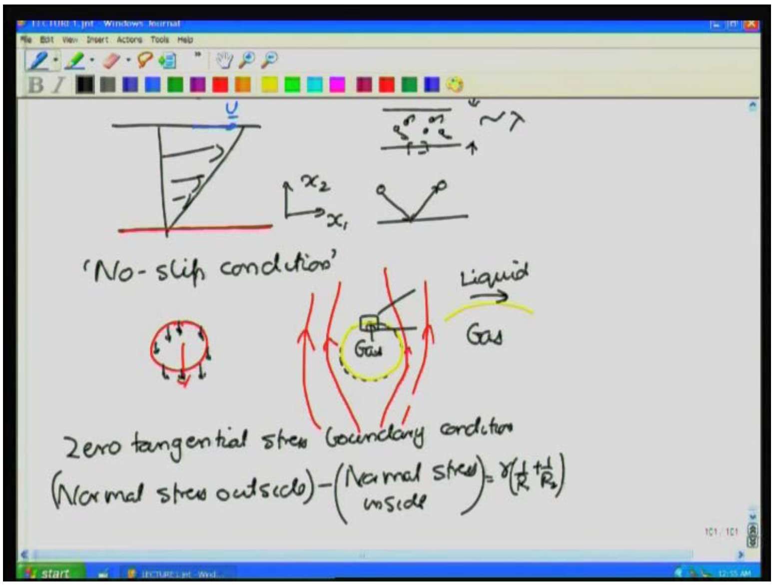

'No-slip condition'

Zero tangential stress boundary condition

(Normal stress outside) − (Normal stress inside) = $\gamma\left(\frac{1}{R_1}+\frac{1}{R_2}\right)$

If I had perfect vacuum here, the probably we would not be able to exert any tangential stresses therefore, the tangential stresses in the liquid has to go to 0 as you approach the surface. So, at a liquid gas interface you have 0, tangential stress boundary condition, there is at the surface itself the tangential stresses have go to 0, as you approach the surface from the liquid side. And the difference in normal stress, the surface itself can exert a normal stress and that is what is called surface tension, the difference between the the the normal force per unit area outside and inside is equal to the coefficient of surface tension times the sum of the principle curvatures. So, the pressure outside so minus the normal stress inside is equal to the surface tension force which is gamma times 1 by r 1 plus 1 by r 2 where 1 by r 1 and one by r 2 are the 2 principle curvatures at the surface.

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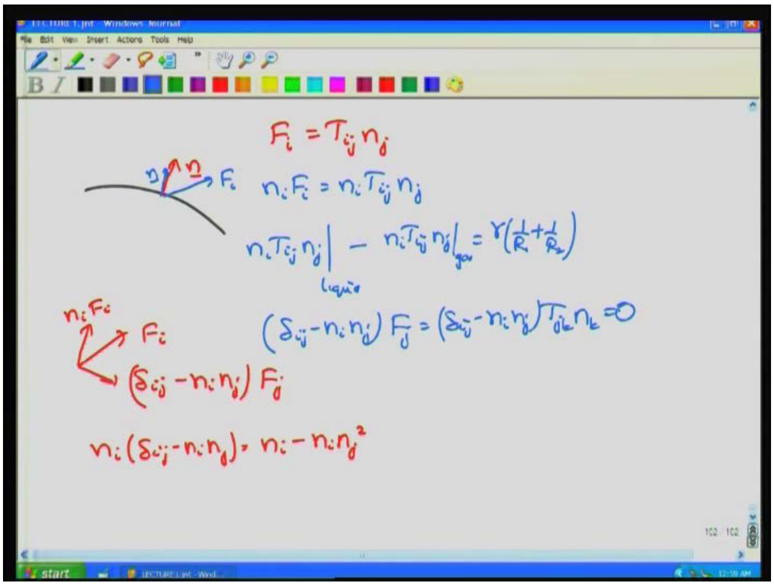

$$F_i = T_{ij} n_j$$

$$n_i F_i = n_i T_{ij} n_j$$

$$n_i T_{ij} n_j \Big|_{\text{liquid}} - n_i T_{ij} n_j \Big|_{\text{gas}} = \gamma \left(\frac{1}{R_1} + \frac{1}{R_2} \right)$$

$$(\delta_{ij} - n_i n_j) F_j = (\delta_{ij} - n_i n_j) T_{jk} n_k = 0$$

$$n_i (\delta_{ij} - n_i n_j) = n_i - n_i n_j^2$$

Let us make this a little bit more definite, I have the surface and I have the unit normal here, a component a force acting in the i direction at this surface force per unit area in the i direction is equal to T i j times n j, the force unit area acting at the surface the force itself were unit area is in some direction F i. The contribution to the force along the normal direction along the normal direction the contribution to the force is going to be equal to n i times F i is equal to n i T i j times n j.

So, the difference between the normal force on the two sides n i T i j n j in the liquid minus n i T i j n j in the gas, this has to be equal to gamma times 1 by r 1 plus 1 by r 2, where 1 by r 1 and 1 by r 2 are the radii in 2 orthogonal directions on the surface, so there is an some normal force balanced relation. What about the component of the force that is perpendicular to this, that is tangential to the surface, which is perpendicular to a normal.

So, if I have a force F i if i have this force fi the force fi, if I take the dot product of this with n i will get n i F i this is the force along the normal there is a remaining force, which is acting perpendicular to the normal. What is this force, this force the most general expression for this force is delta i j minus n i n j times F j, this is a vector could be in any direction along the plane. You can see that if I take this and dotted with n i will get 0, n i times delta i j minus n i n j gives me n i times n i n j square n i times is equal to n i minus n i n j square n j square is n dot n which has to be 0, because n is a unit vector therefore, this has no component along the unit normal it is always perpendicular to unit normal. So, the component of the stress in the liquid perpendicular to the unit normal or the tangential direction in the liquid phase has to be equal to 0, at a liquid gas interface therefore, this can be written as delta i j minus n i n j t F j is equal to delta i j minus n i n j T j k n k this has to be equal to 0.

A component of the force tangential to the surface in the liquid side has to be equal to 0, because the viscosity of the gas is very small, so this is how you would enforce the force the stress boundary conditions at a liquid gas interface. Next class we will continue with this we will briefly review this material the derivation of the equations, and then will go on to analyzing how to solve this equation in different limiting cases, convection dominated diffusion dominated as was the case in our previous fundaments of transport processes one.

Thank you.