Fundamentals of Transport Processes II
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Lecture - 11
Angular Momentum Conservation Equation

Welcome to lecture number eleven of the course on fundamentals of transport processes - 2. In the last lecture, we were just deriving the mass momentum conservation equations. We had defined the stress and we were just going through the angular momentum conservation equation.

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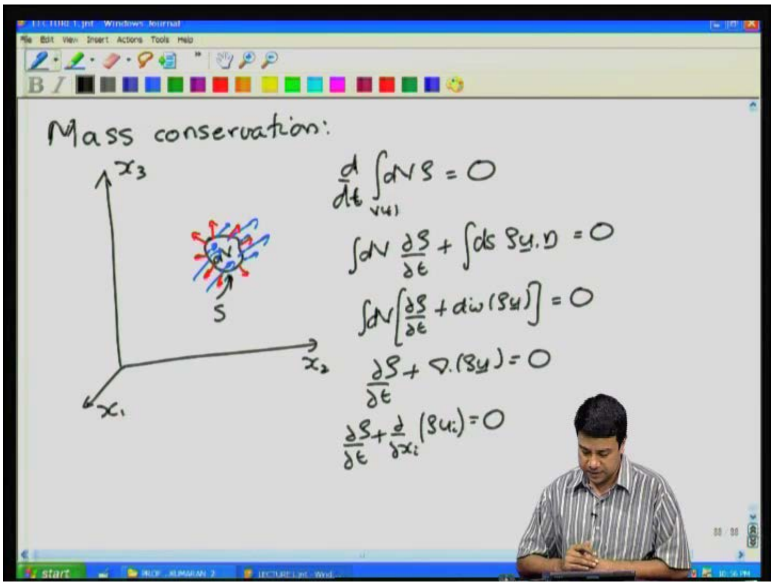

So, let us just briefly review what we did in the last lecture mass conservation start up with a differential volume in this coordinate system. This differential volume d v with the surface s differential volume is a material volume that is I have a fluid which is moving and there is a fluid velocity which is specified at each point within the fluid. So, I have a fluid velocity field and this volume is defined to be a material volume that is all points from the surface move with the same velocity as the fluid velocity at that point. So, if I have some fluid velocity field all points on the surface move with the same velocity as the fluid velocity at that particular point on the surface.

Consequence of that is that points that are inside this volume continue to remain inside this volume points that are outside continue to remain outside points that are on the surface continue to remain on the surface this is the point to where to go from inside to outside. That means, that it has to cross the point on the surface and if it crosses the point on the surface then the velocity is not single valued at that particular location. So, the requirement that the fluid velocity filed the single valued function means that points cannot cross the surface what is inside continuous to remain inside what is outside continuous to remain outside.

So, for this differential volume since material cannot cross the surface the mass conservation condition is simply that the total mass within this differential volume has to remain unchanged in time it should be independent of time. So, the mass with in this volume should be independent of time that is d v times rho is the mass present in the volume rate of change of that mass has to be equal to 0.

We transform using the Leibnitz rule if you recall by accounting for the fact that has this volume is moving certain regions are coming in because the volume is sweeping them in as it moves and certain regions are between left behind. If you take this two into account this can be transformed in to partial rho by partial t plus integral of the surface of rho times u dot n is equal to 0, where n is the outward unit normal to the surface local at every point. You have a outward unit normal that is defined at the surface that outward unit normal is n this can be simplified using the divergence theorem to give integral d b partial rho by partial t plus divergence of rho u is equal to 0.

Integral of surface of some vector dotted with n is equal to the integral of the entire volume of the divergence of that vector. In this case that vector is the whole thing the density times the velocity and this has to be true at each and every point within the volume which means that my mass conservation equation becomes partial rho by partial t plus del dot rho u is equal to 0. If I were using cartesian coordinates, I just write the simply as d rho by d t plus partial by partial x i of rho u i is equal to 0. However, if I use spherical or a cylindrical coordinate system I have a definition of divergence and we saw how to get that, so we just substitute that definition of divergence here.

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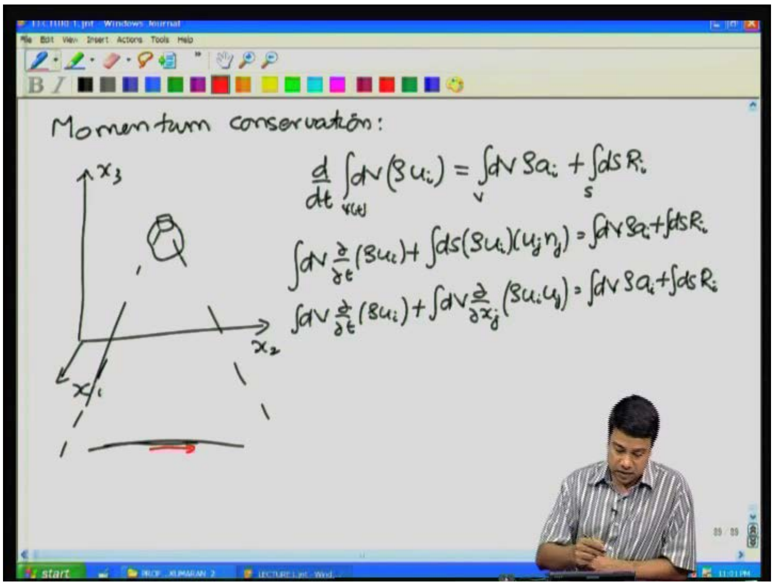

Momentum conservation rate of change of momentum is equal to some of a applied forces for this moving differential volume what is inside continuous remain same, what is outside continuous to remain outside. So, the rate of change of momentum of that material element is equal to sum of the forces that are excited on that material element the rate of change of momentum is d by d t integral over the volume. Volume itself is function of time rho times u i the density times the mean velocity integrated over the volume gives me the momentum d by d t of that is the rate of change of momentum that is equal to the sum of forces acting on this volume.

Two kinds of forces body forces and surface forces body forces are proportional to the total volume integral d v rho times a i, where a is the acceleration vector and rho is the density plus surface forces over the surface where r is a surface force. Examples of body forces gravitational forces centrifugal forces, all of these are proportional to the total volume. So, if the total mass is equal to that the total mass within the volume in rho times d v times the acceleration the acceleration vector is the gravitational acceleration vector for gravitational forces omega square by r in terms of in omega square r in the case of centrifugal forces and so on.

So, these are all proportional to the total volume of the fluid, so I call volumetric forces. The other types of forces are surface of forces these forces exerted at the surface by the fluid outside on the fluid inside these corresponded to the momentum diffusion process the transpose of momentum due to diffusion. These are the surface forces and obviously just as in the case of mass and heat diffusion the fluxes are related to the gradients in the concentration temperature field. In this case, this momentum flux across the surface momentum transfer per unit area per unit time will be related ultimately to the gradients in the velocity of the momentum fields.

Now, you can do the Leibnitz rule for the left hand side as usual and what you get is that integral d v partial by partial t of rho u i plus integral d so f u dot n times rho u i have already used index i for the vector momentum, note that this is a vector momentum equation contains three components. So, I get the vector momentum times u dot n u j n j is equal to integral d v of rho a i plus integral d s of the surface force, so that is the total momentum conservation equation for the second term. On the left you can use the divergence theorem once again to get integral d v partial by partial t of rho u i plus integral d v of d by d x j note that the unit vector n had index j.

So, the divergence has to be with respect to that same index of rho u i u j is equal to integral d b rho a I plus integral of the surface of the surface force. Now, last class I have done Cauchy's constructions for you in order to show you how R i can be expressed in terms of a stress tenser and the local unit normal basically R i is the force exerted if I expand out a small region over here. If I expand out the small region over here R i is the force exerted by the fluid outside on the fluid insider i is the force exerted by the fluid outside on the fluid inside this could of course, be in any direction.

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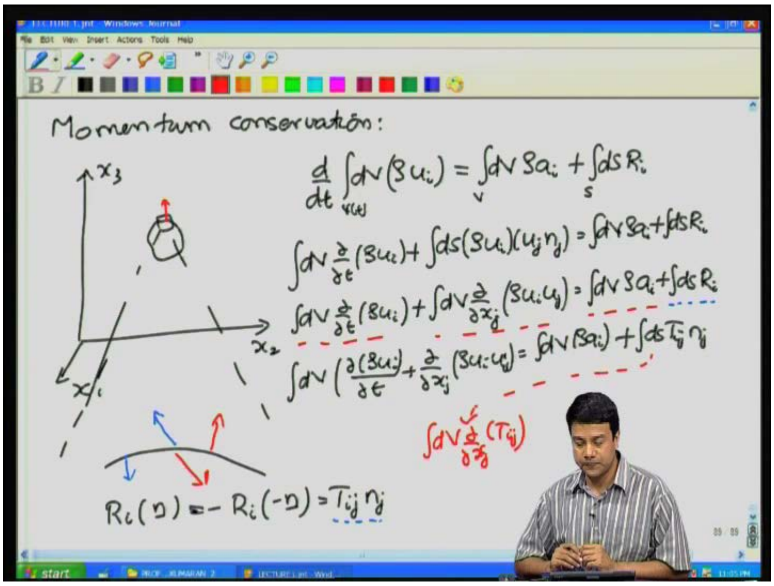

In the last class I shown you it is tangential force, but it does not has to be the force exerted by the fluid outside on the fluid inside last class I showed it is a tangential force it is not necessarily have to be. So, the force can be in any direction the force exerted by the fluid outside on the fluid inside. Now, for the fluid inside the outward unit normal is in this direction is going from the fluid inside to the fluid outside this would be outward unit normal is outward.

However, the force exerted by the fluid outside on the fluid inside has to be equal in opposite to the force exerted by the fluid inside. On the fluid outside Newton’s third law, so that be equal and opposite forces exerted on both fluids acting at the surface. Therefore, this fluid inside has to exert an equal and opposite force on the fluid outside for the fluid outside the unit normal is acting inverse for the fluid outside the unit normal is acting inverse.

Therefore, if I reverse the direction of the unit normal the direction of the force at the surface reverses what acts on the fluid inside is opposite of what acts on the fluid outside and going from outside to inside is just reversing the unit normal. Therefore, the surface force has to satisfy the relation the surface force acting at a surface with unit normal n is equal to minus the force acting at a surface with unit normal which is minus n.

This of course, suggests that the force can be expressed as the stress tensor times a unit normal because if the force can be expressed as the stress ten times a unit normal. Then reversing the direction of the force automatically reverse, I am sorry reversing the direction of the unit normal automatically reverses the direction of the force acting on the surface. Note that this is a force per unit area and t i j is the second order tensor call the stress tensor using Cauchy construction.

In the last class, I shown you that this is actually an exact relationship the reason is because in this equation if you look at all of these terms they are all volumetric terms. They all proportional to the volume in the limit as the volume goes to 0 these terms will be proportional to the volume of the length cubed. This last term here is proportional to the surface area and the only way that you can maintain a balance is if this last term can also be converted into the volumetric term in the limit has the volume goes to 0 and if this last term is can be converted into volumetric term then all terms are proportional to volume.

Therefore, you will get an equation that remains extensive as in the limit as the volume goes to 0 and it can be converted into volume only if I have any equation of this form because I have something dotted with n over a surface is equal to the divergence of that thing. Integrate over the volume that is that is the rational alpha this kind of a stress tensor.

So, with this kind of a stress tensor the equation, that I get integral d b partial of rho u i by partial t plus partial by partial by x j of rho u i u j plus the surface term integral d b d s d s t i j and j. This last term here i can also write it as a surface integral as d v partial by partial x j of t i j. So, I write that as a volume integral and once I have that then all terms are proportional to the volume this equality has to be satisfied any differential volume that you take. Therefore, it has to be satisfied at each and every point within the fluid.

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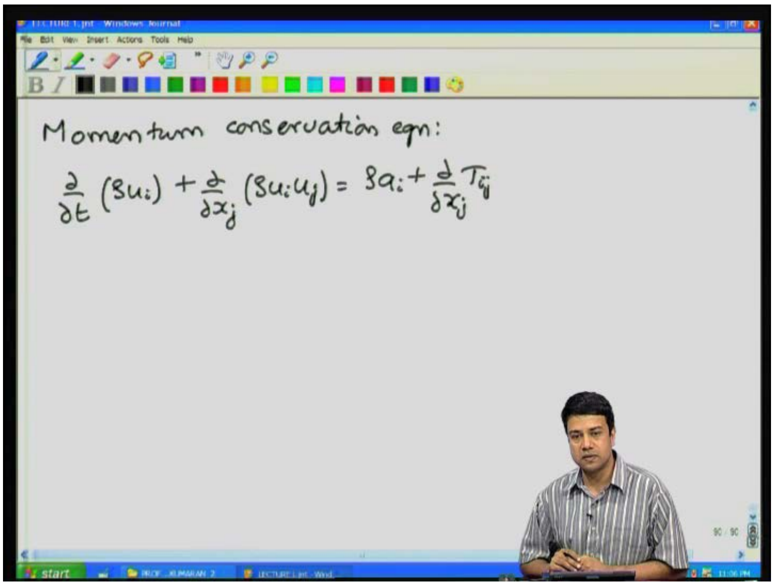

So, obviously if the equality satisfied at each and every point within the fluid. My momentum conservation equation becomes partial by partial t of rho u i plus d by d x j of rho u i u j is equal to rho a i plus partial by partial x j of t i j t i j is now the stress tensor force per unit area in the i direction acting at a surface force was unit normal is in the j direction.

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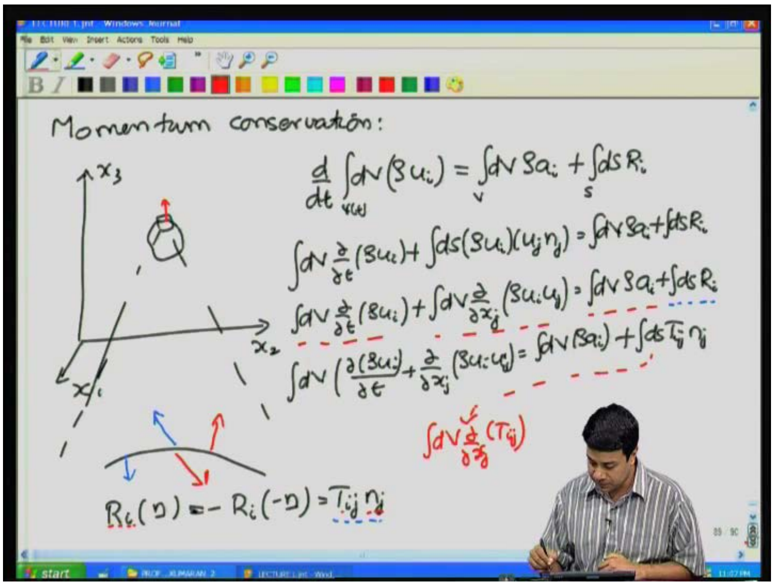

Go back to the definition the force direction is high; in this expression the force direction is high. So, the first index i gives me the force direction the second index gives me the direction of the unit normal here j. So, t i j is the force per unit area in the i direction acting at a surface force was unit normal is in the j direction.

The x force per area on the x direction acting at a surface was unit normal in the x direction d x y force per unit area on the x direction acting at a surface was unit normal is in the y direction outward unit normal important to keep in mind. There will be cases were you will see this written as minus t i j and j that is because in those textbooks they have defined the force with respect to the invert unit normal here we have used the outward unit normal.

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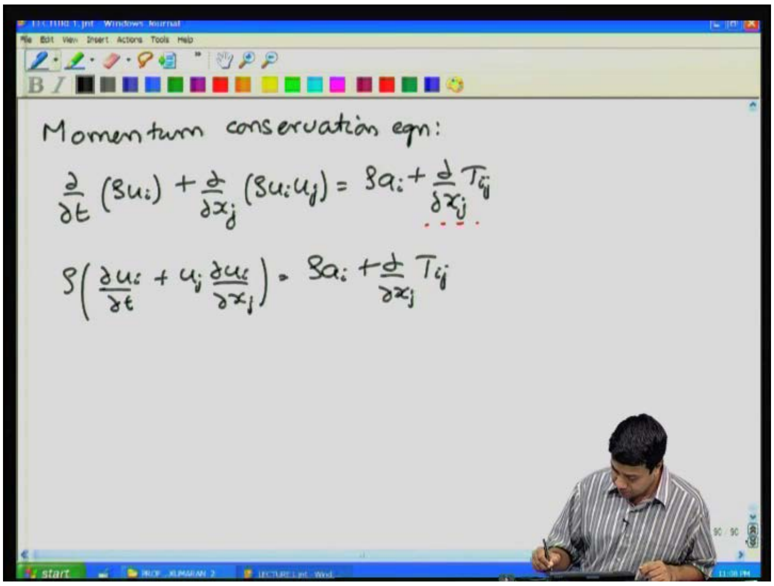

In cases where you see this as a negative sign in front of this last term here that is because they have defined the force with to respect to the invert unit normal. Since, the divergence theorem is related to the outward unit normal will use this notation for without loss of generality.

If it is defined with respect to the invert unit normal the stress term here will change sign, but also the constitute of relation the equation for the stress in terms to the strain rate there will be one sign change there as well in such way that the final equations the mean unchanged. I had also shown you that this can be written by combining this with the mass conservation equation, this can be written as rho times partial u i by partial t plus u j partial u i by partial x j is equal to rho a i plus d by d x j of t i j.

So, that was my momentum conservation equation. Next, we had gone through and almost finished the angular momentum conservation equation I will go through that once again because if you are seeing it for the first time it is a little difficult to understand, but gives us an important lesson about the nature of the stress tensor. So, that is why I will go through that in some detail.

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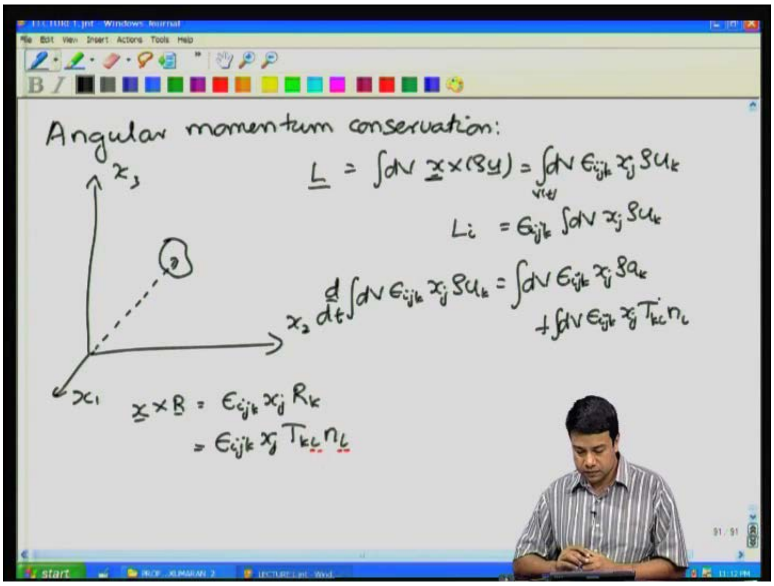

The angular momentum with respect to any origin in this case we can take the origin of the coordinate system without loss of generality, but the angle of momentum equation can be defined with respect to any origin, but that origin has to remain the same as this volume is moving around. So, the angular momentum I will define, it is a vector is equal to integral d v of x cross rho u where x is the displacement vector from the origin in this case this I should write this as yeah a vector in indicial notation. I will write this is integral d v of epsilon i j k x j rho u k and epsilon is the anti-symmetric tensor is the same everywhere in space. So, it does not depend up on volume, so in general I can take the epsilon out of this volumetric along this.

I can write this as the angular momentum with index i is equal to epsilon i j k integral d v of x j rho applying the Leibnitz rule for this rate of change of angular momentum within this differential volume is equal to the some of the applied torques. That is the Leibnitz rule for this, I am sorry that is the angular momentum conservation equation for this particular case. Therefore, the angular momentum conservation equation statement is going to bed by d t integral d v epsilon i j k x j rho u k is equal to the sum of the torques acting on this volume torques are due to two kinds of forces. One is the volumetric forces the body forces the other is the surface forces.

So, the body forces are of the form d v epsilon i j k x j rho times the acceleration plus a surface component which is equal to integral d v epsilon i j k x j times r k r was the surface force that is r. If as x cross r vector x cross r vector can be written as epsilon i j k x j times r k R itself is equal to t dot n R itself is equal to t dot n where t is the stress tensor.

So, this I can write it as epsilon i j k x j t k l n l, note that this for this dot product R R is equal to t dot n. I have to use a different index because there are many indices that I have already appeared two times and once I have to use a separate index. In this case it will be this index l which appears two times to denote this dot product so this is equal epsilon i j k x j t k l n l so that is the statement of the angular momentum conservation equation.

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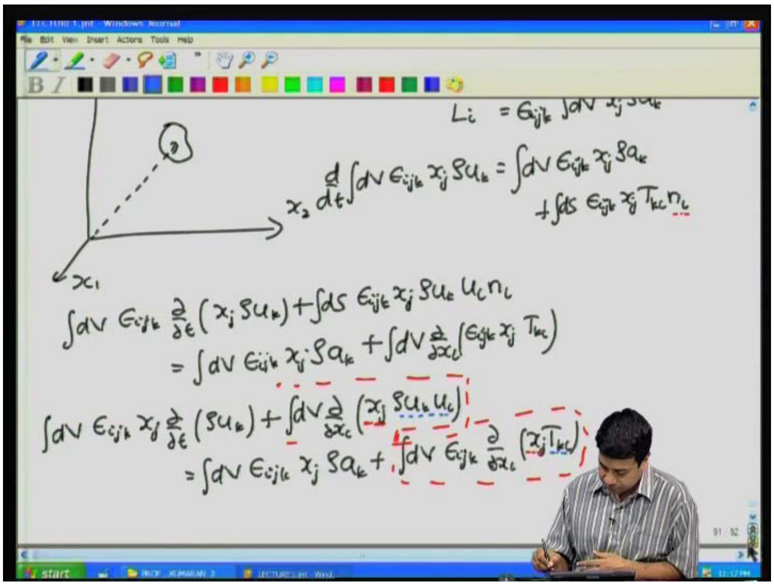

As usual, we use Leibnitz rule to simplify this, so I get integral d v epsilon i j k partial by partial t of x j rho u k plus integral the surface d s of epsilon i j k x j rho u k integral d s of this whole thing times u dot n partial by partial t of this whole thing that is epsilon i j k x j rho u k plus d s of this whole thing times u dot n. So, that for that other u dot n, I have to use a separate index now, so I will use u l n l we have to use separate index for that other u dot n. And this right hand side can also be simplified epsilon i j k x j rho u k plus integral, sorry this should be a surface this should be a surface integral and this surface integral also I have to convert into a volume integral.

So, that convert that into volume integral partial by partial x l the divergence has the same index as the unit normal in the surface integral the divergence has the same index as the unit normal in the surface integral times epsilon i j k x j t k l. So, that is the entire statement of this angular momentum conservation equation, so we can simplify this epsilon is independent of time and space x is independent of time. So, for the first term I can simplify it as integral d v epsilon i j k x j partial by partial t of rho u k plus the second term is a divergence of convert the surface integral to a volume integral using the divergence theorem.

In this case the unit normal has index n and therefore, my divergence also should have index n so I should get partial by partial x l of x j rho u k u l. So, this will end up being equal to integral d v epsilon i j k x j rho a k plus integral d v partial by partial x l this term to be taken out x j times t k i call, so that is the final step. Now, in these two I am going to use the chain rule for differentiation in these two terms in these two terms I will use the chain rule for differentiation to simplify the terms in particular. I will I will expand the derivative into two parts one of which is the derivative of this one this x j here and this x j here the second is the derivative of what is remaining.

The second is the derivative of what remains and all of these involve partial by partial x l of x j partial by partial x l of x j that is equal to either 1 or 0. If j and l are equal that is equal to 1 and if j and l are not equal its equal to 0. So, this is are familiar identity tensor delta j l partial x j by partial x l is 1 f j is equal to l 0 j is not equal to l therefore, it is equal to be identity tensor delta j l.

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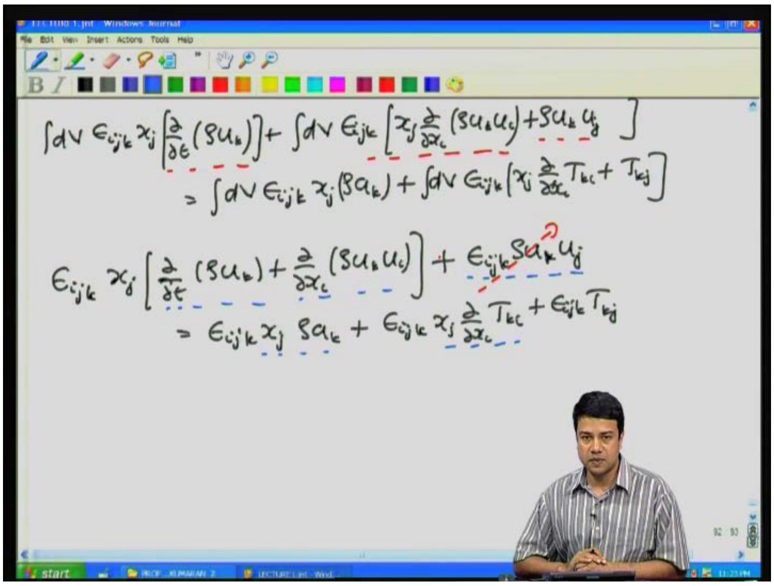

So, let us see expand and simplify integral d v epsilon i j k x j partial by partial t of rho u k plus integral d v there should be an epsilon i j k here, so we must put that in, so this symmetric tensor is in there. So, I have epsilon i j k in to first is x j partial by partial x l of rho u k u l that is the first term plus the second term which can takes the derivative of x l with respect to x j that is just equal to delta j l.

So, I get plus rho u k u l delta j l on the right hand side I have integral d v epsilon i j k x j rho a k plus integral d v once again I do in differentiation using the chain rule in terms of those two terms x j and t k l. The first one will just be x j partial by partial x l of t k l plus t k l delta l j because partial by partial x l of x j is equal to delta l j times t k l.

Now, this also has to be true for each and every differential volume that that means that this expression has to be satisfied at each point within the flow. So, this also has to be true at each and every differential volume this has to be satisfied at each and every point within the flow the brief simplification here delta j l times u k u l is nonzero only when j is equal to l. Therefore, delta j l times u k u l can be replaced by u k u j because delta j l is non zero only when j is equal l. So, in the expression instead of u l i can replace it by u j and remove the delta, so I just replace u l by j and remove the delta function.

Similarly, t k l times delta l j replace l by j and remove that delta function and I get t k j good, now this is true for each and every differential volume that means that this must has to be true at each and every point within the flow rate. So, if I just write down the since if this volume integral which is true for every volume that means, that is true at every point within the fluid. Therefore, the integrand itself has to be if I remove the volume integral the integrand itself in the equation has to be satisfied at each every point within the fluid. So, collecting terms first one is I will get epsilon i j kx j into partial by partial t of rho u k plus partial by partial x l of rho u k u l.

That is these two first terms here, this one and this one and then I have this last term here, so that becomes just plus epsilon i j k rho u k u j keep as a last term here. On the right hand side, I have epsilon i j k x j rho a k plus epsilon i j k x j partial by partial x l of t k l plus epsilon i j k t k j after removing the volume integrals and I should have an epsilon in this term as well. So, that we put the epsilon in plus t k j good if you look at this term here epsilon i j k is anti-symmetric in the indices the j and k. If I reverse j and k I get the negative of the value that it originally was u k times u j is symmetric, because the one two component which is u 1 u 2 is the same as the two one component which is u 2 u 1.

If I interchange j and k, I get exactly the same result. So, I have this double dot product of an anti-symmetric tensor epsilon with a symmetric tensor rho u k u j and as we had discussed earlier the double dot product of a symmetric and anti-symmetric tensor that double dot product is equal to 0. So, this term now is equal to 0because it is the double dot product of an anti-symmetric tensor that is the anti-symmetric tensor in j and k and the symmetric tensor which is u k times u j. You should look at all of these other terms all the other terms there are this four terms which are identical to the terms which were present in my momentum conservation equation.

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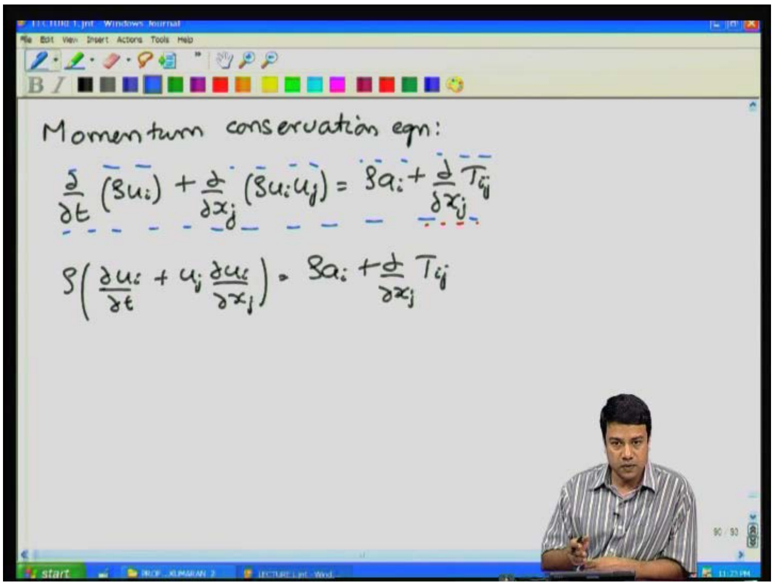

First four terms are identical to what was present if you recall in this momentum conservation equation I have epsilon i j k x j times rho u k partial, partial t of rho u k plus partial by partial x j of rho u k u j plus rho a k plus the divergence of this stress. These four terms are equal to each other the left hand side and the right hand sides are equal, so which means that these four terms on the left and the right hand sides.

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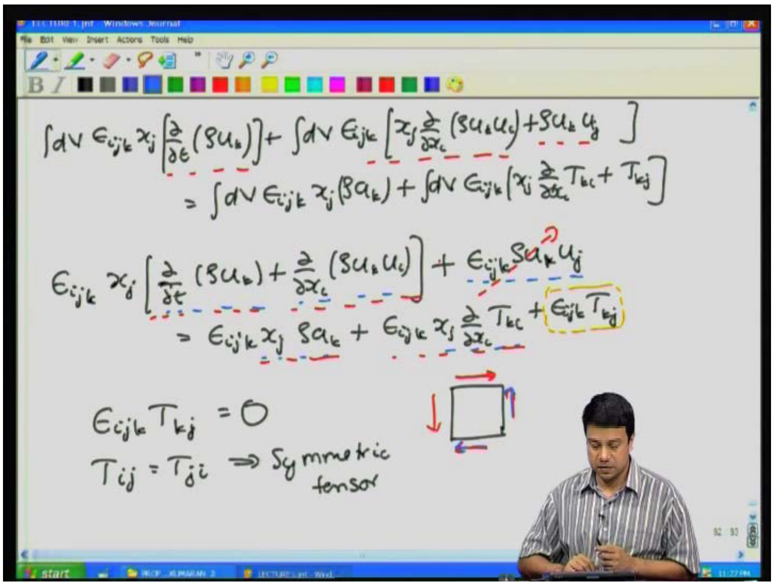

One two three and four have to be equal left and right hand side with just these four terms has to be equal what that means is that this last term here is identically equal to 0. If the momentum conservative equation is to be satisfied because the momentum conservative equation that I just taught you here is just equal to epsilon i j k x j times the mass conservative equation plus this additional term in the mass conservation equation itself adds up to zero. Therefore, this additional term also has to equal to 0. So, I require that epsilon i j k t k j is equal to 0 for angular momentum to be concerned and that has an important application epsilon is an anti-symmetric tensor that double dotted with t has to be equal to 0.

That means that t is a symmetric tensor the stress tensor is a symmetric tensor in order to satisfy angular momentum conservation at least that t e i j equal to t j i it is a symmetric tensor this is always true. These are always a symmetric tensor unless for some reason there are some microscopic torques that are acting on the system. In most general cases in most fluids there is no microscopic torque. Therefore, t will always be a symmetric tensor there are some fluids which are for example, magnetoriological fluids where under application of simultaneous electrical and magnetic fields.

You may actually generate microscopic torque in which case it would not be a symmetric tensor, but those cases are rare in most of the cases, where we do not have microscopic torques acting in the fluid t will always be a symmetric tensor the stress tensor is always symmetric. That is the reason that in text books, when you see this differential volume being drawn. The force balance on this differential volume is usually drawn this way why I said drawn this way because in this manner the torque is actually, I am sorry the stress is actually symmetric.

Of course, the stress on top and below the stress on top here and the stress below here have to be in opposite directions, because on you change the direction of the unit novel the direction of the force changes. However, I also have this additional torques over here, there is one here and there is one in the opposite direction here and is drawn this way. As you see there is no rotational component here in this particular case when t x y is equal to t y x there is no rotational component. Let us discuss that in some in little further detail.

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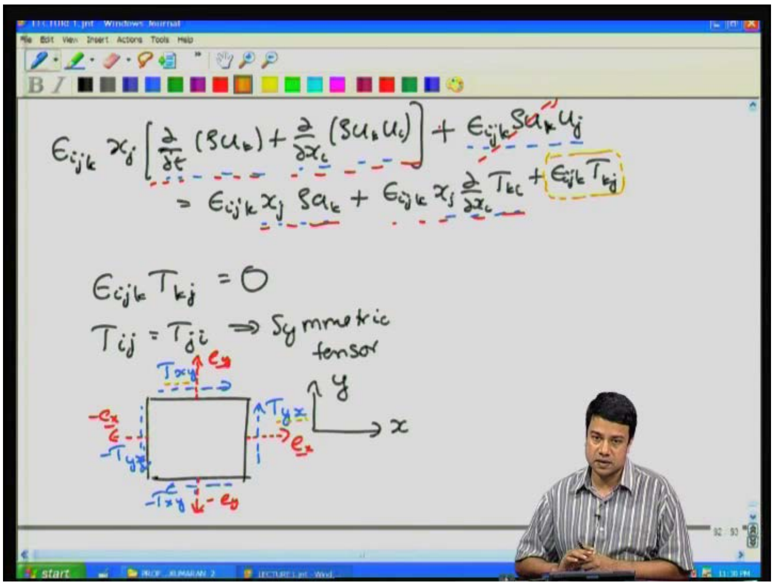

Let us take a differential volume the coordinate system x and y we usually take x 1 and x 2 for simplicity. In this case I will take x and y to make it to connect up to something that you are familiar with now on the right surface the force per area in the y direction acting at a surface whose unit normal is in the x direction the outward unit normal is in the x direction, e x that means that the force between unit area in the y direction is t x y. On the left hand side the unit normal is in the minus e x direction and of course, the stress when you reverse the unit normal the stress gets reversed. So, this becomes minus t x y how about the top and bottom at the top surface the unit normal is in the plus e y direction.

The top surface the unit normal is in the plus e y direction the force per unit area acting at this surface in the x direction is the shear component of the stress this is equal to t, I made a mistake here the unit normal direction is x and the force direction is y. So, this should be t y x and this is minus t y x this is t x y and this is the unit normal is in the minus e y direction and this is once again minus t x y and the symmetry. The rate of deformation tensor that means that this t x y is equal to t y x.

It is symmetric only if t x y is equal to t y x implication is that there no net torque on this differential volume because you can see that that the forces on the top and bottom tend to rotate this in the clockwise direction, whereas the forces in the right and left tend to rotate this on the anti clockwise direction. If the forces are equal there will be no net torque on this differential volume that is the important implication of this stress tensor being symmetric.

That is why typically in textbooks you will see the stresses the shear stressed written in this fashion without any rotation and that is because from angular momentum conservation there is no net torque on the system there is no microscopic torque on the system. The net torque on any differential volume has to be exactly balanced, otherwise you will violate angular momentum conservation and this net torque is balanced, if the stress tensor is symmetric.

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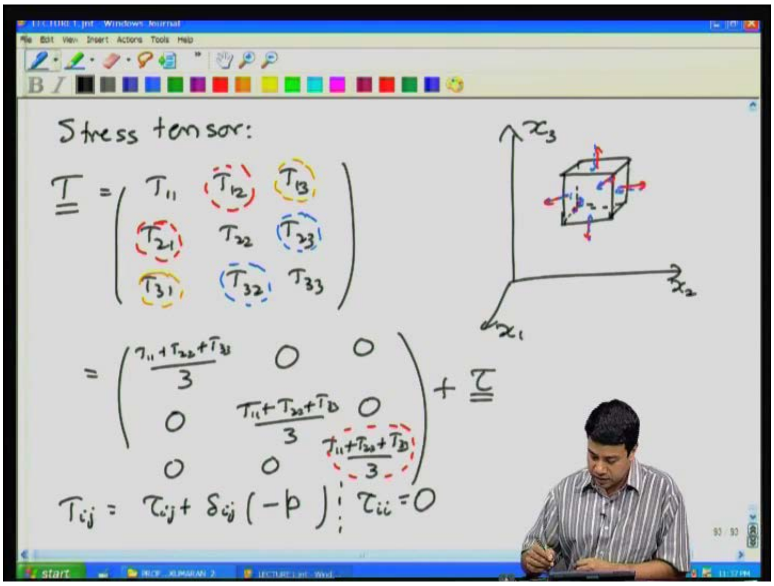

So, now let us look at the stress tensor the second order stress tensor t x x t x y i t s symmetric. Therefore, these two are equal these two are equal and these two are equal six independent components symmetric stress tensor and we did the rate of deformation tensor. We said that the rate of deformation tensor can in general be expressed, you can any matrix can be decomposed into a symmetric and anti-symmetric. The symmetric further can be decomposed into an isotropic and a symmetric trace less tensor the symmetric trace less tensor the trace that is the sum of the diagonals is equal to 0.

In this particular case the stress tensor is already symmetric. Therefore, it can be decomposed into two parts, one is an isotropic tensor and the other is the symmetric traceless tensor. So, this I will write it as an isotropic tensor which is t 1 1 plus t 2 2 plus t 3 3 by 3 plus a second matrix which I will call the symmetric trace less part, I will call it as tau which be the shear stress. So, two parts one is the isotropic part and the other is the symmetric trace less part tau physically, what do these mean?

If I have a differential volume here the isotropic part is the some of the diagonal elements t 1 1 is the force per unit area in the x 1 direction acting at a surface which unit normal is in the x 1 direction t 2 2 is the force unit area in the x 2 direction acting at a surface which unit normal is in the x 2 direction. Similarly, for t 3 3 that means that these represents components of the force which acts in the same direction as the local unit normal.

So, these are forces that acting along the same direction as the unit normal is front back up down and left right on each of these phases the forces are acting in the same directions as the unit normal and tau is the component of the force which does not have on average. There is no net force acting along the direction of the unit normal’s, if you add at the sum of the force is acting along the direction of the unit normal’s in tau you will get basically 0. In our indicial notation I can write this as t i j is equal to tau i j plus an isotropic part delta i j times 1 by 3 t k k t k k repeated index that means there is a summation no unit vector, so t k k is t 1 1 plus t 2 2 plus t 3 3.

Therefore, one-third t k k is what I have here t 1 1 plus t 2 2 plus t 3 3 divided by 3 that times the identity tensor is this isotropic part and the remaining is the symmetric trace less part the symmetric trace. Less part has the property that it is the trace which is this is equal to 0that the sum of the diagonals of the symmetric trace less part is equal to 0, what do this physically mean. If a fluid was just static for fluid there was no flow within the fluid just imagine static volume of fluid in this fluid there is force acting on the surfaces on all surfaces. There is a force that force is equal in all directions and it is called the pressure.

So, at a static head of fluid and if I go some were within that fluid and I put in a small differential volume and I measure the forces acting on all surfaces of the differential volume all those forces will be equal they will be acting inwards. They will be equal to the local pressure times the surface area as you know the hydrostatic pressure is equal and tracks equally in all directions at each point. If you have a surface to measure that hydrostatic pressure, the force always acts perpendicular to that surface. So, the pressure is defined as a force that it acts inward at each location where as we have defined our forces as acting outward.

Therefore, this one-third t k k is just equal to the negative of the pressure because the pressure at each surface acts inward we have defined the stress with respect to the outward unit normal. So, this just in this one-third of t k k in a static fluid is just equal to the pressure with a negative sign because the pressure is defined to be positive when it acts inward where as I defined my stress tensor to be positive when it acts output. So, this pressure is present even in a fluid which is not moving it is a static component of that stress when the fluid moves in general, if the fluid is not moving.

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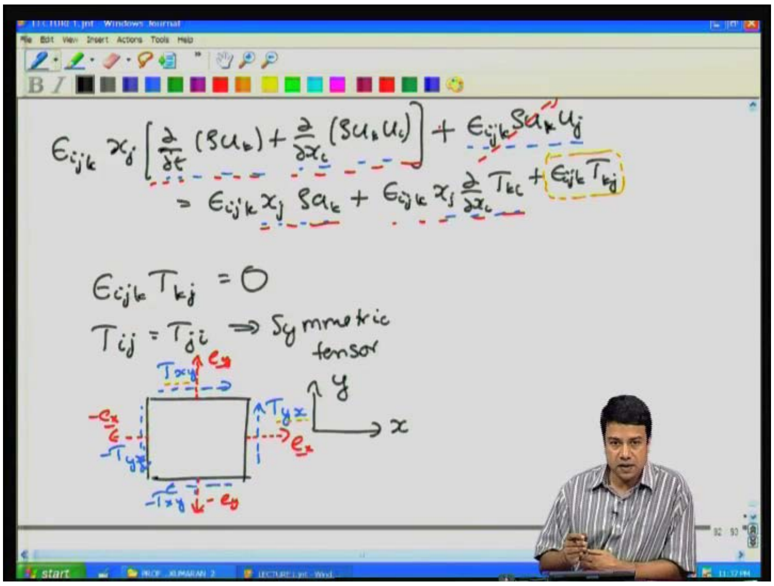

When the fluid is not moving of course, one cannot get stresses that act tangential to surfaces because its only when there is fluid flow in this momentum transport across the surface can I get forces that are tangential to surfaces. So, only when there is fluid flow will I get these shear stresses acting at surfaces in the absence of fluid flow, I just have a normal pressure acting at each surface that pressure always action inwards.

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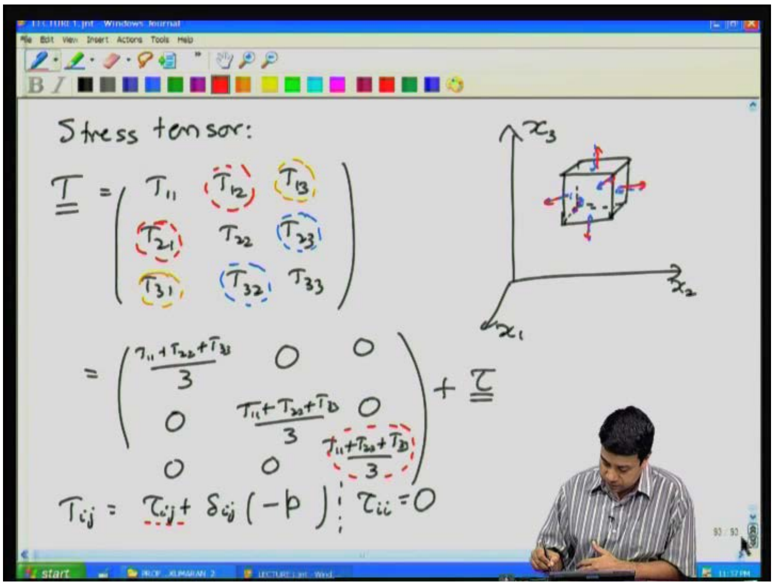

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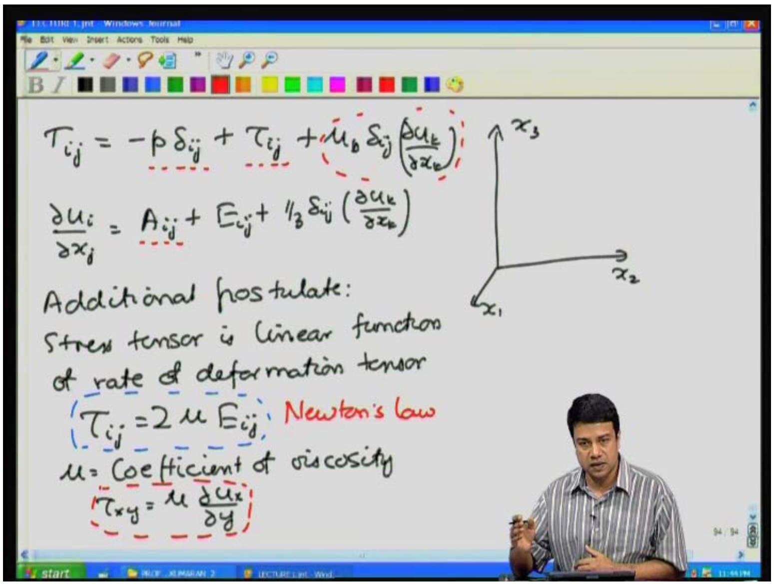

So, the second component that I had here is the component of the stress due to fluid flow. So, to reiterated i j the stress tensor can be written as minus p times delta i j where p is the pressure isotropic part of the rate of deformation tensor plus the second part tau i j. The second part of course, depends up on the rate deformation within the fluid. Therefore, a constitutive relation will relate this tau i j to the gradients in the velocity field the constitutive relation will relate this to the gradient to the velocity field. In a similar manner to the flux mass flux will related to gradients in concentration field heat flux may related to gradients in temperature field.

Secondly, this tensor is already symmetric this tensor is already symmetric the stress tensor as well as the symmetric trace less part are still symmetrical. This is anti-symmetric component the rate of deformation tens or had components symmetric anti-symmetric, I am sorry symmetric trace less anti-symmetric and isotropic. So, partial u i by partial x j has three components which was a i j plus e i j plus one-third delta i j times the divergence of velocity. There, were three components and as you can see each of these represented in specific forms of deformation each of these transform differently under rotation or translation of the coordinates.

So, each of these represents a specific form of deformation anti-symmetric part represents one particular form of deformation that is the rotational deformation. As I said in the rotational deformation, there is no change in the relative distances between material points when you have a solid body rotation about an axis there is no change in the distance between material points and because of that there is no deformation. So, stresses cannot depend up on this anti-symmetric part the stress could in general depend up on the symmetric trace less and the isotropic part it will generally depend up on the symmetric trace less and the isotropic part this tensor tau i j symmetric trace less.

If we make the additional postulate additional postulate at the stress tensor is linear function of rate of deformation tensor it is a linear function of the rate of deformation tensor then tau i j cannot depend up on anti-symmetric part. It cannot depend up on the isotropic part because the isotropic part is isotropic where this tau i j has to be trace less. Therefore, there is only one way that can relate this stress to the rate of deformation tensor if it is linear.

That is it is proportional to that tensor itself not to its higher order products that is e i j times, I am sorry I could create higher order products which as products such as product such as e i k times e k j. For example, that would be non linear in the rate of deformation tensor if I postulate that this stress has to be linear in the rate of deformation tensor then the only way I can get that is for the symmetric trace less part. I will have tau i j is equal to some constant times e i j because both the left and the right hand side are symmetric trace less tensors.

I cannot have it, proportional to the isotropic part because the trace of that is nonzero this coefficient is usually written as two times mu where mu is equal to coefficient of this viscosity. This coefficient of viscosity recall relates the stress to the rate of deformation and this is the equivalent of the Newton's of viscosity of viscosity written in tensor form you are usually used to writing this. As for example, tau x y is equal to mu times partial u x by partial y if I have an x and y coordinate system orthogonal coordinate system this form of the constitutive relation is not frame invariant in the sense that.

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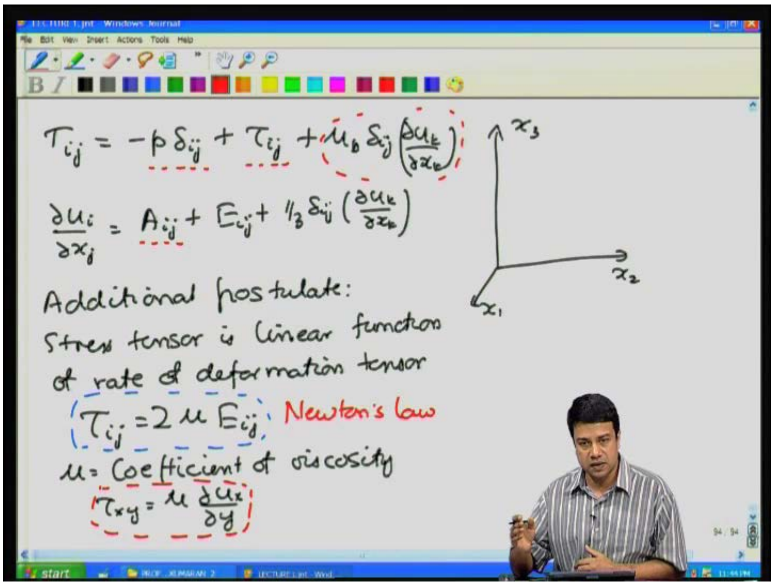

If I change rotate coordinate reference frames this will change this equation will change. It is its its specific to a particular coordinate system where the flow is only in the x direction the velocity gradient is only in the y direction where as this tensor relationship that I have here is coordinate frame invariant it works in any coordinate system that you choose to use. So, this is the Newton's law the constitutor relation for the stress tensor relation between the momentum fluxes in the gradient in the velocity field. I could also have an addition an isotropic contribution due to deformation I could also have in addition to the pressure, I could have an isotropic contribution to the stress tensor due to the isotropic deformation.

That is often written in terms of what is called the bulk viscosity this additional term apart from the fluid pressure here apart from the fluid pressure an additional term which is related to the divergence of the velocity. That is plus a bulk viscosity times the divergence of the partial this is an isotropic additional contribution due to deformation. An additional contribution to the viscosity due to the bulk radial expansion of compression. So, these two constitute the constitutive relations for what is called a Newtonian fluid.

There is the stress rate of deformation relationship the stress the isotropic the symmetric trace less part of the stress is related to the symmetric trace less part of the rate of deformation tensor through the viscosity coefficient mu. You could in general have an isotropic contribution to the stress due to radial expansion of compression and this pressure term here. This is the usual pressure term which in the case of normal fluids will be given by some kind of an equation of state in normal fluids this will be given by an equation of state.

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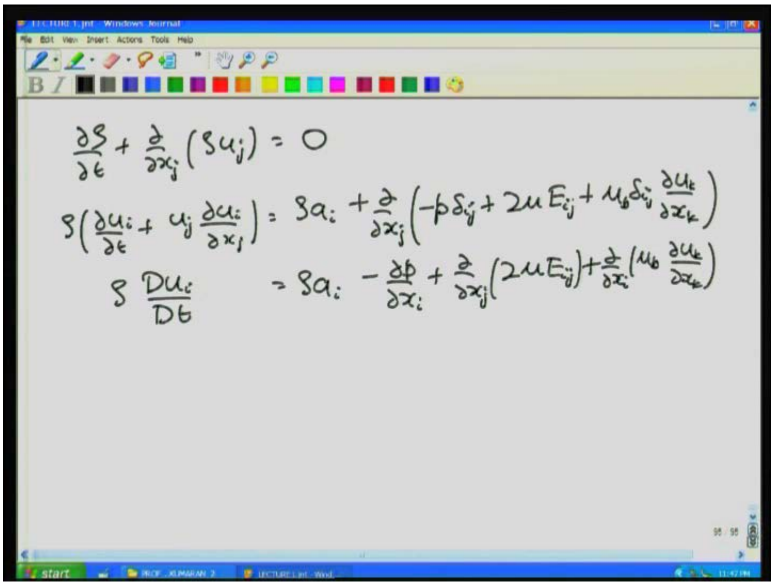

So, combining these mass and momentum conservation equations partial rho by partial t plus partial by partial x j of rho u j is equal to 0. That is the mass conservation equation and partial by is equal rho a i plus partial by partial x j of the stress which we are separated into three components minus p delta i j the pressure. The static pressure, which is present even when there is no flow plus 2 mu times e i j plus, in general the bulk viscosity component plus in general bulk viscosity component. You can simplify this a little bit this is just equal to rho a i minus partial p by partial x i plus partial by partial x j of 2 mu e i j plus partial by partial xi of mu b partial u k by partial x k.

The left hand side as you will readily recognize is just the substantial derivative of the velocity field the left hand side is just the rho times d u i by d t. So, these are these are the conservations equations for a simple Newtonian fluid one could of course, have more complicated forms of the constitutive relation. In the case of non Newtonian fluids in that case the stress is not necessarily a linear function of the rate of deformation tensor it could depend in some more complicated way on the rate of deformation tensor.

However, no matter how the relations are framed one has to ensure that they satisfy the frame invariance. That is if you translate or rotate coordinate systems the resulting relationship should not depend up on the specific choice of coordinate systems basically the difference between the Newton's laws that I have written for you in tensor form here.

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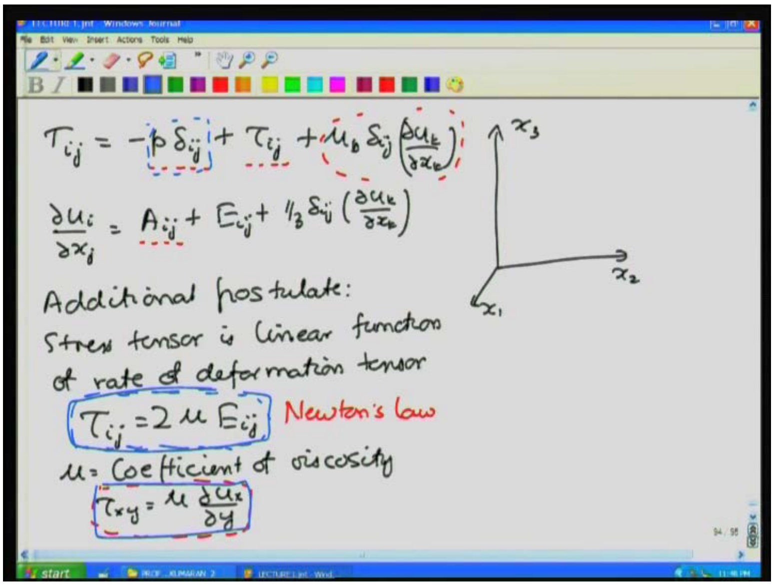

I have written for you in tensor form here, so this one and the one that you would supplemently write for a simple one dimensional flow. So, the way I have written the Newton's law in tensor form is frame invariant and similarly, it has to be frame invariant no matter how one formulates an equation for a non Newtonian fluid. So, we will briefly see how to formulate equations for non Newtonian fluids in the next lecture. Then we will progress to completely specify the equations of motion for a Newtonian fluid itself in the limit in the incompressible limit where the density is not changing with position.

The rest of the lectures will be restricted only to the incompressible limit the incompressible limit is valid in most cases where the flow is not very fast where the velocity is small compare to the speed of sound. In most applications that we deal with this condition is actually satisfied the velocity is actual small compared to the speed of sound.

Therefore, you can always use an incompressible equation to describe the flow. So, we will briefly look at non Newtonian constitutive relations in the next lecture and then go on to defining our equations and defining what are the boundary conditions that we need to satisfy these equations so will continue that in next lecture, we will see you then.