Fundamentals of Transport Processes II
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Lecture - 10
Momentum Conservation Equation

Welcome to this lecture number 10 of our course on fundamentals of transport processes, where we had just completed the derivation of the mass conservation equation and we were about to start the derivation of the momentum conservation equation. So, we had used vector identities and tensor identities. In order to derive that mass conservation equation in a simple manner, without reference to the underlined coordinate system that is used to analyze the problem.

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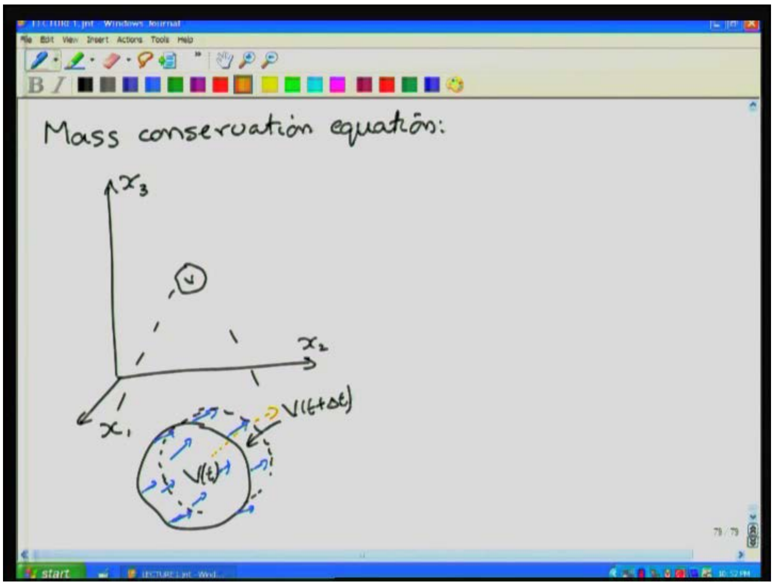

So, in that sense the derivation that we did treated vectors and tensors as objects in themselves, without reference to a coordinate system. Let us just briefly review that first before we go to the momentum conservation equation. So, the idea of the derivation of the mass conservation equation was as follows, if I have some volume v let me expand that out some volume v this is within the fluid. So, there is a velocity field that is defined at each point within the volume and outside. Of course, it is a velocity field defined at each point within the volume as a velocity field defined at each point on the surface.

So, this is a fluid velocity field that is defined at each point it may vary with position, but there is a well-defined fluid velocity field. Now, if I consider the surface of this volume to be moving at the same velocity as the fluid at that surface. The fluid velocity at that surface is equal to the velocity with, which the surface moves. So, in that sense this volume is now a function of time, it moves with time along with the fluid the Lagrangian reference frame, if you work moving with the mean velocity of the fluid. So, at some later time, this volume will be at some other location depending up on the fluid velocity on the surface, this volume will be at some other location.

So, at the initial time say t the volume was at this location at some later time t plus delta t it has shifted. Whatever, fluid material points fluid molecules say whatever fluid material points that are moving at that same velocity as the mean fluid velocity. Those that were inside the volume will continue to be inside the volume, those that were on the surface will continue to be on the surface and those that were outside will continue to be outside. That is because if we consider, if a if a material element want to move from inside to outside this volume a material element want to move from inside to outside this volume. It would have to cross one of the material elements on that surface, in order to move outside, but obviously if it goes from inside to outside it has to cross the surface. If it crosses the surface it implies that the velocity has two values at that particular position at that time because the one from inside is moving outside. So, it is moving faster than the surface material element.

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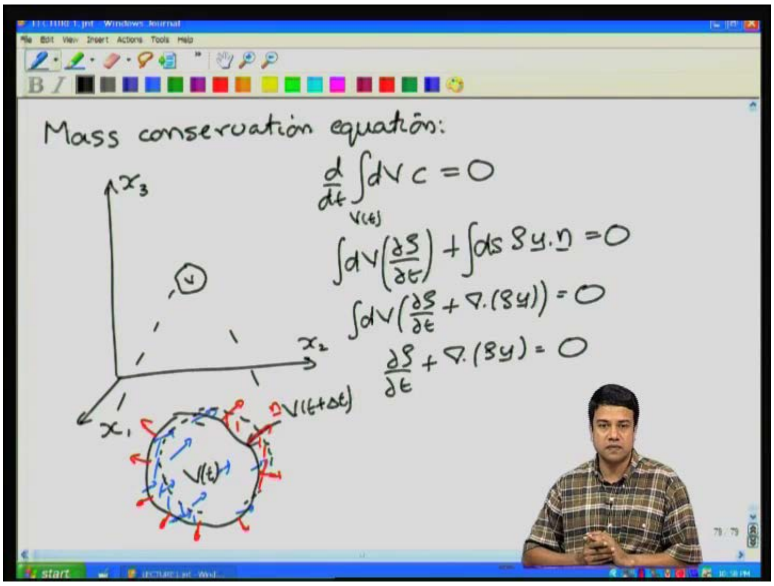

Therefore, it has to cross the surface that means that the velocity has to be two valued at that location. However, the velocity vector as we said is a function of position and time it has only a single value at each position and at each time. Therefore, material elements cannot cross from inside to outside, so what remains inside continues to remain inside what remains outside continues to remain outside and what is on the surface continues to remain on the surface? Therefore, if I want to write a mass conservation equation since, what was inside continues to remain inside that means that, the mass within this volume as it is moving is independent of time.

The mass within this volume is d by d t of the integral of volume s is the density where this volume is now the function of time. The rate of change of mass within the volume is equal to 0 or the mass within the volume is a constant, so that is the mass conservation condition. Now, this was written for a moving volume element where as we always define our velocity density etcetera with respect to a fixed coordinated reference. Therefore, we have to translate what is in this moving element into something that is fixed in space. So, we have a moving element and there is some change the change in the mass within that element has to be equal to 0 as this moves there are volumes will in front, which it occupies after this time which enter this differential volume. There are once at the back, which leave, so the change in mass within this volume is called to be both due to what has entered as well as what has left as well as due to the change in density within this volume itself. The density within this volume is changing in time it is a function of position and time.

So, because of that there will be a change in mass in addition some mass that is ahead is coming in some mass that is behind is left behind because the volume has vacated that position. Therefore, that sum of what comes in what has been left behind plus the change in the density within this volume itself due to their I am sorry, the change in mass within this volume itself due to change in density the sum of all of those three has to be equal to 0.

The change in mass due to the change in density within the volume itself is of course, just the integral of the volume of partial rho by partial that is the rate of change of mass within this volume due to the change in density itself. The sum of the total mass that comes in and leaves because this volume is moving, if I defined the outward unit normal to the surface at each location as n I defined the outward unit normal at each location as

n. Then the distance moved by a patch on the surface is going to be equal to the velocity dotted with n u dot n is called to be, the distance moved by a patch of the on the surface. Therefore, the volume that comes in is going to be equal to d s the patch surface area times u dot n, which increases the volume if u dot n is positive. So, that the velocity vector is in the same direction as the outward unit normal then there is an increase within this volume.

If it is negative the patch is the surface has left behind the volume and therefore, the change in volume is negative however u dot n also is negative. So, that automatically it takes care of just saying is it as u dot n automatically tells you whether it has come in or left. So, on that basis we had shown that this second part due to the volume that is coming in and the volume that is leaving is integral d s rho times u dot n that has to be equal to 0. Mass conservation equation use the divergence theorem to write this as integral d v of partial rho by partial t plus the divergence of rho u is equal to 0 simple use of divergence theorem. Once, that is done we know that this even though it is a volume integral over a small volume this has to be satisfied for each and every volume. In fact, even if I shrink this volume to a point limit as delta v goes to 0 this condition is still satisfied that means that it has to be satisfied at each point during the fluid.

So, the mass conservation equation becomes partial rho by partial t plus del dot rho u is equal to 0 that is the fluid mass conservation equation. Briefly, we had also applied it to the concentration equation for mass transfer and the energy equation for heat transfer. In that case the concentration the mass of solute within a differential volume. This is kind to be equal to integral d v of the concentration of the solute d by d t integral d v of the concentration concentration is mass of solute per unit volume.

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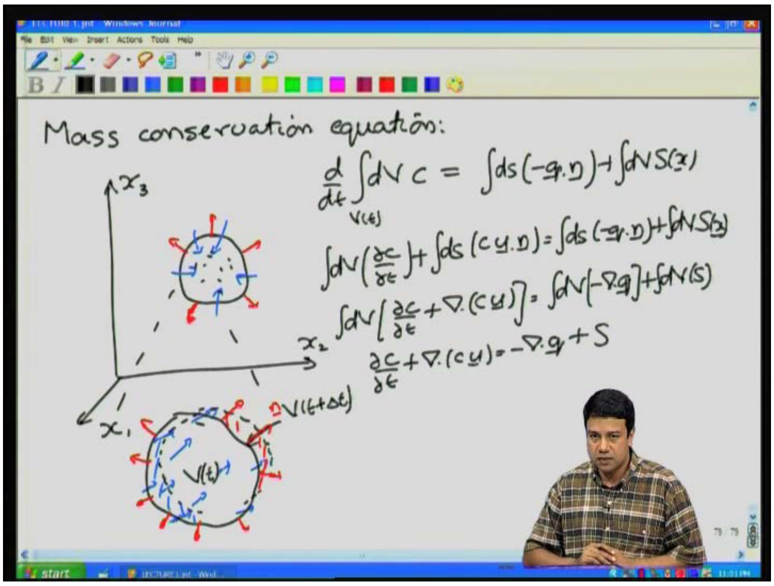

So, there is a total mass of solute within this volume of course, that is not 0, that is not 0 because you could in general have a flux of the solute coming in or going out of this volume due to temperature due to concentration gradients. So, for example if I had this volume here, if I had this volume here, there could be a flux of the solute which tends to increase the concentration within this volume. There could be a flux of solute, which tends to increase the concentration within this volume. It increases the concentration, if the flux is inwards that is it is opposite to the outward unit normal in all of these I define the outward unit normal as n. So, there is an increase in the mass within this volume if the flux is inwards so the flux is defined as positive which is inwards where ass it is opposite to be a outward unit normal.

So, the net increasing in mass within this volume is going to be integral over the surface of minus q dot n. So, if q is along the direction of n there is a decrease in mass within this volume, if q is opposite to the direction of the outward unit normal there is an increase in mass within this volume. These was all that I had done in the previous lecture, but in general if you had some sources of mass due to say some reaction or something within this volume. If I had a source of mass due to the reaction within this volume I could have incorporated quite easily, which just becomes integral d v times the source u which is a function of position.

So, I could some reactor concentration reacting to give you a product as is the source of mass per unit volume per unit time integrated over time, gives me the source of mass per I am sorry, integrated over volume gives me the source of mass. The amount of mass increase in this volume per unit time; so this will be the equation if I had sources or sinks if it is the source it will be positive if it is a sink it will be negative. Now, this I can use my Leibnitz rule, once again integral d v partial c by partial t plus integral over the surface d s of c u dot n is equal to integral d s of minus q dot n plus integral d v of the source. Now, I have two surface integrals which have both some vector dotted with the unit normal integrate over the surface. I can convert both of them into volume integrals using the divergence theorem. And you will get integral d v partial c by partial t plus the divergence of c u is equal to integral d v of there is a negative sign here minus divergence of q plus integral d v of s the source. Once again, this has to be true for each and every differential volume.

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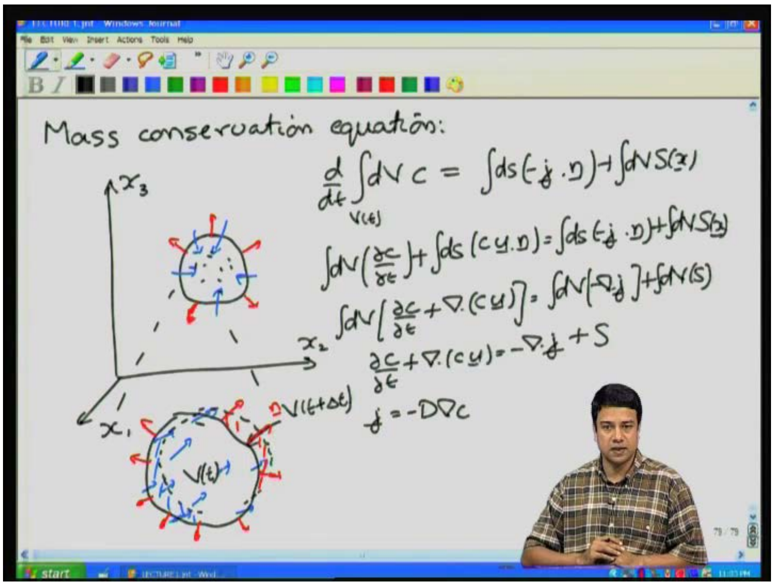

So, my concentration equation becomes partial c by partial t plus del dot c u is equal to minus del dot q plus s where s is the source term. And if I use my constitutive relation for once again I wrote the heat flux instead of the mass flux. So, I will just replace this by j to avoid confusion minus j. And once again if you use the constitutive relation j is equal to minus d grad c, you get d del square c for the divergence minus divergence of j. So, this is the simple extension we had derived these equations in rather laborious way in fundamentals of transport processes, one getting a fixed volume and then looking at what comes in and what goes out.

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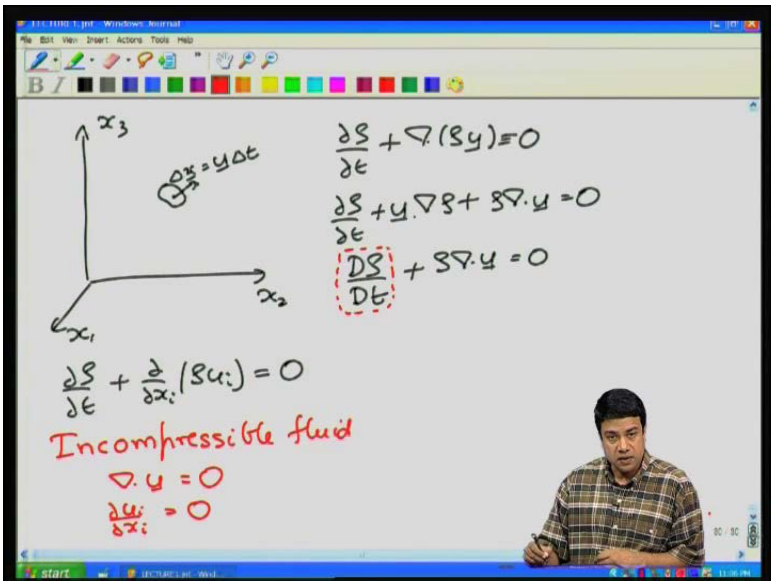

Whereas, if you just look at moving differential volumes that is the big advantage here. You treat vectors as objects of themselves considerate on the moving reference frame, where the mass has to be conserved and translate or shift or to a transformation of that in moving reference frame on to a fixed reference frame and you get the conservation equations quite easily. So, my conservation equation was partial rho by partial t plus del dot rho u is equal to 0. I can expand it out to write it as partial rho by partial t plus u dot grad rho plus rho del dot u is equal to 0 and the first two terms put together is just the substantial derivative in a reference frame moving with the mean velocity of fluid. So, this is you know the definition d rho by d td rho by d t is a substantial derivative rho at x 1 plus u 1 delta t x 2 plus u 2 delta t x 3 plus u 3 delta t, t plus delta t minus rho at x 1, x 2, x 3 t divided by delta t. So, in the reference frame that is moving with the same mean velocity as the fluid. So, there is the difference in the density at two locations, which are separated by delta x is equal to u delta t.

For future reference it will also be convenient for me to write this in indicial notation in which case my conservation equation will be partial rho by partial t plus d by d x i of rho u i is equal to 0 repeated index dot product divergence reduces to a scalar. In case the fluid is incompressible what is meant by incompressible is that d rho by d t is equal to 0 the substantial derivative is 0 in a reference frame moving with the fluid velocity. If the fluid is incompressible then for an incompressible fluid you have the divergence of the velocity is equal to 0 or partial u i by partial x i is equal to 0. If you recall we had defined the string function in two dimensions for an incompressible fluid whenever divergence of a vector is equal to 0 it can always be written as the curl of some other vector and reduce that to define string function in two dimensions.

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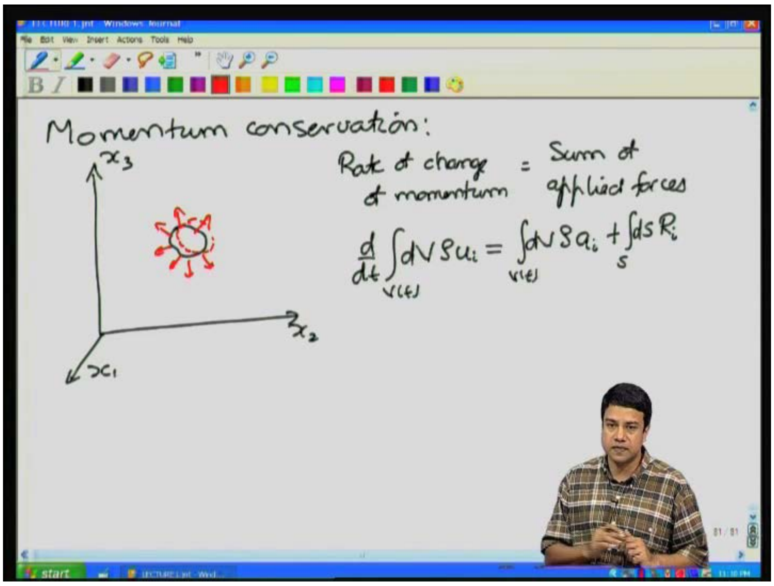

So, that is the mass conservation equation next we will go on to the momentum conservation equation. The fundamental principle and the momentum conservation equation is Newton law rate of change of momentum is equal to sum of applied forces, only thing is rate of change of momentum. We got to define on a moving volume element some volume element, which had some later time goes to some other location. So, a basic principle rate of change of momentum applied forces rate of change of momentum is equal to sum of applied forces rate of change of momentum in this moving differential volume is d by d t integral of v t d v times. The mass it is the mass times the velocity mass is d v times rho and the local velocity is u note I am using indicial notation here I have used u i because u itself is a vector.

So, this momentum has three components and I am writing the equation for each of these three components separately, except that I am not expanding out the components I am just writing u i as the representative one of the components. One pre index implied there is one summation one unit vector, this is equal to the sum of the applied forces. The applied forces in general could be of two kinds; one is what is called a body force that is a body force, which acts on the entire volume of the fluid examples, are gravitation force, so if I have a volume then the gravitational force is equal to the mass within that volume times g and centrifugal force, which is also proportional to the volume itself.

So, these are volumetric forces they are proportional to the entire volume of fluid. So, this will be of the form integral d v times the mass times the acceleration right this is an acceleration. Acceleration is also a vector a i is the acceleration vector for example, if the gravitational force was acting vertically downwards in this figure that I have this a I will be equal to minus g times e 3 in this figure, but we do not have to consider a specific form of the acceleration itself. We will just consider that as a general acceleration due to a body force. So, that is the first part and the second is you could also have forces that are applied on the surface of this volume.

So, let me call this as a surface force it is an integral over the surface d s times the force per unit area r i integral over the surface d s of the force per unit area r i. Surface force of course, is the flux of momentum across the surface the flux of momentum across the surface which is resulting in a force, which is applied on the surface. Except that r i is defined as the force exerted on the surface whose outward unit normal is n. So, this surface force is similar to the surface flux that we had for mass conservation equation or the surface flux for the energy conservation equation. So, this is the surface flux in the momentum conservation equation and you will expect that we will have to write a constitutive relation for this surface flux soon or a later we will come back to that.

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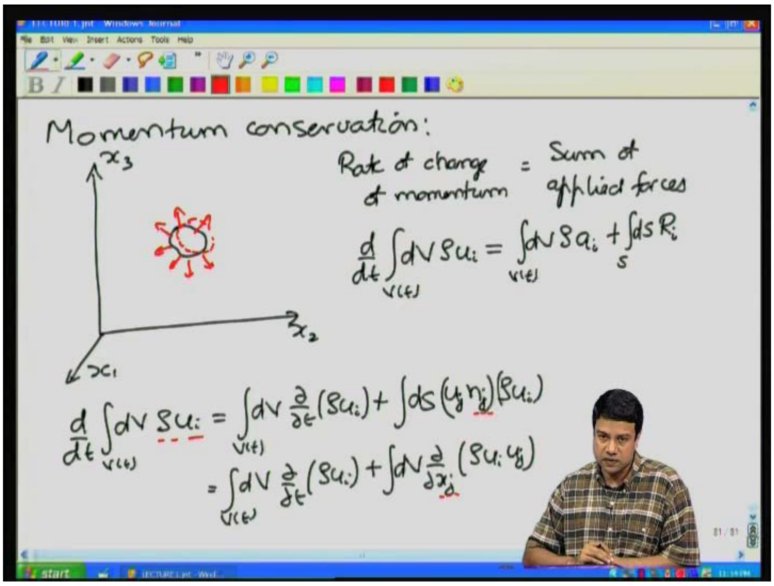

The body forces of course, specified for you if you know what the gravitation acceleration is you know what the body force is or if you know what the centrifugal acceleration is you know what the body force. Whereas, the surface force is due to the deformation of the fluid just as the mass flux was due to the gradient and concentration the heat flux was due to gradient and temperature. You would expect this surface flux to be related in somewhere to the gradient and momentum or the gradient and velocity.

We will come back to that first how do we use the apply the Leibnitz rule for this left hand sided by d t integral d v of rho u i integral over the volume the Leibnitz rule. One part due to the change in momentum within the volume itself that is going to be equal to d v partial by partial t of rho u i plus. The second part due to the volume that comes in and goes out as this differential volume is moving, because there is some fluid that is coming into this differential volume as it is or some locations coming into this differential volume, as it is moving some locations have been left behind. So, this is going to be equal to u dot n times the quantity itself for the mass conservation equation.

This is equal to rho times u dot n because u dot n times d s was the volume that came in momentum. Similarly, rho u i at that location times u dot n this is where it is useful to have indicial notation because that u dot n I can write it in indicial notation as u j n j. Repeated index dot product j is different from the index i, which is the direction of the momentum itself. It is different from the index i, which is the direction of the momentum itself. So, on the second term surface integral of n dotted with something that thing is now a second order tensor because n is now dotted with rho u i u j it is dotted with the second order tensor.

So, integral of n dot something is equal to the integral over the volume of the divergence of that same thing does not matter, whether it is a vector or tensor I can always take the divergence of a second third order tensor as I had shown you in the previous lectures. So, this just becomes equal to integral over the volume d v partial by partial t of rho u i plus integral over the volume of partial by partial x j of rho u i u j.

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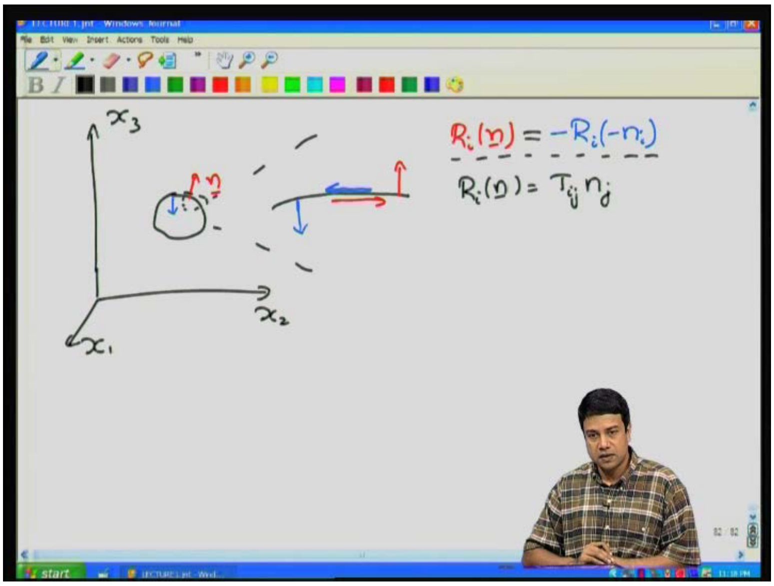

Note that the index that I use for the divergence has to be the same index as that for the unit normal n dot something integral over the surface is equal to the divergence of that same thing integrated over the volume. So, I have to use the same index in both those places. So, this is the Leibnitz rule as applied to this particular case the momentum conservation equation and so this is the left hand side of the equation the rate of change of moment on the right hand side. I have the sum of applied forces one is volume surface force the other is the surface force. As I said the surface force is due to momentum flux let us look at that a little more carefully.

If I have a volume with some unit normal n the surface force R i is the force that is exerted on the surface of this volume there is the force exerted by the outside fluid on the inside fluid. So, if I can expand out this little differential volume here for you this is the surface it has an outward unit normal and the outside is exerting a force on the fluid inside. The fluid outside is exerting a force on the fluid inside, which is changing the momentum of the fluid inside and that is what is coming into the momentum conservation equation.

So, the fluid outside is exerting a force on the fluid inside, so this is the force exerted by the outside fluid on the inside fluid. I am just using straight for that this is the force exerted by the fluid outside on the fluid inside, but we know from Newtons third law that the fluid inside exerts an equal and opposite force on the fluid outside. Therefore, the fluid inside will exert an equal and opposite force in outside. This is the force that is exerted by the fluid inside on the fluid outside, for the fluid outside the outward unit normal for the fluid outside. The outward unit normal is actually pointing into this volume that is the outward unit normal for the fluid outside. So, the outward unit normal for the fluid outside is in this direction it is pointing into the volume. Therefore, if I have a surface force for a given volume for a given unit normal n this is the unit normal for the outward unit normal for the fluid inside.

If I have a surface force for that if I consider the opposite the force exerted by this fluid on the outside fluid. I have interchanged the direction of the unit normal and the direction of the force also changes. If I reverse the direction of the unit normal in which I will I am looking for the force exerted by the fluid inside on the fluid outside the direction of the force reverses. Therefore, r i for a unit normal n is equal to minus r i for the unit normal minus n. This dependence on the unit normal suggest that I can write R i of n as T i j n j you can see that with this specific form T i j n j where T i j is now defined at a location it does not depend upon the unit normal any more it is defined at each location in space independent of the unit normal. If I can write it this way it satisfies this reversibility criteria it satisfies these reversibility criteria.

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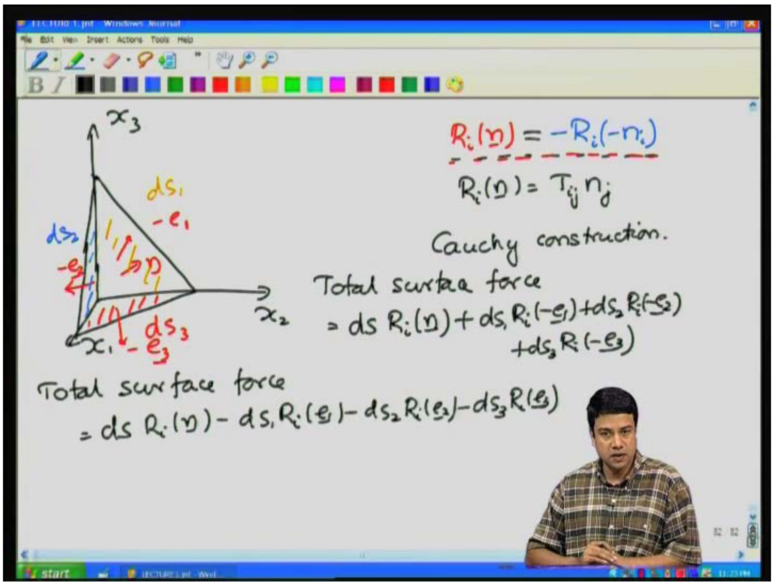

You can show that this criteria is actually valid for any differential volume using what is called the Cauchy construction and the way it works is as follows. Let us small tetragonal volume element of fluid that looks something like this. So, let me just draw it larger so that it is easier to see let us take this small tetragonal volume element of fluid. So, it is bounded by four surfaces it is bounded by four planar surfaces; one is the x 1, x 2 plane x 2, x 3 plane x 1, x 3 plane and this slanted surface in the surface that is an angle to all of these three planes. So, this surface that is at an angle has an outward unit normal, which is coming out of the plane. So, this has an outward unit normal that is coming out of the plane perpendicular to this surface.

The other three surfaces have outward to be in normal is in the minus e 1, minus e 2 I am sorry this is minus e 3. This is in the minus e 2 directions and minus e 1 direction the minus e 1 normal is pointing behind minus e 2 in the left and minus e 3 downwards. So, these are the four outward units normal for this surface so what is the net force due to these surface forces on all of these for surfaces the net force. So, let me first write down the total areas that are perpendicular to each of these.

So, I will call as the area in the x 1, x 2 plane I will call it as d s 3 because this is perpendicular to the s 3 direction. Similarly, this area in the x 1, x 3 plane I will call it as d s 2 the area on the x 2, x 3 plane I will call it as d s 1 that is behind there d s 1 and this area that is at an angle to all three planes I will call it as d s I said r was the force exerted per unit area. So, the total force exerted due to the surface forces will be equal to d s times the force per unit area at a surface whose unit normal is in the plus n direction. So, this is n where normal that is coming out of this angle surface, so that is d s times R i of n plus the contribution along these three planes R i of minus e 1 plus d s 3 times R i f minus e 3 that is a total surface force. I also have this condition that when I reverse the unit normal when I reverse the unit normal the direction of the force changes. Therefore, R i of minus e 1 is equal to minus R i of e 1 r i of minus e 2 is equal to minus R i of e 2 and so on.

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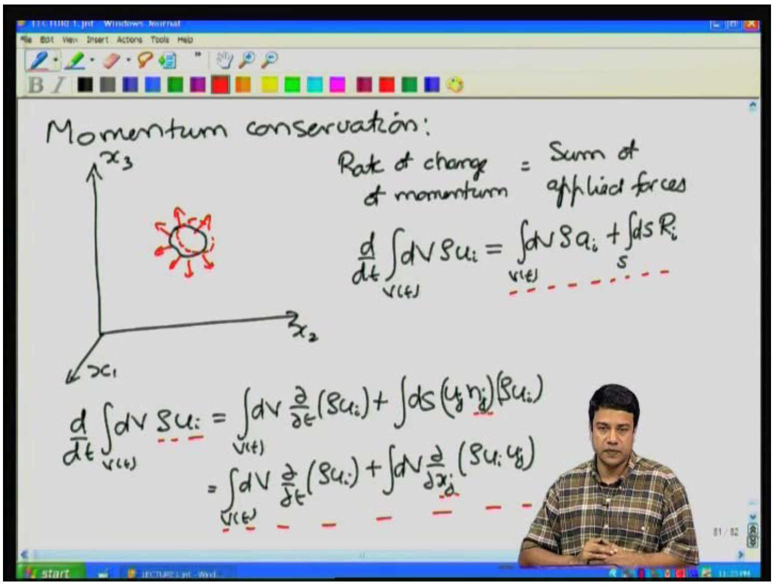

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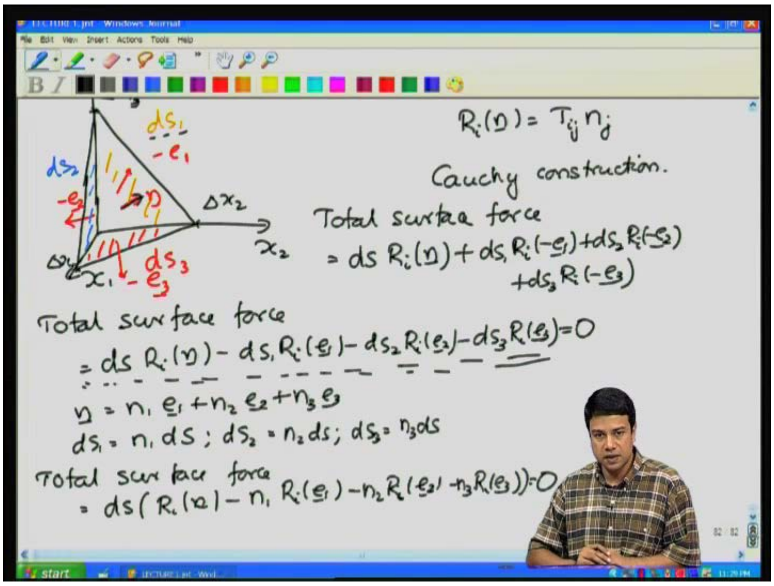

So, therefore I can write the total surface forces d s R i of n minus d s 1 R i of e 1 minus d s 2 R i of e 2 minus d s 3 R i of e 3 that is the total surface force. Of course, on this volume there is a surface force in addition that there are volumetric forces as well if you look at back at the conservation equation that we had this two are volumetric forces. They are there is a rate of change of momentum is extensive it depends up on the total volume. Similarly, this body force is extensive it depends up on the total volume where as this depends only up on the surface area. So, my momentum conservation equation in general contains these three terms two of which depend up on volume and the third depends up on the surface area.

Now, if you take the limit thus these all go to 0 so this is delta x 2, this is delta x 1 this is delta x 3. If you take the limit as these all go to 0 you can see that the volumetric terms the rate of change of momentum as well as the body force. They will decrease as the cube of the length because the volume is proportional to delta x 1 times delta x 2 into delta x 3. So, the limit as these three go to 0 the volumetric terms will vary as the cube of the length will go to 0 as delta x cubed. Surface terms on the other hand will go to 0 as delta x square as delta x goes to 0 delta x cubed goes to 0 much faster than delta x square because the ratio of the surface to volume goes to infinity as delta x goes to 0.

So, as this volume goes to 0 delta x cubed goes to 0 much faster than delta square, which means that I can the momentum conservation equation with these volumetric. The surface terms will continue to be valid in the limit as delta x goes to 0, only if all three terms go to 0. Obviously, the volumetric terms are going to 0 as delta x cubed surface terms are going to 0 as delta x square. So, the only way that you will be able to retain a balance is if is this thing goes to 0 as delta x goes to 0. In other words, I should be able to reduce the surface force into a volumetric form that is the only way that I can ensure that because the surface terms go to 0 as delta as x square and the volumetric terms go to 0 as delta x, x cubed the ratio goes to infinity.

Unless for each and every differential volume element this thing goes to 0 in the limit as delta s goes to 0, then the terms promotional surface area will go to 0 and you have still be left with volumetric terms. Now, so if this has to go to 0 if this has to go to 0, then what is the condition as let us as the condition is equal to 0. Now, you know that this unit normal n 1 I mean this unit normal n to the surface has three components n 1 e 1 plus n 2 e 2 plus n 3 e 3 n 1 is the angle made by the unit. Normally, with the x 1 with the x 1 direction n 1 is the angle made by the unit normal of the x 1 direction. Now, this angle surface area perpendicular to n 1 the surface on the x 2, x 3 plane that is this surface with the unit normal in the e 1 direction d s 1 is perpendicular to the e 1 plane.

Therefore, since the angle between n vector and e 1 is the same as the angle between d s and d s 1. I can write d s 1 is equal to n 1 d s, because the angle between d s 1 and d s, d s 1 is in the x 2, x 3 plane d s is the actual slatted surface. The angle between those two is the same as the angle between the unit normal to those 2 n and e 1 angle between them the cos theta of the angle between n and e 1 is just n 1 that also has to be equal to d s 1 by d s. Similarly, d s 2 is equal to n 2 d s and d s 3 is equal to n 3 d s the angle between the surface s and the x 2, x 3 plane in the x 1, x 2 plane is the same as the angle between n and e 3 and so on. So, I can use this to simplify and I will write this entire expression as total surface force is equal to d s into r i of n minus n 1 r i of e 1 minus n 2 r i of e 2 minus n 3 r i of e 3 is equal to 0.

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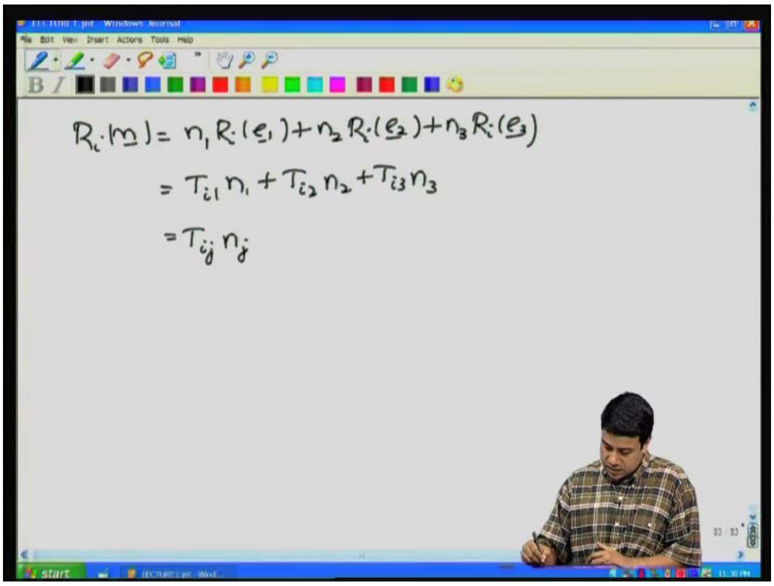

Note that n 1 R i of e 1 is the force acting at a surface perpendicular to the x 1 direction, R i is e 2 is the force acting at a surface perpendicular to the x 2 direction.

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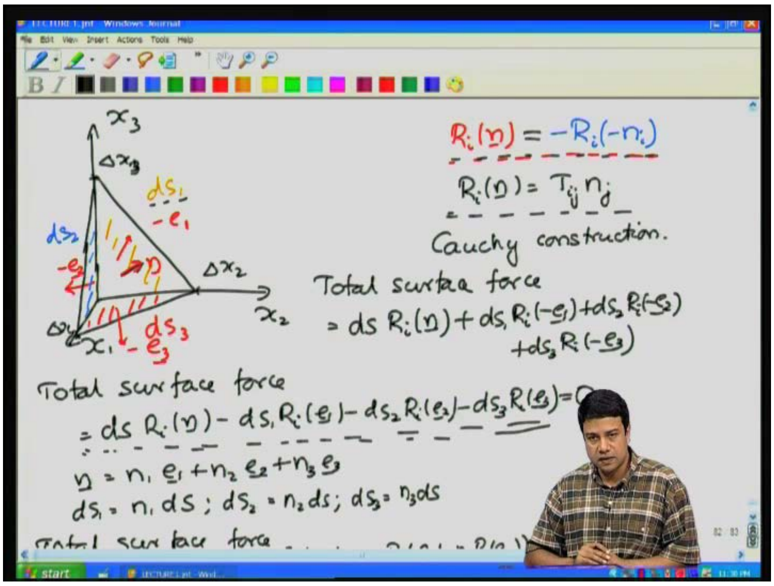

R i of e 3 is the force acting perpendicular to the x 3 direction and these three have to equal to 0. That means that R i of n is equal to n i n 1 R i of e 1 plus n 2 R i e 2 plus n 3 R i of e 3, R i of e 1 force in the i direction acting at a surface whose unit normal is in the e 1 direction.

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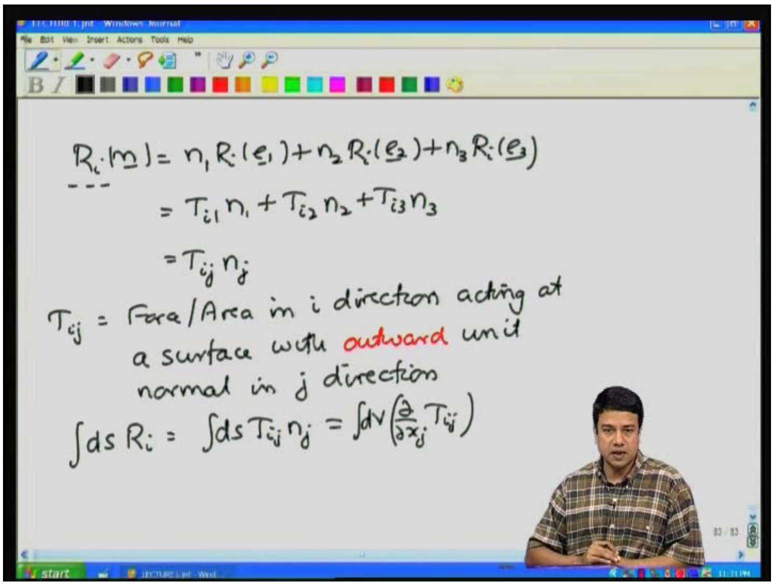

This I can write it as T i 1 where T i 1 is the force in the i direction acting at a surface whose unit normal is in the 1 direction T i 1 times n 1 plus T i 2 times n 2 plus T i 3 n 3. This indicial notation is just the second orders tensor dotted with the unit normal. So, as I said I can always write the force acting at the surface in this form as a second order tensor dotted with the unit. Normal T i j is the stress tensor as I have defined it over here T i j is the stress tensor t i j is equal to force per area in i direction after all i was the direction of the force there.

Force per area in the i direction acting at a surface with outward note that we had defined the forces with respect to outward unit normal in this in the j direction. So, that is the stress tensor surface force can always be written as T i j times n j. So, when I write the surface force as T i j times n j then the integral over the surface d s of R i is equal to integral d s of T i j and j. This can now be reduced to a volume integral using the divergence theorem d s partial by partial x j of T i j. I am sorry, this is the volume and if I can reduce it to volume integral all the other terms in the conservation equation also decrease proportional to the volume. Therefore, the balance the dimensionality of the entire equation is maintained.

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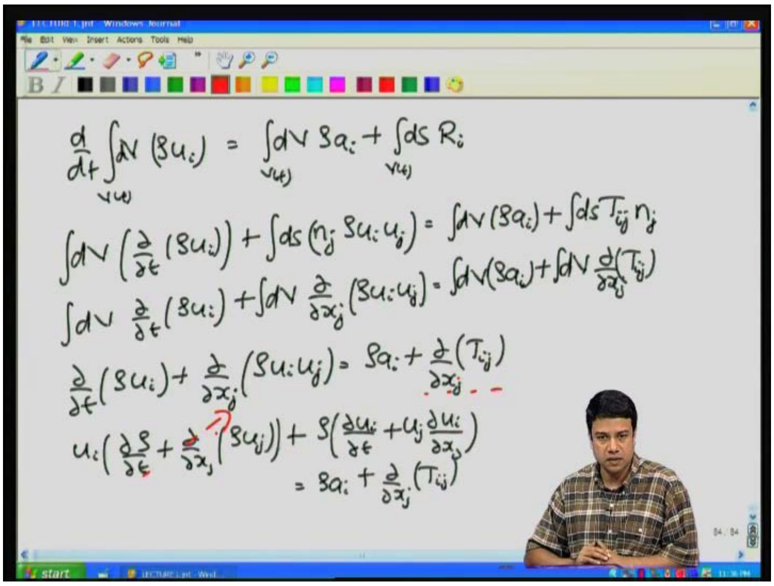

So, this gives me an expression for the surface force back to my momentum conservation equation integral v of t integral d v of rho u i is equal to integral d v of rho a i plus integral d s of R i surface. Expand this out using the Leibnitz rule on the left hand side integral d v of partial by partial t of rho u I plus integral over the surface d s of n j rho u i u j is equal to integral d v of rho a i plus integral d s. Now, we write this in terms of the stress tensor dotted with the unit normal T i j n j i have two surface integrals here of the unit normal dotted with something integrated over the surface convert them both into volume integrals. Convert them both into volume integrals, everything now is a volume integral, and this has to be true for any volume it has to be true in the limit as the volume goes to 0. And what that implies is that the momentum conservation equation is partial by partial t of rho u i plus partial by partial x j of rho u i u j.

So, that is the momentum conservation equation if I write it back in vector notation, it will be partial by partial t of rho u vector plus divergence of rho u is equal to rho a vector plus divergence of the second order tensor. I prefer not to write it in this form because this for example, is confusing does not tell you whether the divergence is to be taken with respect to the first index or the second index where as written in this form. This is very clear that the divergence is to be taken with the second index I can expand this out. So, the first two terms I can expand it out as u i into partial rho by partial t plus partial by partial x j of rho u j plus rho just using the product rule for differentiating these equations.

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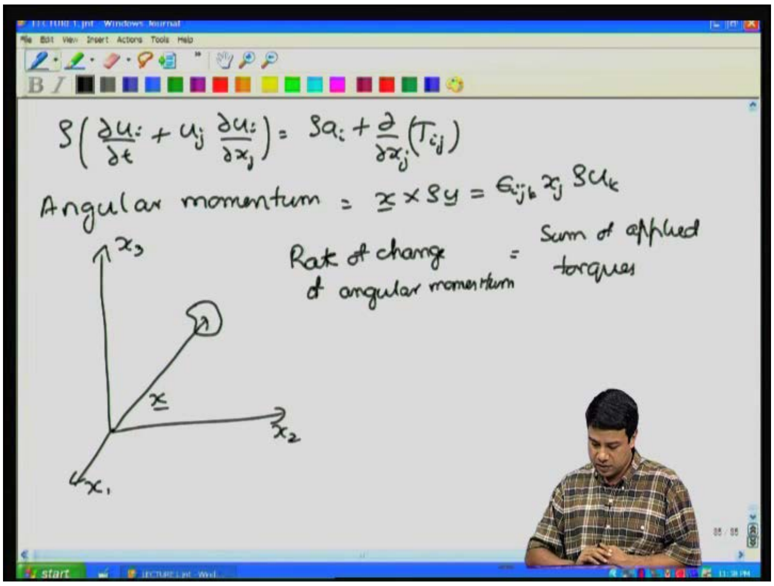

So, this is equal to rho a i plus partial by partial x j of T i j and of course, partial rho by partial t plus partial by partial x j of rho u j partial rho by partial t equals divergence of rho u it is just equal to 0 from the mass conservation equation. So, this is equal to 0 from the mass conservation equation therefore, this momentum equation can also be written as rho into partial u i by partial t plus is equal to T i j is the stress tensor T i j is the stress tensor. So, that is the mass conservation equation I am sorry that is the momentum conservation equation in the x direction. Now, from the momentum conservation equation it does not tell us exactly what form of the stress tensor is for that we have to write on a constitutive relation. However, you can get some information about the properties of this stress tensor by looking at the equation for the conservation of angular momentum.

Angular momentum r s cross f I am sorry, r cross the linear momentum. So, I can write angular momentum as if I have some coordinate system the angular momentum of this entire volume about some fixed reference frame. Along some fixed coordinate has to be concerned to that law of generality I can take this coordinate as the origin of my coordinate system. Then I could I would define angular momentum as the distance from the origin the vector displacement from the origin crossed with fluid. So, equal to x cross rho u where x is the displacement vector from the origin in indicial notation I can write this as epsilon i j k x j rho u k. The angular momentum conservation states that the rate of change of angular momentum is equal to sum of applied torques, the rate of change of angular momentum is equal to sum of applied torques.

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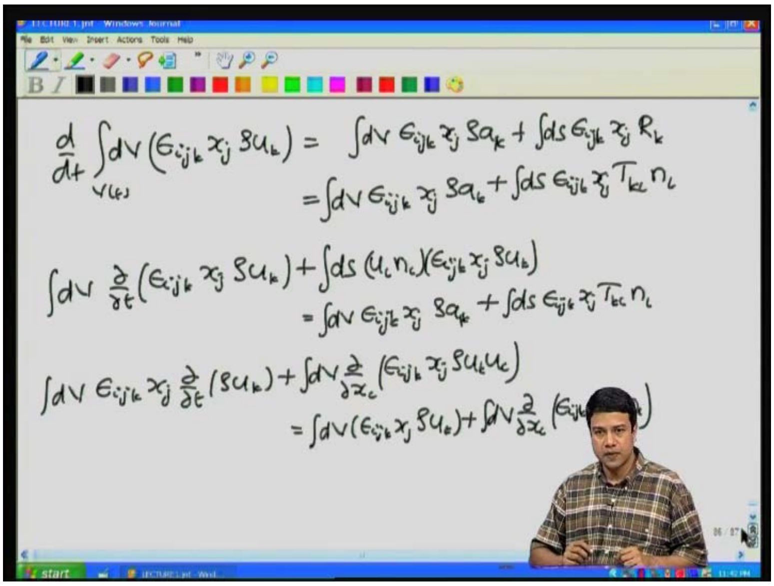

So, rate of change of angular momentum d by d t integral over the volume, which is a function of time d v of epsilon i j k x j rho u k is equal to integral over the volume of epsilon i j k x j times rho a k that is the torque due to the body force. The torque due to the body force plus the torque due to the surface force d s of x j times the surface force r k not that I have taken the cross product. I have taken epsilon i j k times x j x so the second index j is on x third index k is on u. Similarly, on the right hand side second index j is on x third index is the acceleration or the force in the final expression.

From my expression for the surface force in terms of the stress I know that this has to be of the form epsilon i j k x j T k l n l that should be of the form t dot n. I have to choose indices which are new for each dot product because if I use the same index j or k for this last dot product that index would appear more than two times and it results to an incorrect equation.

So, simplify the left hand side first so I know how to simplify this quite easily using the Leibnitz ruled d v partial by partial t of epsilon i j k x j rho u k plus integral over the surface of u dot n times this whole thing u dot n times this whole thing. I have to use a new index here for u so this becomes u l n l times epsilon i j k x j rho u k is equal to integral over the volume d v of epsilon i j k x j rho a k plus integral d s epsilon i j k x j T k l n l.

Convert the surface integrals into volume integrals using the divergence theorem and also use the fact that epsilon is independent of time x is also independent of time because position and time are independent coordinates. So, I will get integral over the volume of epsilon i j k x j partial by partial t of rho u k plus integral over the volume of partial by partial x l of epsilon i j k x j rho u k u l is equal to integral d v of epsilon i j k x j rho u k plus integral d v of partial by partial x l of epsilon i j k x j t k l.

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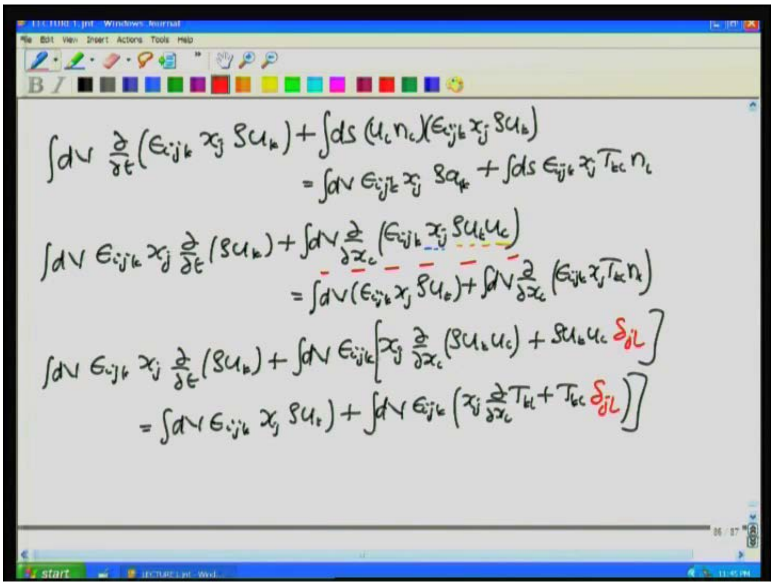

In this equation I use chain rule for differentiation I use chain rule for differentiation and I will get integral over the volume of epsilon i j k x j partial by partial t of rho u k plus integral d v epsilon i j k. Now, in this term here I have one derivative this and the other derivative of what is rho u k u l. So, I can separate out those two so I will write this as epsilon i j k times x j partial by partial x l of rho u k u l plus rho u k u l partial by partial x l of x j. On the right hand side I have integral d v epsilon i j k x j rho u k plus integral d v once again I use the chain rule for differentiation epsilon i j k into x j partial by partial x l of t k l plus T k l partial x j by partial x l. Now, in this partial by x j by partial x l the partial x if j is l and l is 1 it is 1 if the j is 1 and l is 2 it is 0, if j and l are the same it is one if j and 1 are different its equal to 0.

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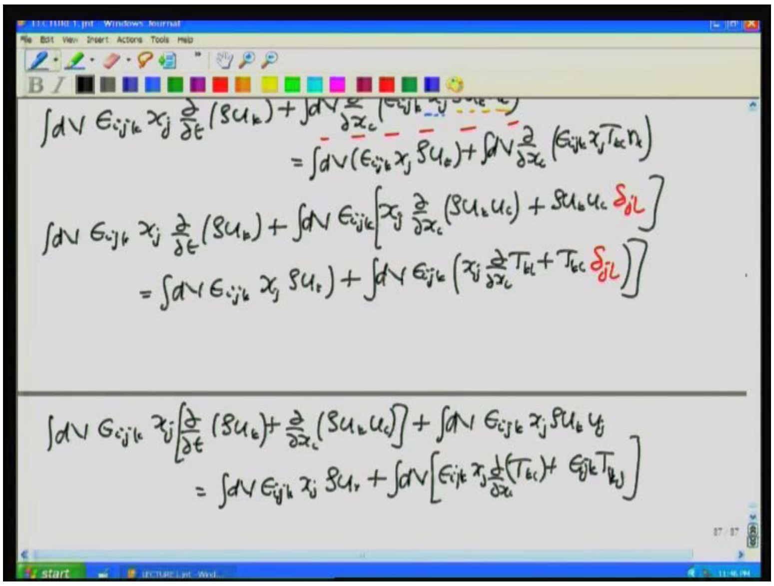

So, this I can write it simply as partial x j by partial x l is just equal to 2 delta j l. Similarly, on the on the left hand side I have partial x j by partial x l, which is also equal to delta j l. So, both of these are equal to delta j l plus plus integral d v delta j l times u k u l rho u k u j. This is my rho u k u j delta j l times rho u k u l is equal to the, left hand side is equal to integral d v epsilon i j k x j rho u k plus integral d v of epsilon i j k x j partial by partial x l t k l plus epsilon i j k T k l times delta l j l is just T k j.

It is been a complication derivation so for, but I have gone through this to make a point an important point at that I want to able to complete with this lecture, but I will continue it in the next lecture. I will go through the same derivation to give you what the final point is b regarding the stress tensor as we will see that this angular momentum conservation equation shows that the stress tensor is symmetric. Once we have that, then we can proceed to find out constitutive of relations with the stress tensors. So, kindly go through this derivation once again and we will continue it in the next.

We will see you then.