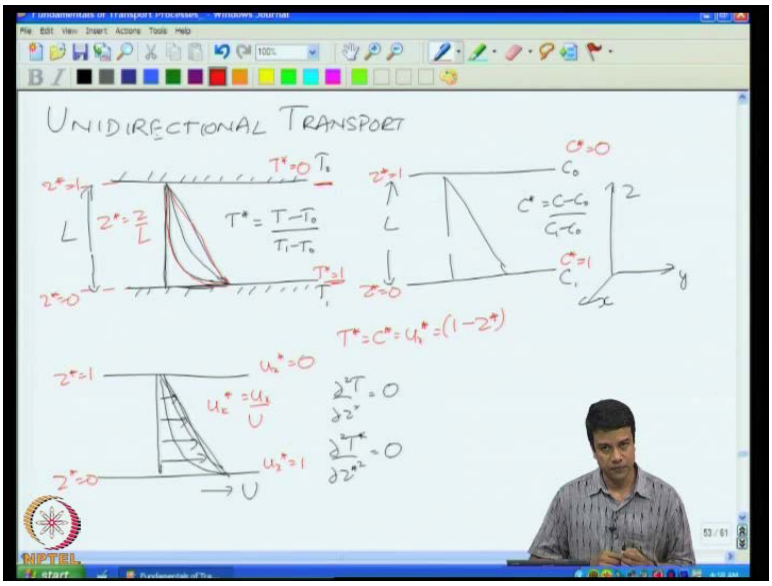
Fundamentals of Transport Processes
Prof. Kumaran
Department of Chemical Engineering
Indian Institute of Science, Bangalore

Module No. # 03
Lecture No. # 09
Unidirectional Transport Cartesian Co-ordinates-II (Similarity Solutions)

The fundamentals of transport processes, this is lecture number nine and last time we were discussing unidirectional flows. Just to remind you of where we started, I first explain to you why transport processes are important in material transformations both physical and chemical. It is not just sufficient to put in the raw materials, put in the heat required, the product need not come out. You need to make sure that the materials get to the place where the transformation actually occurs and that is where the discussion of transport phenomena is so important. We looked at dimensional analysis and empirical correlations based on dimensional analysis and I try to give you some physical inside into what those dimensionless groups mean.

And we talked about diffusion, the molecular phenomenon of diffusion and how that translates into diffusion coefficients, why the values of the diffusion coefficients are the numbers that they are in gases and liquids. And then, we got down to the business of solving actual problems in transport phenomena. The simplest configuration you can consider is unidirectional transport. That means the transport occurs only in one direction and that is why the problem is greatly simplified.

(Refer Slide Time: 00:02)



UNIDIRECTIONAL TRANSPORT

$T^* = \frac{T-T_0}{T_1-T_0}$

$C^* = \frac{C-C_0}{C_1-C_0}$

$u_x^* = \frac{u_x}{U}$

$T^* = C^* = u_x^* = (1-z^*)$

$\frac{\partial^2 T}{\partial z^2} = 0$

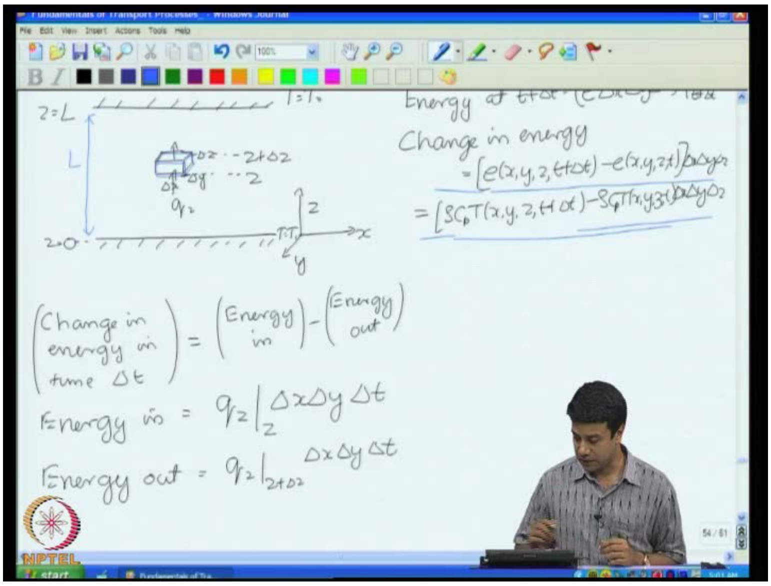
$\frac{\partial^2 T^*}{\partial z^{*2}} = 0$

And in what is called a Cartesian coordinate system. The three coordinates are x y and z and unidirectional transport in a Cartesian coordinate system means that it, the transport is taking place only along one direction and the simplest example in Cartesian coordinates is the transport between two flat plates and we looked at various instances of heat and mass transfer between two flat plates both at steady state and at unsteady state.

So in the simplest example of heat transfer; you have two plates one of which is at a higher temperature and the other of which is at I am sorry this is at a lower temperature and this is at a higher temperature. Because of this temperature difference energy is transported from the higher to the lower temperature and that results in a temperature gradient. In the final steady state the temperature gradient is going to be linear, it is going to be a linear function of position you'll have a linear variation in temperature between the two plates which means of the flux is a constant because flux is proportional to the derivative of the temperature. However, one could also consider unsteady situations for example, you have this entire system at one constant temperature T naught and instantaneously you change the temperature of the bottom surface to T 1 and the temperature is going to change as a function of time. It is going to look something like this. At very early times and as time becomes larger and larger you are going to get a flat profile in the long time limit. Same thing with concentration you can have two plates; one at a lower concentration, one at a higher concentration and it is going to be a flux from one plate to the other. At steady state when there is nothing; changes in time that flux has to be a constant and that concentration has to be linear function of position between the two plates.

We looked at a momentum transport problem. In this case one plate is stationary, it has no velocity the other plate is moving with a constant velocity. In the final steady state of course, the velocity profile is linear and the momentum flux or the shear stress is going to be a constant throughout. But, we also want to look at what happens when there is a variation in time and that was the purpose why we started looking at what are called shell balances.

(Refer Slide Time: 04:41)



Change in energy
$= [e(x,y,z,t+\Delta t) - e(x,y,z,t)]\Delta x \Delta y \Delta z$
$= [\rho C_p T(x,y,z,t+\Delta t) - \rho C_p T(x,y,z,t)]\Delta x \Delta y \Delta z$

$$\begin{pmatrix}\text{Change in}\\ \text{energy in}\\ \text{time } \Delta t\end{pmatrix} = \begin{pmatrix}\text{Energy}\\ \text{in}\end{pmatrix} - \begin{pmatrix}\text{Energy}\\ \text{out}\end{pmatrix}$$

$$\text{Energy in} = q_z\big|_z \Delta x \Delta y \Delta t$$

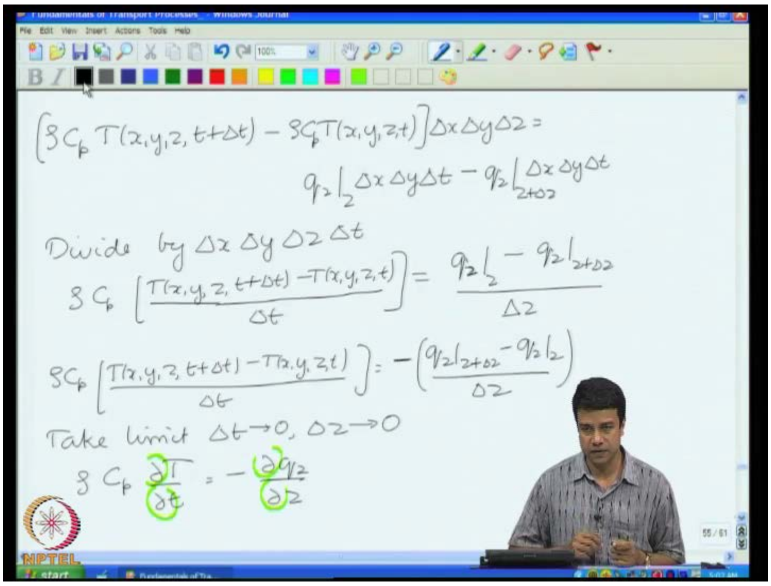
$$\text{Energy out} = q_z\big|_{z+\Delta z} \Delta x \Delta y \Delta t$$

The ideas as follows; we write a balance equation for a small differential volume within the fluid. We assume that the length the thickness of this volume as well as the area of this volume is small compared to the macroscopic scale which in this case is the total length theorem. It has to be small compared to l but, it has to be large enough that the continuum approximation is valid. In other words it has to be large enough that it contains a large number of molecules going back to our earlier discussion of the continuum approximation.

If it contains a large number of molecules then you can define continuum fields. In this case the energy density field. So because of the of the of the of the gradient in the difference in temperature there's going to be energy coming in and going out of this volume. The bottom surface of this volume is hotter than the top surface therefore, the flux at the bottom will there will be a flux coming in at the bottom and a flux going out at the top. Energy conservation condition basically set the change in energy within this time delta T is equal to energy in minus energy out simple. Just a balance telling you that there can be no creation or distraction of energy. This is valid only when there is no a source or syncs. If you imagine a chemical reaction which was generating heat there would be a source. If there is an endothermic reaction which absorbed heat there would be a sync of energy. If none of these is present you just have a fluid which neither generates or consumes energy then the balance condition basically says the change in energy in the time delta T is equal to energy in minus energy out.

And we calculated explicitly the change in energy within a time delta T the energy within this volume is equal to the specific energy energy per unit volume at this point x y z times the small volume around it. Therefore, the change in energy was written as e at x y Z T plus delta T e at T x y Z T plus delta T minus e at x y Z T divided by times the volume itself and that we wrote in terms of the specific heat and the temperature here. The fluxes in and out is equal to the heat flux. The flux now is in the Z direction because there is a variation temperature only in the Z direction. So, the energy in this q Z times delta x times delta y that is the, q Z is the flux energy per unit area per unit time. q Z times delta x delta y is the energy in per unit time which is the flux times the area and then multiply that by delta T to find out how much is coming in within that time interval delta T put all of these together.

(Refer Slide Time: 07:54)



$$\left[\rho C_p T(x,y,z,t+\Delta t) - \rho C_p T(x,y,z,t)\right]\Delta x \Delta y \Delta z = q_z|_z \Delta x \Delta y \Delta t - q_z|_{z+\Delta z} \Delta x \Delta y \Delta t$$

Divide by $\Delta x \Delta y \Delta z \Delta t$

$$\rho C_p \left[\frac{T(x,y,z,t+\Delta t) - T(x,y,z,t)}{\Delta t}\right] = \frac{q_z|_z - q_z|_{z+\Delta z}}{\Delta z}$$

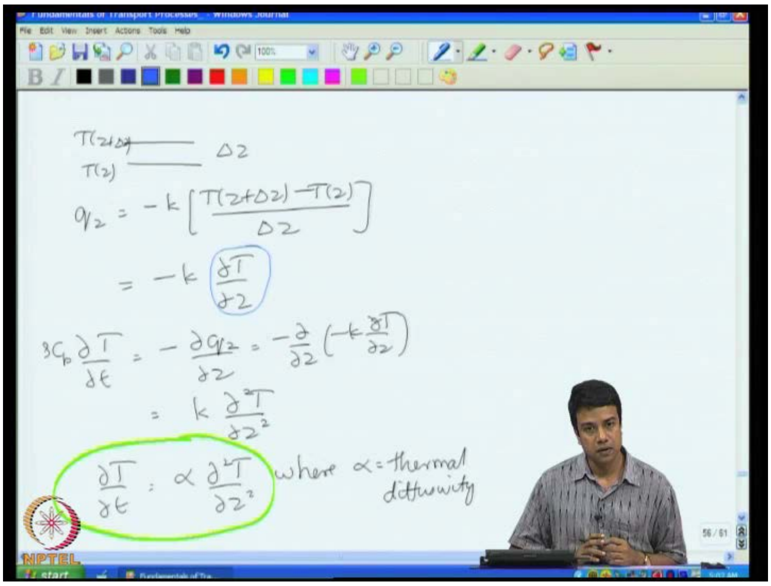
$$\rho C_p \left[\frac{T(x,y,z,t+\Delta t) - T(x,y,z,t)}{\Delta t}\right] = -\left(\frac{q_z|_{z+\Delta z} - q_z|_z}{\Delta z}\right)$$

Take limit $\Delta t \to 0$, $\Delta z \to 0$

$$\rho C_p \frac{\partial T}{\partial t} = -\frac{\partial q_z}{\partial z}$$

And we got a balance equation which was basically of this form rho c p into d T by d T is equal to minus partial q Z by partial z. So this basically tells you that the rate of change of specific energy within that volume is equal to the rate at which energy comes in minus the rate at which it is going out. The difference in fluxes across the two surfaces the bounding surfaces.

(Refer Slide Time: 08:20)



$T(z+\Delta z)$

$T(z)$

Δz

$$q_z = -k\left[\frac{T(z+\Delta z)-T(z)}{\Delta z}\right]$$

$$= -k\boxed{\frac{\partial T}{\partial z}}$$

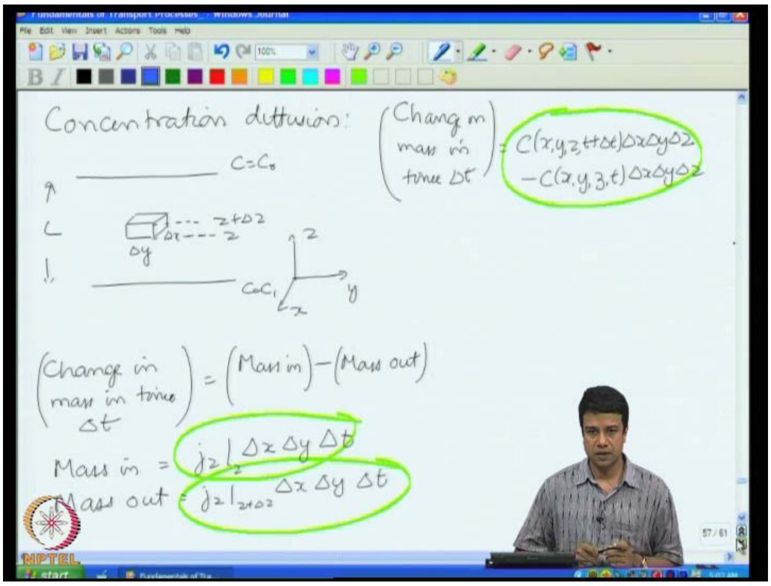
$$\rho C_p \frac{\partial T}{\partial t} = -\frac{\partial q_z}{\partial z} = -\frac{\partial}{\partial z}\left(-k\frac{\partial T}{\partial z}\right)$$

$$= k\frac{\partial^2 T}{\partial z^2}$$

$$\boxed{\frac{\partial T}{\partial t} = \alpha\frac{\partial^2 T}{\partial z^2}}$$ where α = thermal diffusivity

And then we need an equation for the flux. Fourier’s law of heat condition q Z is equal to minus k partial T by partial Z. Note the partials here. These are important. It implies that you’re taking the variation in temperature with Z keeping all other coordinates the same. That is you are varying Z a little bit and finding out the difference in temperature keeping x y and T a constant. And from that we got this diffusion equation for the temperature where alpha is the thermal diffusivity.

(Refer Slide Time: 08:56)



Concentration diffusion:

$$\begin{pmatrix}\text{Change in}\\ \text{mass in}\\ \text{time } \Delta t\end{pmatrix} = \boxed{\begin{matrix} C(x,y,z,t+\Delta t)\Delta x\Delta y\Delta z \\ -C(x,y,z,t)\Delta x\Delta y\Delta z \end{matrix}}$$

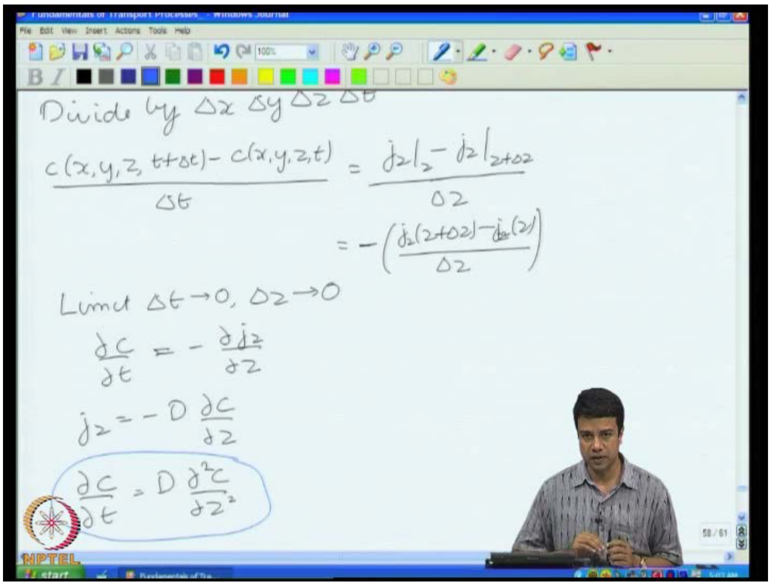
$$\begin{pmatrix}\text{Change in}\\ \text{mass in time}\\ \Delta t\end{pmatrix} = (\text{Mass in}) - (\text{Mass out})$$

$$\text{Mass in} = \boxed{j_z|_z \Delta x \Delta y \Delta t}$$

$$\text{Mass out} = \boxed{j_z|_{z+\Delta z} \Delta x \Delta y \Delta t}$$

Did exactly the same thing for mass. Instead of T we have c the concentration, instead of alpha we have d the diffusion coefficient and you get the exact same equation.

(Refer Slide Time: 00:02)



Divide by $\Delta x\, \Delta y\, \Delta z\, \Delta t$

$$\frac{c(x,y,z,t+\Delta t)-c(x,y,z,t)}{\Delta t} = \frac{j_z|_z - j_z|_{z+\Delta z}}{\Delta z}$$

$$= -\left(\frac{j_z(z+\Delta z)-j_z(z)}{\Delta z}\right)$$

Limit $\Delta t \to 0$, $\Delta z \to 0$

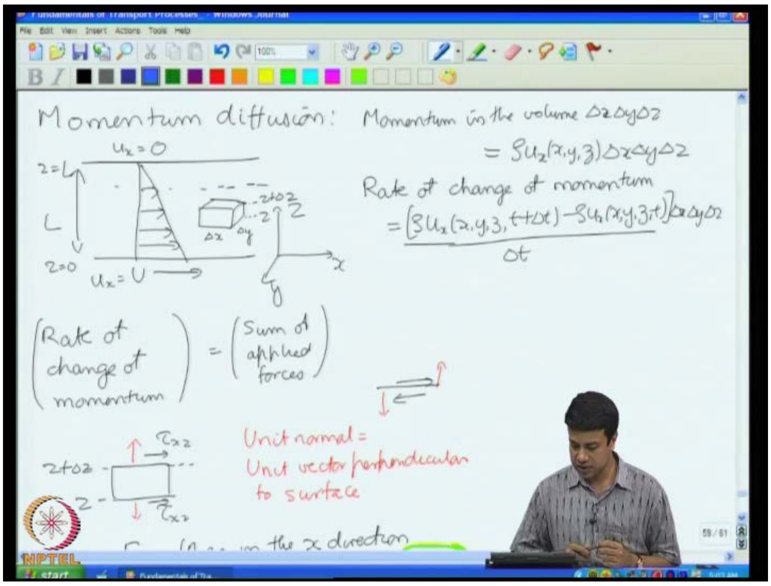
$$\frac{\partial c}{\partial t} = -\frac{\partial j_z}{\partial z}$$

$$j_z = -D\frac{\partial c}{\partial z}$$

$$\boxed{\frac{\partial c}{\partial t} = D\frac{\partial^2 c}{\partial z^2}}$$

The exact same equation except that concentration is substituted for temperature; mass diffusion coefficient is substituted for the thermal diffusion coefficient. In all of these we assume that the diffusion coefficients both from mass and the thermal diffusion coefficients are not functions of position. If they are functions of position it gets a little more complicated. We will assume that the fluid properties are uniform so the thermal conductivity, the mass diffusivity does not depend upon position.

(Refer Slide Time: 09:47)



Momentum diffusion:

Momentum in the volume $\Delta x \Delta y \Delta z$

$= \rho u_x(x,y,z)\Delta x \Delta y \Delta z$

Rate of change of momentum

$$= \frac{[\rho u_x(x,y,z,t+\Delta t) - \rho u_x(x,y,z,t)]\Delta x \Delta y \Delta z}{\Delta t}$$

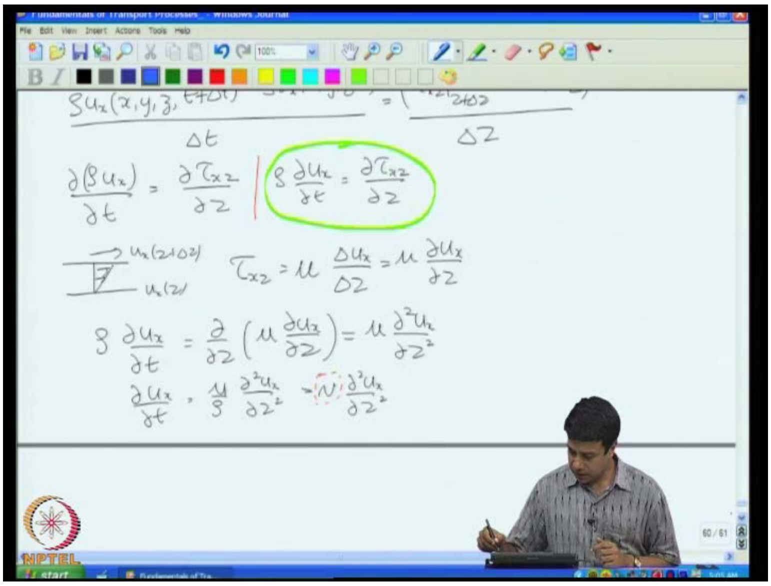
$$\begin{pmatrix}\text{Rate of} \\ \text{change of} \\ \text{momentum}\end{pmatrix} = \begin{pmatrix}\text{Sum of} \\ \text{applied} \\ \text{forces}\end{pmatrix}$$

Unit normal = Unit vector perpendicular to surface

in the x direction

The form of the equation for momentum diffusion was slightly different. It came out of Newton's law third law, Newton's laws of motion rate of change of momentum. In a differential volume is equal to the some of the applied forces and there I had defined for you in some detail what is the stress tau x z?

(Refer Slide Time: 10:46)



$$\frac{\rho u_x(x,y,z,t+\Delta t)}{\Delta t} = \frac{(\ \ )_{z+\Delta z}}{\Delta z}$$

$$\frac{\partial(\rho u_x)}{\partial t} = \frac{\partial \tau_{xz}}{\partial z} \quad \boxed{\rho \frac{\partial u_x}{\partial t} = \frac{\partial \tau_{xz}}{\partial z}}$$

$$\tau_{xz} = \mu \frac{\Delta u_x}{\Delta z} = \mu \frac{\partial u_x}{\partial z}$$

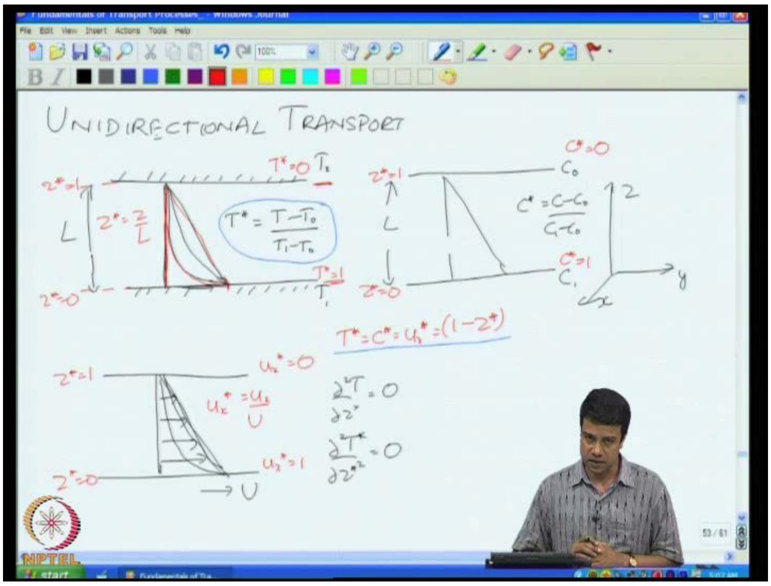
$$\rho \frac{\partial u_x}{\partial t} = \frac{\partial}{\partial z}\left(\mu \frac{\partial u_x}{\partial z}\right) = \mu \frac{\partial^2 u_x}{\partial z^2}$$

$$\frac{\partial u_x}{\partial t} = \frac{\mu}{\rho}\frac{\partial^2 u_x}{\partial z^2} = \nu \frac{\partial^2 u_x}{\partial z^2}$$

It is the force per unit area in the x direction acting at the surface with outward note outward. You are at normal in the Z direction and I also told you that if the outward unit normal is in the minus Z direction the force will be minus tau x Z because if you interchange if you if you if you reverse the direction of the unit normal the direction of the force also changes according to Newton’s third law.

And then we have, we basically use the balance condition that the rate of change of momentum is equal to sum of the applied forces to obtain an equation for momentum transfer which once again looks identical to the equations for heat and mass transfer. Except instead of concentration you have the velocity u x. Instead of mass diffusion coefficient you have the kinematic viscosity or momentum diffusivity nu; ratio of viscosity and density.

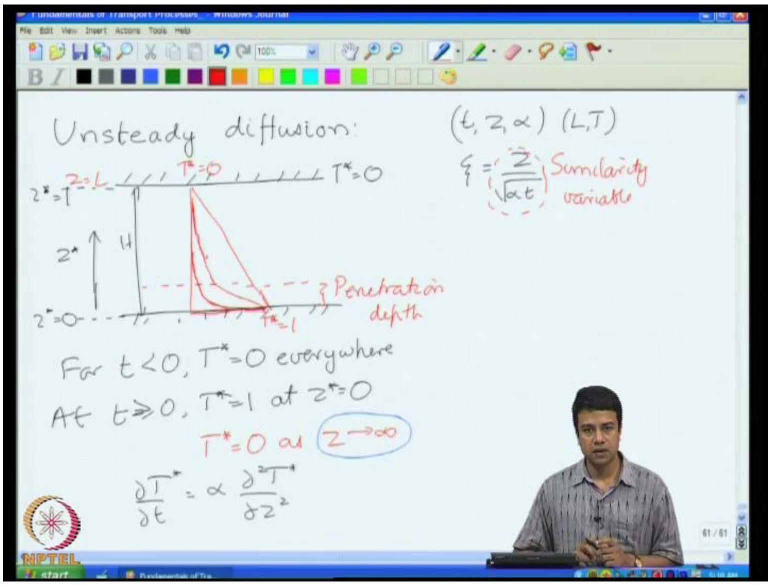
(Refer Slide Time: 11:25)



And we talked a little bit about scaling. I said that rather than work with these un scale coordinates alpha I am sorry T u x and c right I could define scaled coordinates. In this case T star is equal to T minus T naught by T 1 minus T naught. T star is equal to 0 on the top surface and T star is equal to 1 on the bottom surface. If I define Z star is equal to Z by l then Z star is equal to 1 on the top surface and Z star is equal to 0 on the bottom surface. Define this way; in all cases the concentration the scaled temperature varies between 0 and 1. The scaled concentration varies between 0 and 1 the scaled velocity varies between 0 and 1.

So the equations are the same except that one has to be put in the appropriate diffusivity the lengths all scale vary between 0 and 1. The the the quantity is being transported. Once again vary between 0 and 1. So, all of them have a common solution. This is the solution at steady state. The linear profile for the temperature concentration and velocity fluids. So, that is where we stopped off the last lecture. Now, we come to looking at the simplest unsteady state problem. That is the problem that I would just explained to you a little earlier if initially the temperature is equal to 0 everywhere and at some time T is equal to 0. The temperature at the bottom surface is increased to one then, what happens? How can you can we measure the temperature field in this unsteady situation? That is the topic for today's lecture.

(Refer Slide Time: 13:22)



Unsteady diffusion:

$(t, z, \alpha)\ (L, T)$

$\xi = \frac{z}{\sqrt{\alpha t}}$ Similarity variable

For $t<0$, $T^*=0$ everywhere

At $t\geq 0$, $T^*=1$ at $z^*=0$

$T^*=0$ as $z\rightarrow\infty$

$$\frac{\partial T^*}{\partial t} = \alpha\frac{\partial^2 T^*}{\partial z^2}$$

So let us go ahead with analyzing that problem. So, back to our problem two plates and I will just work in scaled coordinate Z star is this axis. I can scale by the thickness whatever the thickness is, I can scale it by the thickness to get this boundary at Z star is equal to 0 and this boundary at Z star equals 1. So, the scale distance varies between 0 and 1.

The boundary conditions in this case are as follows for T less than 0 right, I will assume that the concentration the I am sorry the temperature T star is equal to 0 everywhere. So, it is equal to 0 here and it is equal to 0 on this surface as well and at T is equal to at T is greater than or equal to 0 T star is equal to 1 at Z star is equal to 0. So, at the initial time instantaneously so initially if I plotted the temperature; the temperature was basically equal to 0 everywhere. Scale temperature I showed you in the last class how to scale it. So, this is the temperature T star is equal to 0 at the initial time. I instantaneously increase the temperature here to T star is equal to 1. So, initially the fluid has not felt the heat coming out of this plate. So, it is just at 0 temperature. Gradually there is going to be a diffusion of heat from the bottom because I have raised the plate of the raised the temperature of the bottom plate.

So, initially I am going to get a temperature profile that looks like this. That temperature profile is slowly expanded as a heat goes further and further up until in the longtime limit. I am going to get back a linear temperature profile. Let me just put that little more carefully. The longtime limit I am going to get this linear temperature profile and I want to find out how the temperature evolves from this initial state to the final linear state before we tackle the problem of the temperature evolution from the initial to the final steady state. Let us look at the very early instants of time when the temperature disturbance due to the bottom plate has not yet gone far into the fluid. So, if the temperature disturbance at the bottom plate generated due to instantaneously increase in the temperature of the bottom plate if it has not gone very very far up then this disturbance is going to be restricted to what is called a penetration depth. It is going to be restricted to a penetration depth which is small compared to the total height the total height was Z is equal to 0 to Z is equal to l. If the penetration depth is small compared to that; that means that the as as as far as the temperature field is concerned, the top surface is very far away. So, I do not have to know the exact location of the top surface. I just have to know that as I go further and further up. The temperature reaches 0 as the distance becomes very large ok.

So my problem rather than T star is equal to 1 at Z is equal to 0 and T star is equal to 0. Everywhere if the penetration depth is small compared to the total height then I can say that T star is equal to 0 as Z goes to infinity. When I go further and further up the temperature approaches in 0. In case the temperature in this case is approaching 0; in this case it is approaching 0 as I go further and further up. So, in that sense I am effectively solving a problem in an infinite domain. The problem reduces to a problem in an infinite domain because the penetration depth to which the the the temperature disturbance due to the bottom surface has been felt is very small compared to the total height.

So, this is the temperature boundary conditions that I will use and the differential equation for the temperature field that we had got was d T by d T is equal to alpha d square T by d Z square. So, this was the differential equation when we initially scaled variables. We scaled it by the total height of this. The distance between the two plates H. We will scale it by the total distance between the two plates. However, now I am considering a situation where the distance to the top plate is large compared to the penetration depth within the flow. So, I am effectively enforcing boundary conditions in the limit as Z goes to infinity. If the penetration depth is small compared to the height it does not matter what the height is. It can be H, it can be 2 H as long as a penetration depth is small. The only requirement for the temperature field is that the scale temperature goes to 0 as H goes to as as Z goes to infinity.

So previously, because I had a finite depth I scaled all length scales by H. In the present case H is no longer a parameter in the problem. So, how do I scale the distance Z? Let us go back and look at all the relevant variables in the problem. One relevant variable of course, is the temperature T star but, that is already dimensionless. T star at scaled as T minus T naught by T 1 minus T naught varies between 0 and 1. No units; it is dimensionless so that cannot be a dimension. That is that is important in this case. I have three others; two are the coordinates, one is the z coordinate, the other is time and the only other thing that I have in the problem is this diffusivity alpha.

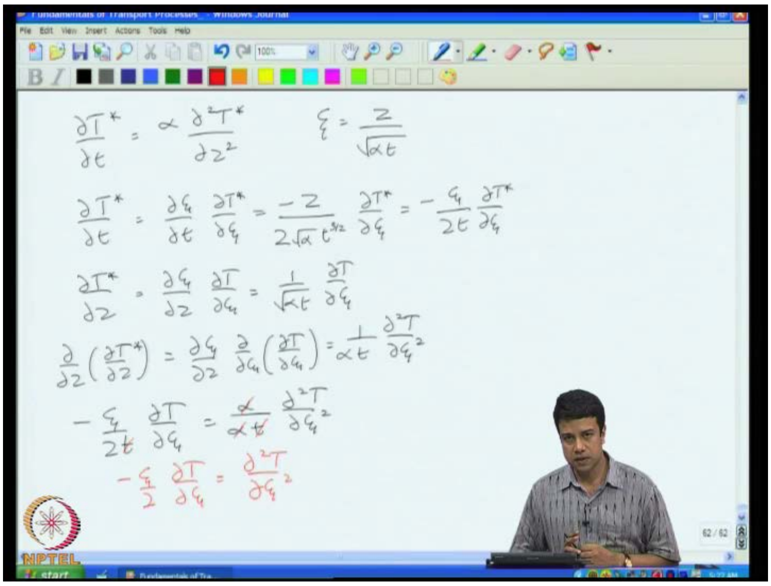
So there are there is one dimensionless variable T star, two coordinates Z and T and 1 dimensional parameter, the thermal diffusivity alpha. So, I do not have any length scale to scale Z by because initially at Z that the distance between the plates was H. But, then if the the the temperature field hasn’t penetrated that far right; it does not matter what H is. The temperature field near the bottom surface is not going to depend upon what the total height is so that is the issue here.

Now, the answer to this problem is as follows. I have three variables T, Z, alpha which have dimensions and there are only two dimensions that they can depend upon. They do not depend upon mass here, they depend only upon length and time. Therefore, from these three coordinates, I can get only one dimensionless group. I told you last class that we tried to we tried to a scale or variables. When we try to solve this problem so if I if I did the problem simplistically what I would do is to scale Z by a height and T by a time. In this case I do not have a height because the penetration depth is small so the the boundary conditions applied at Z going to infinity. I do not have a scale for time. Either out of these three; I can get only one dimensionless group and that dimensionless group is a similarity variable. That dimensionless group is a similarity variable Z by root alpha t.

So just from dimensional analysis; if the penetration depth is small so that I am applying a boundary condition at infinity the progression of the temperature field with Z and with time are linked and they are they are linked in such a way that the progression depends only upon this similarity variable or similarity coordinate and that effectively reduces the problem from two independent variables Z and T to just one independent variables which is a dimensional necessity. I am sorry which is a dimensional necessity because there are no other dimensionless groups that I can form in the problem.

So, if the temperature field depends only upon this a dimensionless similarity variable then I should be able to express my entire equation in terms of that dimensionless similarity variable alone. So, what I need to do is to express Z the derivative with respect to Z. In terms of the derivative with respect to psi the derivative with respect to T also can be expressed in terms of the derivative with respect to psi. Put that into this equation and if my premise is correct that it depends only through this variable psi and not individually through T Z and alpha. The final equation that I get should not contain any of these parameters T Z or alpha. It should be only a function of psi alone so that is that is the basic idea ok.

(Refer Slide Time: 24:29)



$$\frac{\partial T^*}{\partial t} = \alpha \frac{\partial^2 T^*}{\partial z^2} \qquad \xi = \frac{z}{\sqrt{\alpha t}}$$

$$\frac{\partial T^*}{\partial t} = \frac{\partial \xi}{\partial t}\frac{\partial T^*}{\partial \xi} = \frac{-z}{2\sqrt{\alpha}\, t^{3/2}}\frac{\partial T^*}{\partial \xi} = -\frac{\xi}{2t}\frac{\partial T^*}{\partial \xi}$$

$$\frac{\partial T^*}{\partial z} = \frac{\partial \xi}{\partial z}\frac{\partial T}{\partial \xi} = \frac{1}{\sqrt{\alpha t}}\frac{\partial T}{\partial \xi}$$

$$\frac{\partial}{\partial z}\left(\frac{\partial T^*}{\partial z}\right) = \frac{\partial \xi}{\partial z}\frac{\partial}{\partial \xi}\left(\frac{\partial T}{\partial \xi}\right) = \frac{1}{\alpha t}\frac{\partial^2 T}{\partial \xi^2}$$

$$-\frac{\xi}{2\cancel{t}}\frac{\partial T}{\partial \xi} = \frac{\alpha}{\alpha \cancel{t}}\frac{\partial^2 T}{\partial \xi^2}$$

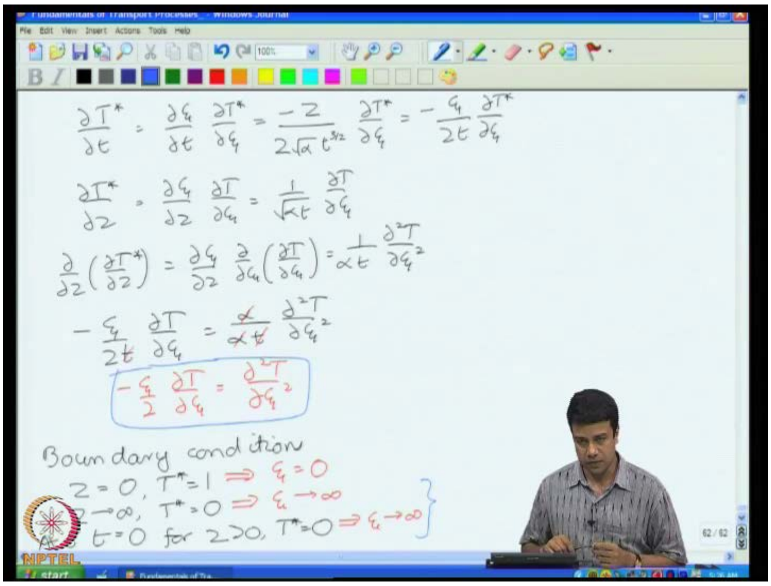
$$-\frac{\xi}{2}\frac{\partial T}{\partial \xi} = \frac{\partial^2 T}{\partial \xi^2}$$

So, let us proceed with the solution. So, I have my equation partial T star with partial T is equal to alpha partial square T star by partial Z square and I have defined my similarity variable psi is equal to Z by square root of alpha T. This is dimensionless because Z has dimensions of length alpha has a length square per unit time. Therefore, alpha T has dimensions of length square and therefore, Z by root alpha T is dimensionless. So, the derivative with respect to time partial T star by partial T is equal to my partial psi chain rule for differentiation partial psi by partial T is equal to minus Z by 2 root alpha 2 power three half's times partial T star by partial psi. Now, Z by root alpha T is psi itself Z by root alpha T is psi itself. Therefore, I can also write this as minus psi by 2 T partial T star by partial psi using chain rule for differentiation.

On the right hand side partial T by partial Z; first take the derivative respect to Z and then with respect to psi by chain rule this is 1 over root alpha T partial T by partial psi. In the second derivative, partial by partial Z of partial T by partial Z is equal to partial psi by partial Z partial by partial psi of and this is equal to 1 by alpha T partial square T by partial psi square. Now, I have to substitute this into the original diffusion equation therefore, I will get partial, I will rub it out, I will get minus psi by 2 T partial, T by partial psi. This is equal to alpha by alpha T partial square T by partial psi square and I can cancel carefully and you can cancel out alpha there and T here to get an equation of the form minus psi by 2 partial T by partial psi is equal to partial square T by partial psi square good. So, just to recall I had previously said that since by dimensional analysis there is only one dimensionless group which involves both the Z coordinate and time I should be able to define this entire, I should be able to rewrite this entire equation in terms of this dimensionless group alone.

Once I have written it that way the equation should not contain individually Z T and alpha; it should only contain this dimensionless variable psi and I do find indeed that the final equation that I get thus contains only this dimensionless variable psi.

(Refer Slide Time: 28:57)



$$\frac{\partial T^*}{\partial t} = \frac{\partial \xi}{\partial t}\frac{\partial T^*}{\partial \xi} = \frac{-z}{2\sqrt{\alpha}\, t^{3/2}}\frac{\partial T^*}{\partial \xi} = -\frac{\xi}{2t}\frac{\partial T^*}{\partial \xi}$$

$$\frac{\partial T^*}{\partial z} = \frac{\partial \xi}{\partial z}\frac{\partial T}{\partial \xi} = \frac{1}{\sqrt{\alpha t}}\frac{\partial T}{\partial \xi}$$

$$\frac{\partial}{\partial z}\left(\frac{\partial T^*}{\partial z}\right) = \frac{\partial \xi}{\partial z}\frac{\partial}{\partial \xi}\left(\frac{\partial T}{\partial \xi}\right) = \frac{1}{\alpha t}\frac{\partial^2 T}{\partial \xi^2}$$

$$-\frac{\xi}{2t}\frac{\partial T}{\partial \xi} = \frac{\cancel{\alpha}}{\cancel{\alpha}\,\cancel{t}}\frac{\partial^2 T}{\partial \xi^2}$$

$$\boxed{-\frac{\xi}{2}\frac{\partial T}{\partial \xi} = \frac{\partial^2 T}{\partial \xi^2}}$$

Boundary condition

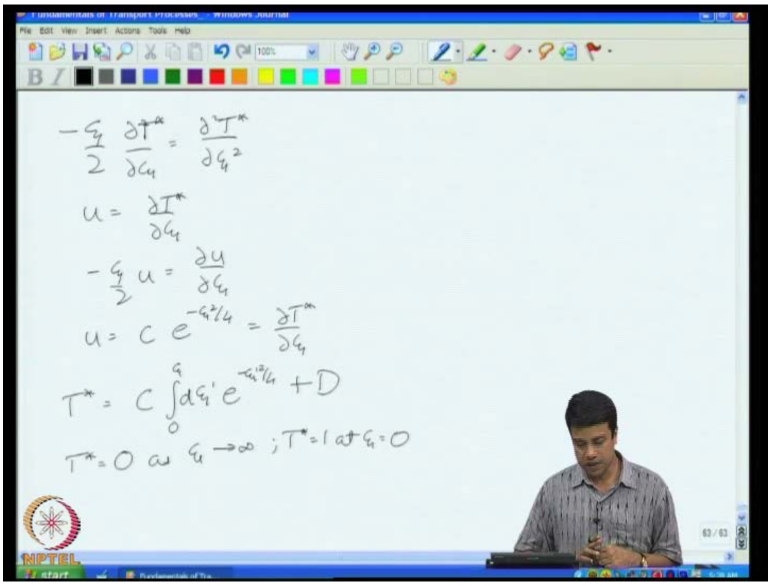
$$\left.\begin{array}{l} z = 0,\ T^* = 1 \Rightarrow \xi = 0 \\ z \to \infty,\ T^* = 0 \Rightarrow \xi \to \infty \\ t = 0 \text{ for } z > 0,\ T^* = 0 \Rightarrow \xi \to \infty \end{array}\right\}$$

But, however if I am to solve this equation; I need boundary conditions. What are the boundary conditions that we had at Z is equal to 0 T star is equal to 1? So, at the bottom plate the temperature T star is equal to 1 at Z is equal to 0. So, that Z is equal to 0 is equivalent to psi equal to 0 because psi is equal to Z by square root of alpha T. The other boundary condition was that in the limit as Z became large Z going to infinity T star was equal to 0. The limit as Z goes to infinity T star was equal to 0 Z going to infinity is equivalent to psi going to infinity psi going to infinity because Z psi is equal to Z by square root of alpha T. That was a third boundary condition rather an initial condition and that was that at T is equal to 0 at the initial time when I just switched on this temperature field at the bottom. Everywhere else within the flow within the fluid the temperature was equal to 0 at the initial time T is equal to 0 when I just switched on the heating from the bottom the temperature was equal to 0 everywhere else apart from that bottom plate.

So, that would imply that at T is equal to 0 for Z greater than 0 T star is equal to 0 right at time T is equal to 0 everywhere in the fluid apart from that bottom plate apart from Z. Z is equal to 0 the temperature is equal to 0 T is equal to 0 psi is equal to Z by square root of alpha T. So, this implies that this has to be applied at psi going to infinity and you can see that when expressed in terms of the psi coordinate; these two boundary conditions are exactly the same. I had started off with a second order differential equation in the Z coordinate, the first order differential equation in time that required two boundary conditions in the special coordinates. One initial condition in time and then I reduced the variables using the similarity transform to an equation just in terms of one variable psi. This equation was only in terms of the variable psi and because it is a second order equation in psi. You can have only two boundary conditions for this equation and therefore, what we find is that one of the boundary conditions and one initial condition of the original problem turns out to be exactly the same when re expressed in terms of psi. So, that we get a consistent solution.

The other thing that I have done here is to convert a partial differential equation with variations in time as well as Z into an ordinary differential equation and that is the big advantage of this method. The similarity solution was defined so that the partial differential equation which I originally had is in transform coordinates and ordinary differential equation and ordinary differential equation is simpler to solve. It is a linear ordinary differential equation and so I can solve it to get a solution for the temperature profile.

(Refer Slide Time: 33:00)



$$-\frac{\psi}{2}\frac{\partial T^*}{\partial \psi} = \frac{\partial^2 T^*}{\partial \psi^2}$$

$$u = \frac{\partial T^*}{\partial \psi}$$

$$-\frac{\psi}{2}u = \frac{\partial u}{\partial \psi}$$

$$u = C\,e^{-\psi^2/4} = \frac{\partial T^*}{\partial \psi}$$

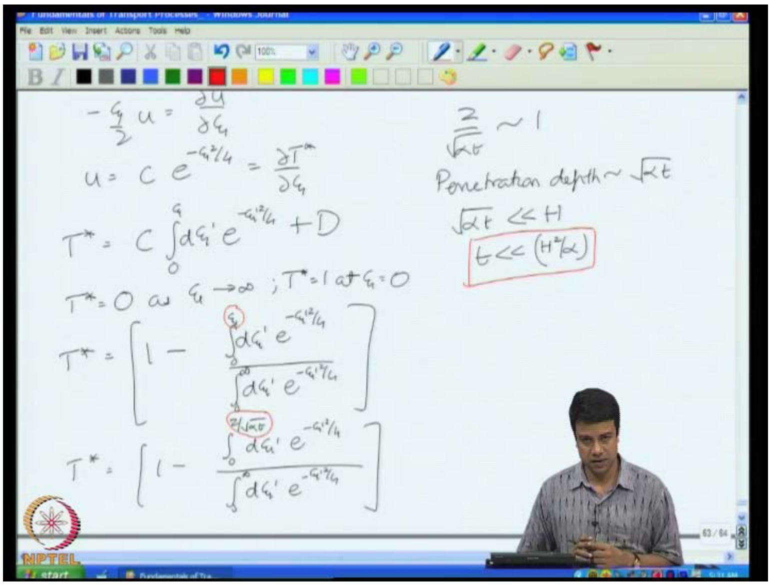
$$T^* = C\int_0^{\psi} d\psi'\,e^{-\psi'^2/4} + D$$

$$T^* = 0 \text{ as } \psi \to \infty \,;\, T^* = 1 \text{ at } \psi = 0$$

So my equation is minus psi by 2 d d T star by d psi is equal to T square T star by d psi square. So, I can solve this. Now in case so if I define u is equal to d T star by d psi then I will get minus psi by 2 u is equal to d u by d psi and u is equal to a constant e power minus psi square by four and this is equal to d T by d psi. Therefore, T is equal to c integral 0 to psi d psi prime e power minus psi prime square by four plus some other constant d.

And I have boundary conditions T star is equal to 0 as psi goes to infinity T star is equal to 1 at psi is equal to 0. Using this, I can easily determine the two constants. Using these two boundary conditions I can determine the two constants in equation.

(Refer Slide Time: 34:33)



$-\frac{\psi}{2} u = \frac{\partial u}{\partial \psi}$

$u = C e^{-\psi^2/4} = \frac{\partial T^*}{\partial \psi}$

$T^* = C \int_0^{\psi} d\psi' e^{-\psi'^2/4} + D$

$T^* = 0$ as $\psi \to \infty$; $T^* = 1$ at $\psi = 0$

$$T^* = \left[1 - \frac{\int_0^{\psi} d\psi' e^{-\psi'^2/4}}{\int_0^{\infty} d\psi' e^{-\psi'^2/4}}\right]$$

$$T^* = \left[1 - \frac{\int_0^{z/\sqrt{\alpha t}} d\psi' e^{-\psi'^2/4}}{\int_0^{\infty} d\psi' e^{-\psi'^2/4}}\right]$$

$\frac{z}{\sqrt{\alpha t}} \sim 1$

Penetration depth $\sim \sqrt{\alpha t}$

$\sqrt{\alpha t} \ll H$

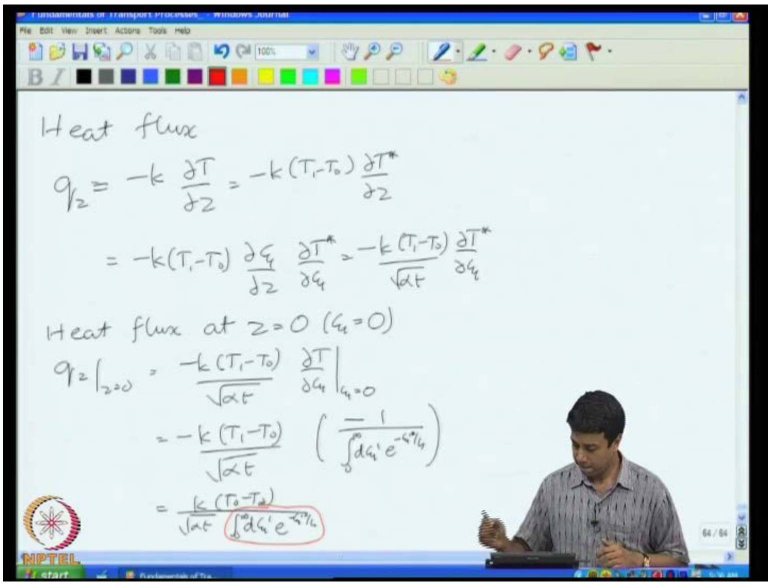
$\boxed{t \ll (H^2/\alpha)}$

So this will give me T star is equal to 1 minus integral 0 to psi d psi prime e power minus psi prime squared by four divided by integral 0 to infinity prime square by four. That is what I would get after I enforce both boundary conditions and I put in the solution the the the similarity transform. Once again that similarity transform you will note comes in only in the limits of this integration. Therefore, in terms of the original variables I can write T star is equal to 1 minus integral 0 to Z by root alpha T d psi prime e power minus psi prime square by four by integral 0 to infinity d psi prime. So, that is the final solution for the temperature field. The temperature field depends on position and time only through this scaled coordinate.

And now we are in a position to answer the question that we had originally asked what is the penetration depth? Clearly the temperature as Z by root alpha T becomes larger and larger in the limit as it becomes large it goes to infinity. The temperature will go to 0, the temperature deviates significantly from 0 only when Z by root alpha T is approximately 1, is of order 1. If it becomes large there is virtually no deviation from 0 therefore, the penetration depth in this case is proportional to square root of alpha times T it goes as square root of alpha times T.

So, the penetration depth is increasing as square root of T as time increases. The next question what did we mean initially when we said that the time is small enough that the penetration depth is small compared to the total height penetration depth is small compared to the total height. Only when square root of alpha T is small compared to the height H or when T is small compared to H square by alpha. So, this is the time period over which you can use a similarity solution when the transport problem looks like the transport into an infinite fluid because the height is large compared to the penetration depth.

(Refer Slide Time: 38:05)



Heat flux

$$q_z = -k \frac{\partial T}{\partial z} = -k(T_1 - T_0)\frac{\partial T^*}{\partial z}$$

$$= -k(T_1 - T_0)\frac{\partial \psi}{\partial z}\frac{\partial T^*}{\partial \psi} = -\frac{k(T_1 - T_0)}{\sqrt{\alpha t}}\frac{\partial T^*}{\partial \psi}$$

Heat flux at $z = 0$ $(\psi = 0)$

$$q_z\Big|_{z=0} = \frac{-k(T_1 - T_0)}{\sqrt{\alpha t}}\frac{\partial T}{\partial \psi}\Big|_{\psi=0}$$

$$= -\frac{k(T_1 - T_0)}{\sqrt{\alpha t}}\left(\frac{-1}{\int_0^\infty d\psi' e^{-\psi'^2/4}}\right)$$

$$= \frac{k(T_0 - T_d)}{\sqrt{\alpha t}\,\boxed{\int_0^\infty d\psi' e^{-\psi'^2/4}}}$$

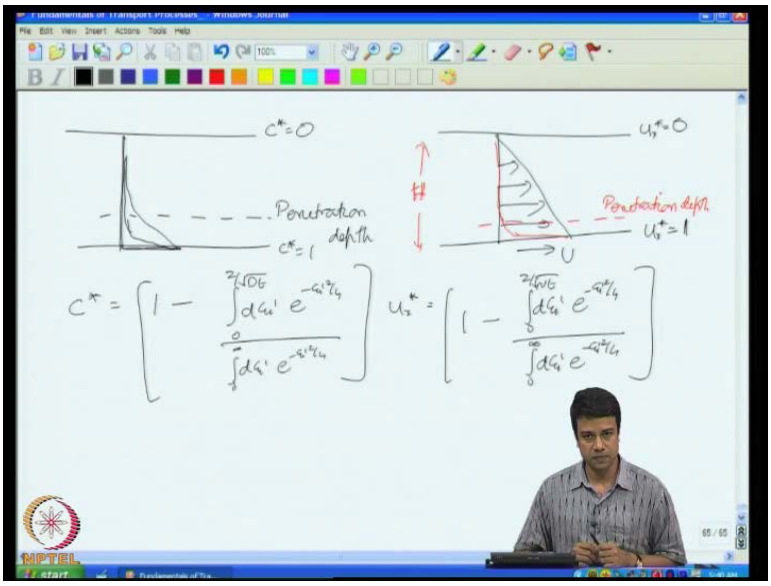
So this gives me the temperature profile at every instant in time, a second relevant question that one can ask is for example, what is the heat flux at the bottom surface? The heat flux q is equal to minus k times d T by d Z k is a thermal conductivity expressed in terms of the scale temperature. This will be equal to minus k into T 1 minus T naught into partial T star by partial Z because T star is T minus T naught by T 1 minus T naught. Therefore, d T star by d Z is d T d Z divided by T 1 minus T 1 good.

Now, this we have to express in terms of the scale variable psi. So, this when expressed in terms of the scaled variable psi becomes T naught into d Z d d psi by d Z times d T star by d psi and d psi by d Z is 1 over square root of alpha T this equals to minus k into T 1 minus T naught by square root of alpha T into d T star by d psi the flux at the bottom heat flux at Z is equal to 0 Z is equal to 0 also corresponds to psi is equal to 0 because psi is equal to Z by square root of alpha T so q Z at Z is equal to 0 is equal to minus k into T 1 minus T naught by root alpha T into d T by d psi at psi is equal to 0.

And now, you can use this solution for T in order to find out what is d T by d psi. You can clearly hear this constant as 1 over integral 0, 2 infinity of d psi prime. So, I use this solution the derivative with respect to psi will just remove this integral sign on the second term and substitute psi is equal to 1 here. So, this is equal to minus k into T 1 minus T naught by square root of alpha T into into minus 1 by integral 0 to infinity d psi prime e power minus psi prime so squared by four. So, this basically is equal to k into T naught minus T infinity by root alpha T integral 0 to infinity d psi prime e power minus psi prime square by four. So, this term here this is just a constant it is a definite integral from 0 to infinity. So, I am getting the heat flux decreasing with time as 1 over square root of T the penetration depth increases the heat flux is the gradient of the temperature. It will go approximately as the temperature divided by the penetration depth as a penetration depth increases the heat flux decreases as 1 over square root of T in this particular problem.

So, to recall we use similarity transform because there were no length or time scales in the problem with which we would scale length and time by and because of that we manage to get a solution which was only in terms of a similarity variable. Now, this similarity transform can also be applied for mass and momentum transport equations in problems. The only thing is that I would substitute the mass diffusivity for the thermal diffusivity. Here in my similarity variable I would substitute the momentum diffusivity kinematic viscosity nu for the thermal diffusivity here in this problem.

(Refer Slide Time: 42:58)



$$C^* = \left[1 - \frac{\int_0^{z/\sqrt{Dt}} d\psi' \, e^{-\psi'^2/4}}{\int_0^{\infty} d\psi' \, e^{-\psi'^2/4}}\right] \quad u_x^* = \left[1 - \frac{\int_0^{z/\sqrt{\nu t}} d\psi' \, e^{-\psi'^2/4}}{\int_0^{\infty} d\psi' \, e^{-\psi'^2/4}}\right]$$

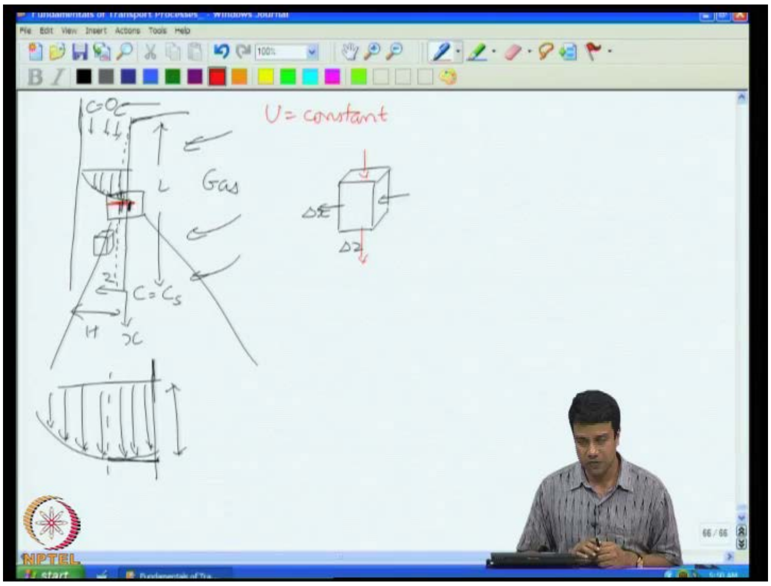
So for example, if I considered mass diffusion problem where there was diffusion from a surface c star is equal to 0 here and c star is equal to 1 here and initially c star was 0 everywhere and then I considered the problem of early times you start off initially with this profile then you you had a profile like this where the penetration depth was small compared to the total height right is exactly the same equation and I can write down the solution c star is equal to 1 minus integral 0 to Z by root d T d psi prime e power minus psi prime square by 4 by integral 0 to infinity.

So, in this case the concentration was 0 everywhere at the initial time T is equal to 0 I said the concentration of the bottom plate alone equal to 1 and then watch the diffusion from there equivalent momentum transport problem. So, I have u x is equal to 0 everywhere at time T equal to 0; I set this velocity into motion with a velocity u in case that u x star which is a scale velocity is equal to 1. In the final state of course, I will have a linear velocity profile but, in the very initial stages the momentum transfer will be limited to a thin region near the base where the penetration depth is small compared to the total height H.

So in this case, I can also write down quite easily what the solution for the velocity is going to be equal to square root of Z by root nu T where nu is the kinematic viscosity. So, you get exactly the same solutions. These are semi infinite domains where you have one surface an an infinite fluid on top and either the concentration temperature of the velocity of the bottom surface instantaneously changed and then you find the diffusion going upwards. So, we do that by this similarity transform. In this case it was a dimensional necessity the the fact that there is only one dimensionless group was a dimensional dimensional necessity.

However, this similarity transform is more general than that we will use it later on when we do boundary layer theory. We will repeatedly use similarity transforms to reduce the equation from a partial deferential equation to an ordinary differential equation.

(Refer Slide Time: 46:25)



And I can show you one example here where such a thing is done. We will consider here the simplest example where we can use the similarity transform not based upon dimensional analysis but, rather based upon physical understanding and that is the mass transfer to a falling film of fluid.

Let us say that I have a film of fluid falling downwards along the wall of of of some contactor and there is gas here, there is gas at the surface. So, at the entrance so the the the there is there is an entrance of towards the fluid then it follows falls as a thin film along the wall and it comes into contact with gas along the wall in the idea is to dissolve this gas into this film of fluid. Now, the film of fluid in general will have some velocity profile like this. We will assume that this width of this film is H and the total length one that it travels is now l and there is this velocity profile very close to the surface.

Now, clearly as the gas in the film at the entrance the concentration of the the the the solute that is being absorbed absorbed from the gas c will be equal to 0. At the entrance as the solution proceeds the concentration increases as you go downwards at the interface itself the concentration is a constant value. It is equal to the equilibrium concentration. The solubility c s in that is in equilibrium with the gas at its particular temperature pressure and concentration so we put a coordinate system here as usual. I will take the depth here as Z depth inverts into the film and this stream wise flow direction as x.

And now, I want to find out what is the concentration in the liquid as function of time my final objective is to get a Nusselt number correlation for this; a correlation similar to the correlations that we had discussed when we did dimensional analysis. Can we get a correlation for this as a function of l, the total length H, the fluid velocity and the diffusivities? So, that will be the final objective this is now not an unsteady problem. It is a steady state problem. However the concentration is now changing as a function of x. So, transport in this case is no longer unidirectional. There is a variation of concentration in the Z coordinate. There is also a variation of concentration in the x coordinate. The reason that I am doing it at this stage where we are discussing unidirectional flows will become apparent a little later on. Turns out that the solution for this configuration is exactly the same that you get for an unsteady flow into an infinite fluid.

Now, as the gas comes in contact you would expect that initially the concentration is at is is is 0. In the fluid as it comes into contact there will be a thin layer of fluid at the surface where the effect of this the dissolution of the gas at the interface is felt this is the equivalent of the penetration depth. In our previous case in that case the penetration depth increases with time as as time progresses the the the solute at the bottom dissolves further and further into the liquid. In this case it is as a function of the x coordinate as as time increases the the the gas at the interface dissolves further and further into the liquid that is trying to dissolve the the gas.

Now, so I have two coordinates here z and x and I have a velocity field I am going to make an assumption and that is that this velocity is a constant as far as the flow is concerned. I am going to make an assumption that u is equal to constant. This is an assumption this is valid if and only if the depth to which the concentration has penetrated is small compared to the total height H. in our In other words, I require that the change in velocity between these two positions has got to be small compared to the magnitude of the velocity itself. So, if I expand out this region alone I have this interface I have this penetration depth here and then I have a velocity profile along this. I have a velocity profile that goes like that my constant velocity approximation is valid only if the change in velocity in this region is small compared to the total velocity itself. If the change in velocity is small; I can assume that the velocity is approximately a constant over the thickness over which there is penetration of the gas similar to our assumption in the previous case that the penetration depth is small compared to the total height I have to come back and justify this ok.

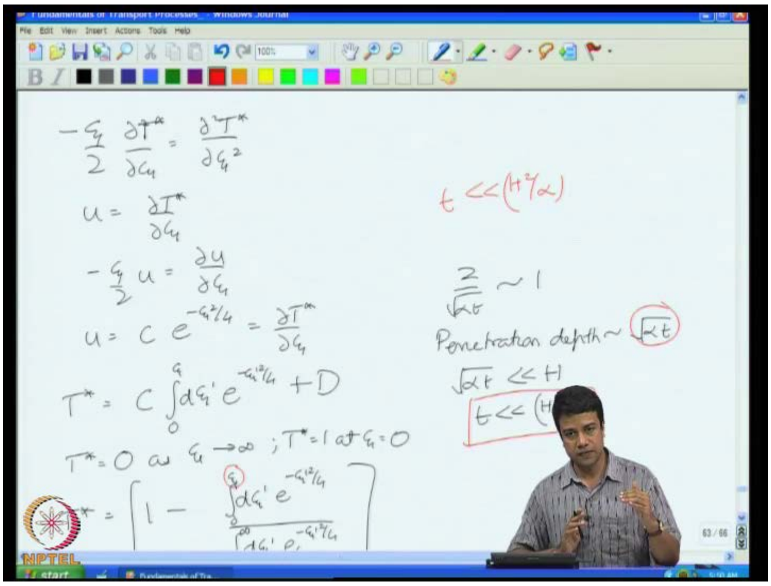
So, the balance equations are the same as before except that now we have the system which is at steady state nothing is changing in time but, there is a change in the x direction. So, I will take a differential volume this is delta Z perpendicular to the film delta x is along the flow direction. A second assumption that we will make, we will neglect diffusion along the x direction there is flow along the direction but, we will assume that there is no diffusion. That is assumption number two and we will come back and look at under what conditions that assumption is valid for the moment. We will assume that there is only diffusion only in the Z direction and there is convective transport because the fluid is flowing downwards there's going to be convection in the x direction and a balance between this convection and diffusion is what is going to give me my differential equation by the shell balance procedure.

So, this is the problem I will do the shell balance in the next lecture and then solve it and get a solution for this and hopefully from that we will get our first correlation between the nusselt number and all of the other parameters in the problem. So, we will see that in the next class. I will briefly review before we leave what we did in the present class. The solution procedure that we used was what is called the similarity solution. That similarity solution in this case it works only when there are no length or times scales in the problem. I had an unsteady problem and Z is equal to 0. I had a boundary condition. Now, I had another boundary condition as Z goes to infinity and an initial condition at T equal to 0, there were no length or time scales in this problem.

And therefore, when I post my differential equation I had a deficit of dimensions to scale my variables with there's no length scale Z going to infinity, no length scale in the problem nothing to scale Z with there was no inherent time scale in the problem. The only thing that I had in the problem was the diffusivity alpha and from the diffusivity alpha and the Z and T. I could get only one dimensionless group. So, this implied that just based upon dimensional analysis there is only one independent dimensionless variable which involves both Z and T and that was related in and that was related as follows.

If this is true then my entire differential equation I should be able to express in terms of psi alone. The differential equation cannot depend independently on alpha and Z and T. It should depend upon psi alone and that is what I got. I got a final equation here which depends only upon psi alone and not individually on Z and T not sufficient for the equations to be reduced to just an equation for psi. If you would be able to reduce the boundary conditions as well and the original problem had two boundary conditions; one initial condition, the problem in terms of the similarity variable should have only two conditions because it is a second order differential equation in psi alone. And I showed you that one boundary condition and one initial condition when expressed in terms of psi reduce to the same condition.

(Refer Slide Time: 56:14)



$$-\frac{\xi}{2}\frac{\partial T^*}{\partial \xi} = \frac{\partial^2 T^*}{\partial \xi^2}$$

$$u = \frac{\partial T^*}{\partial \xi}$$

$$-\frac{\xi}{2}u = \frac{\partial u}{\partial \xi}$$

$$u = C e^{-\xi^2/4} = \frac{\partial T^*}{\partial \xi}$$

$$T^* = C\int_0^{\xi} d\xi' e^{-\xi'^2/4} + D$$

$$T^* = 0 \text{ as } \xi \to \infty \text{ ; } T^* = 1 \text{ at } \xi = 0$$

$$T^* = \left[1 - \frac{\int_0^{\boxed{\xi}} d\xi' e^{-\xi'^2/4}}{\int_0^{\infty} d\xi' e^{-\xi'^2/4}}\right]$$

$$t \ll (H^2/\alpha)$$

$$\frac{z}{\sqrt{\alpha t}} \sim 1$$

Penetration depth ~ $\boxed{\sqrt{\alpha t}}$

$$\sqrt{\alpha t} \ll H$$

$$\boxed{t \ll (H}$$

Once that was done then I had a way to solve the problem and from that we found that the penetration depth is increasing as square root of alpha times T and obviously if you had a finite channel the analysis that we did will be valid only when T is T is small compared to H square by alpha. At these very initial times this analysis will be valid when T becomes comparable to H square by alpha then the influence of the of the boundary condition of the top plate is felt and this analysis is no longer valid and I showed you what the solution was and the heat flux decreases at T power minus half and I told you that you can get exactly the same solutions for heat and mass for mass and momentum transfer I have set up this problem for you. We will continue this in the next class and try to get a correlation for the nusselt number. We will see you in the next lecture.