Fundamentals of Transport Processes

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Module No. # 07
Lecture No. # 40
Summary

Welcome to this the final lecture in our course on fundamentals of transport processes and I would like to use this lecture to briefly review the main lessons that we have learnt in this course. I first started off trying to go through the kinds of analysis that I used for unit operations, in that case we have correlations which basically relate the dimensionless fluxes the nusselt number, the Sherwood number, the friction factor or the drag coefficient to other dimensionless groups. We looked at various kinds of dimensionless groups the most important ones that we use basically give a ratio of convection and diffusion. Ratio the Reynolds number which is ratio of momentum convection and diffusion. The peclet number ratio of mass and heat convection and diffusion.

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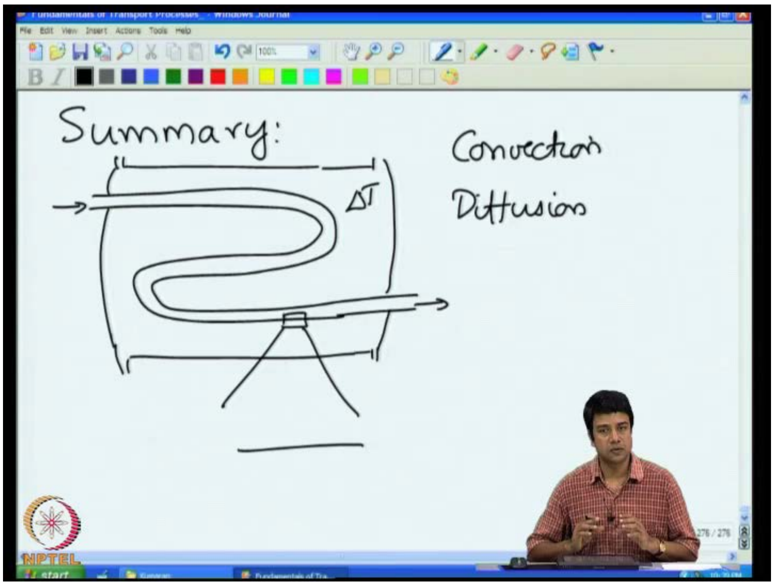

As well as other dimensionless numbers which give the ratio of diffusivities of different quantities for example, the Schmidt number ratio of momentum diffusion to mass diffusion or the Prandtl number the ratio of momentum diffusion to heat diffusion. All of the dimensionless groups that we have all contain ratios of these diffusions. So, for example, we had correlations in unit operations for nusselt numbers for heat transfer from a pipe which involved the Reynolds number the Prandtl number the Sherwood number correlations were diffusion from a particle. The friction factor for the flow in a pipe 16 by R e and these are all macroscopic quantities.

They basically tell you how the total heat transfer from the entire heat exchanger they tell you basically how the total heat transfer from the entire heat exchanger depends upon the average temperature difference between the shell side and the tube side. So, these are correlations that are written down for average quantities. What we wanted to do in this course is actually to write down microscopic transport equations which basically gave some idea microscopically how transport was taking place very close to a particular surface and then use averaging in order to find out the total transport rates.

So, rather than looking at the entire heat exchanger with the shell side (No audio from 03:12 to 03:21) and a tube side with some average temperature difference between the two. What we wanted to do was to look in detail at the surface itself and find out locally what is the difference what is the heat flux due to the local difference in temperature between the two sides and then use this local information in order to find out what is the total temperature heat flux throughout the entire heat exchanger. In this process there are two main mechanisms involved one is what is called convection and the other is diffusion. Convection is transport of heat mass momentum due to the the the flow of the fluid itself.

So, in this particular case there is convection at the inlet and there is convection at the outlet. So, convection is bringing in for example, cold fluid into the heat exchanger and convection is taking out hot fluid from the heat exchanger so, convection is responsible for transport of mass heat and momentum along the direction of the flow. Diffusion is transport which takes place due to the fluctuating motion of the molecules, whenever you have a non 0 temperature there is always thermal energy in the molecules. So, these molecules are in the state of rapid thermal fluctuations and these fluctuations can also transport mass heat and momentum.

So, our idea was to actually find out the competition between convection and diffusion locally at every point within the flow and from that to determine the performance of the entire system as a whole. So, there are two issues here, one is how does a balance between convection and diffusion determine the fluxes locally of heat mass and momentum. The second issue is from the knowledge of the local dependence of convection and diffusion on the temperature differences and so on. How can we determine the total heat transport to get the correlation between the nusselt number and the peclet number for example? So, convection is transport along the direction of the fluid velocity.

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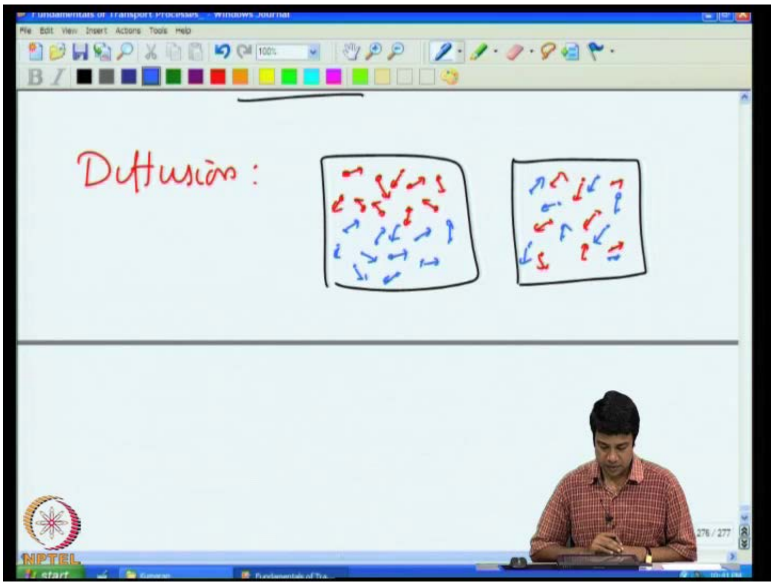

Diffusion :

Diffusion: is transport due to the fluctuating motion of the molecules so, if I had a container in which I initially had molecules of one type and molecules of another type and each of these types of molecules are in a state of fluctuating motion. (No audio from 06:22 to 06:33) So, I have two types of molecules which are separated and each of these are in a state of fluctuating motion after some time the system is going to get mixed.

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There is going to be an equalization of the concentration of the two types of species across the entire system and the reason is going to get mixed is because of the fluctuating motion of the molecules.

There is no net motion here the centre of mass does not move there is no net fluid flow. However because of the fluctuating motion there is going to be an equalization of the concentration field and this process is the process of diffusion and we looked at in detail the diffusion in gases. If you recall we looked at mass diffusion in gases in some detail. We had a concentration gradient so, if you plot concentration as a function of this height set.

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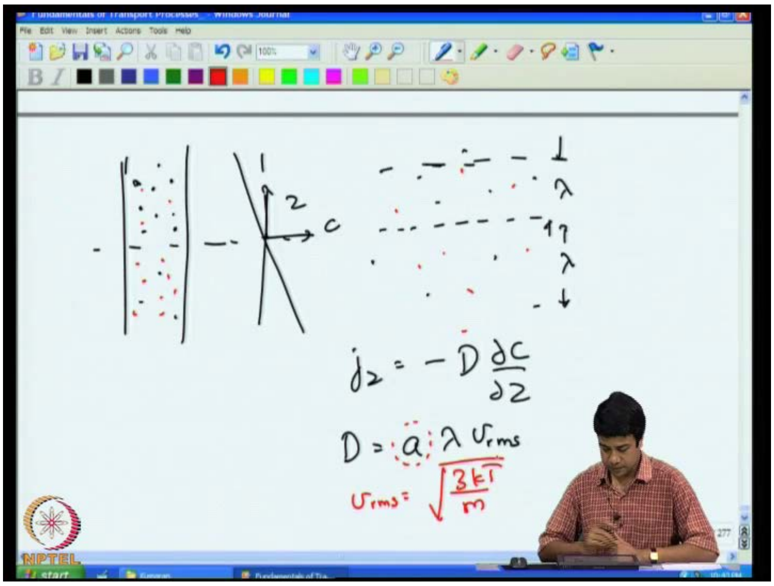

I had more molecules of one type here and molecules of another type here and if you wait for some time there is a net transport of the red molecules upwards and the blue molecules downwards. Physically the reason in gases at least is because if I have one particular plane and I have a concentration gradient across that plane. The molecules that are coming downwards are coming down on average from a distance of the mean free path above the surface whereas, the molecules that are going upwards are going on average from a distance mean free path below the surface.

Between locations one mean free path above and one mean free path below the surface there is a difference in concentration that difference in concentration is going to result in a net flux mass transported per unit area per unit time. We had we we had actually obtained an expression for the flux based upon the concentration gradient and if you recall the flux let me write this minus D times d C by d Z where D is equal to some constant times the lambda times the root mean square velocity of the molecules. Where a is some constant of order one we had seen the the the form of that particular constant, but a is some constant of order one times the mean free path times the fluctuating velocity of the molecules.

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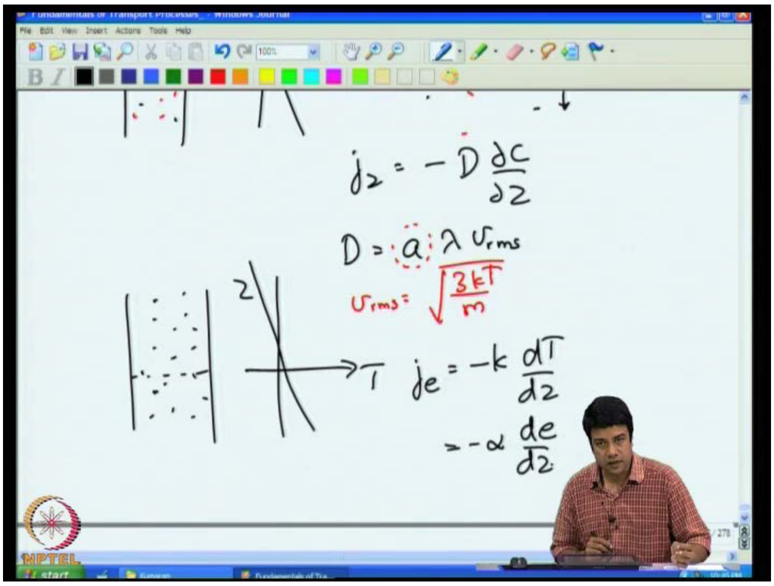

$$j_z = -D\frac{\partial c}{\partial z}$$

$$D = a\,\lambda\,v_{rms}$$

$$v_{rms} = \sqrt{\frac{3kT}{m}}$$

$$j_e = -k\frac{dT}{dz}$$

$$= -\alpha\frac{de}{dz}$$

Where V r m S is equal to square root of 3 k T by m. So, this diffusivity is a velocity times a length it has dimensions of length square per unit time and this is the diffusion coefficient for mass in all gases the diffusion coefficient will go approximately as the mean free path times the root mean square fluctuating velocity of the molecules in a manner similar to mass diffusion one can also define heat diffusion. In this particular case there is a gradient in temperature, the temperature is a linear function of position and therefore, the molecules on top have a lower fluctuating energy than the molecules below there is a temperature gradient. So, molecules on one side have a lower fluctuating velocity on the other side they have a higher fluctuating velocity.

The method for calculating the diffusion coefficient is the same as in the case of mass diffusion, a molecule crossing a surface comes from a distance of the order of one mean free path above the surface the molecule below the surface comes from a distance of the order of one mean free path below the surface and because of that there is going to be a net transport of energy even when there is no net transport of mass. Because faster moving molecules are coming down they are carrying their energy down slower moving molecules are moving upwards and they are carrying the lack of energy upwards and because of that there is going to be a net transport of energy and that energy flux can be written as minus k times d T by d Z where k is the thermal conductivity this can also be written in terms of the diffusivity minus alpha times d e by d z. Where alpha is the thermal diffusivity is equal to k by rho C .

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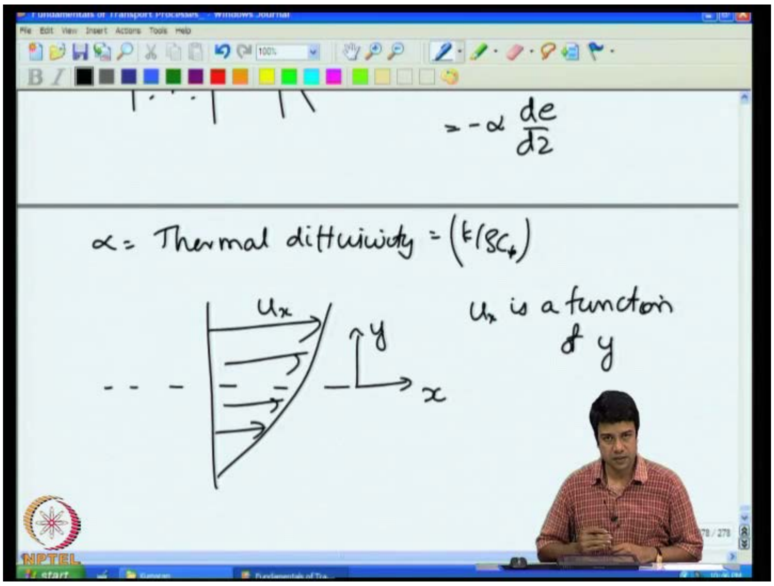

$= -\alpha \frac{de}{dz}$

α = Thermal diffusivity = $(k/\rho C_p)$

u_x is a function of y

So, this is for the diffusion of energy and finally, the diffusion of momentum we consider a surface across which there is a net velocity. So, this is the x direction this was the y direction and I had a velocity u x which was a function of y, u x is a function of y. So, u x is the mean velocity of the molecules and the mass times u x is the mean momentum of the molecules and this velocity will result in a transport of momentum and there is physical reason is the same across the surface.

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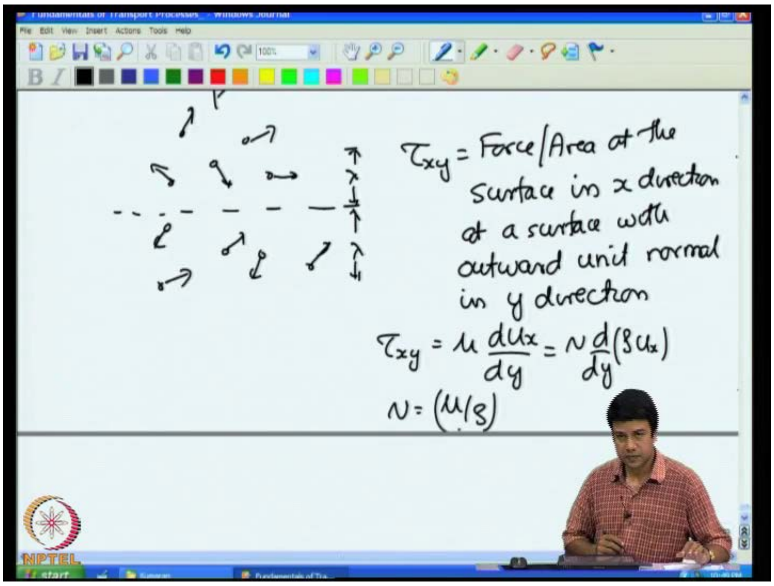

τ_{xy} = Force/Area at the surface in x direction at a surface with outward unit normal in y direction

$$\tau_{xy} = \mu \frac{du_x}{dy} = \nu \frac{d}{dy}(\rho u_x)$$

$$\nu = (\mu/\rho)$$

So, is the mean velocity is the function of position molecules above the surface have on average a higher mean velocity than the mean velocity at the surface molecules below have lower. Molecules that are coming downwards are coming from a distance of the order of the mean free path above the surface whereas, molecules going above are coming from a distance of the order of the mean free path below the surface, that means molecules that coming downwards have a net higher momentum molecules going upwards have a lower momentum that results in a net momentum transport from above to below.

In the case of momentum transport we had defined tau x y is equal to force per area at the surface in x direction at a surface with outward unit normal in y direction. So, this is a little different from what we use for mass and heat transfer mass and heat transfer typically the within a region the mass will increase if the flux is directed along the inward unit normal the mass will increase and the flux is directed along the inward unit normal whereas, for the stress we define the force with respect to the outward unit normal that is the convention. So, we use the same convention and this can be written as tau x y is equal to mu times the d u x by d y. I can also write this as the kinematic viscosity times D by D y of rho times u x where nu is equal to mu by rho .

Once again the kinematic viscosity is a momentum diffusivity it has dimensions of length square per unit time the thermal diffusivity has dimensions of length square per unit time the mass diffusivity also has dimensions of length square per unit time. So, these three are the constitutive relations for mass momentum and energy so, this is the constitutive relation the ficks law for mass transfer this is the fouriers law for heat transfer in case the flux is the function of the gradient in the temperature and finally, we have the newton’s law for the stresses for momentum transfer.

All of these have a common form transport of a quantity per unit area per unit time is equal to the diffusion constant times the variation in the density of that quantity per unit length. So, rho times u x is the momentum density concentration is the mass density and e is the energy density so, the transport of a quantity per unit area per unit time is equal to the diffusion coefficient times the gradient of the density of that quantity mass momentum or energy and all of these have identical forms in all cases the mass momentum and energy diffusivities in gases are all proportional to the mean free path times the fluctuating velocity of the molecules in gases that is because, the transport mechanism is the same in all gases mass momentum and energy transport take place only due to the physical motion of the molecules in gases, because the distance between molecules is large compared to the molecular size.

In liquids that is not strictly speaking true the transport the mass diffusion is a slow process. Because for a molecule for mass diffusion to take place that has to be the physical motion of molecules. A molecule has to move from one location to the other for diffusion to take place. In liquids the distance between molecules is comparable to the molecular diameter and therefore, if this particular molecule is to move in one particular direction all other molecules have to move out of the way and that requires what is called cooperative motion. Molecules have to move in a cooperative manner for the transfer for the mass transfer of one molecule and because of this it is a very slow process the assumption of the mean free path times the molecular velocity does not quite give you the correct coefficient in this case.

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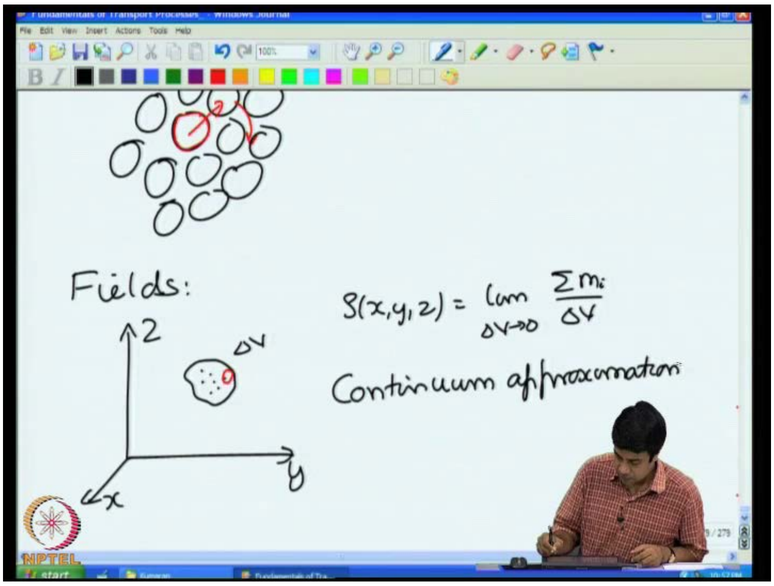

Fields:

$$\rho(x,y,z) = \lim_{\Delta V \to 0} \frac{\sum m_i}{\Delta V}$$

Continuum approximation

In gases all diffusion coefficients under for molecules such as hydrogen, nitrogen oxygen at standard room temperature and pressure u S 10 power minus 5 in metre square per second. In liquids they go typically as 10 powers minus 9 for small molecules and even lower 10 powers minus 11 to minus 13 for larger molecules. Momentum transport does not require physical motion of molecules, momentum can be transported even due to intermolecular forces and because of that the momentum transport in liquids is typically much faster than mass transport. The Schmidt number could be of the order of a 100 or so, in most liquids.

Energy transport once again it depends upon the mechanism of energy transport in liquid metals for example; the transport is due to the outer electronic shells around the molecules. Which are shared among all the molecules because of that the transport is a very fast process and due that the prandtl number is actually very small in liquid metals whereas, in non conducting liquids the transport takes place due to the physical rotational or vibrational motion of the molecules and there it is relatively slow process?

So, on this basis we had rationalized why the diffusion coefficients have the values that they have in liquids and in gases. The important a picture to keep in mind is this picture of the fluctuating motion of the molecules transporting either mass momentum or energy. Once this fluctuating motion of the molecules is isotropic it takes place with equal velocity in all directions. Convection is not an isotropic process it is directional it takes place only along the direction of flow. So, we have these two effects one is diffusion which takes place with equal magnitude in all directions and convection which has a definite direction to it and the balance between these two is going to give us the transport rates in real systems.

So, this basically tells us if we take a small little differential volume how the transport is going to takes place within that volume. The next step is from this small differential volume how do we look at an entire system and that involves a process of integration. Before that we have to define fields what we are dealing with here is what are called fields in our analysis procedure we will be dealing with fields that is fields of mass momentum and energy and one has to define these fields carefully .

At the microscopic level any volume that you take is made up of molecules is made up of molecules and the density field the density field for this differential volume delta V rho at x y z is equal summation of the mass of all the molecules within this differential volume by the volume itself. So, you find out what is the total mass of molecules of all the molecules within this differential volume divide by the volume itself and you take the limit as the volume goes to 0 in the limit as the volume goes to 0 this is the density at a fixed point in space this definition of course, has a problem in the sense that if I make the volume smaller and smaller as the volume becomes comparable to the molecular to the size of a molecule itself. So, volume becomes comparable to the size a molecule itself then either I have a molecule there or I do not have a molecule there.

So, this definition will not work if the size of the differential volume in considering is comparable to the size of a molecule itself. The differential volume that I consider has to be large compared to the molecular volume in order for this definition to make sense so, that I have a large number of molecules within this differential volume and it does not really matter whether one comes in or goes out this makes only a small change to the total number of molecules within the system so, long as the differential volume is large compared to the molecular size such that I have a large number of molecules within this differential volume I can use this continuum approximation. Where I consider the density to be a continuous field defined at every point within the flow this volume has to be small compared to this volume has to be large compared to the molecular size.

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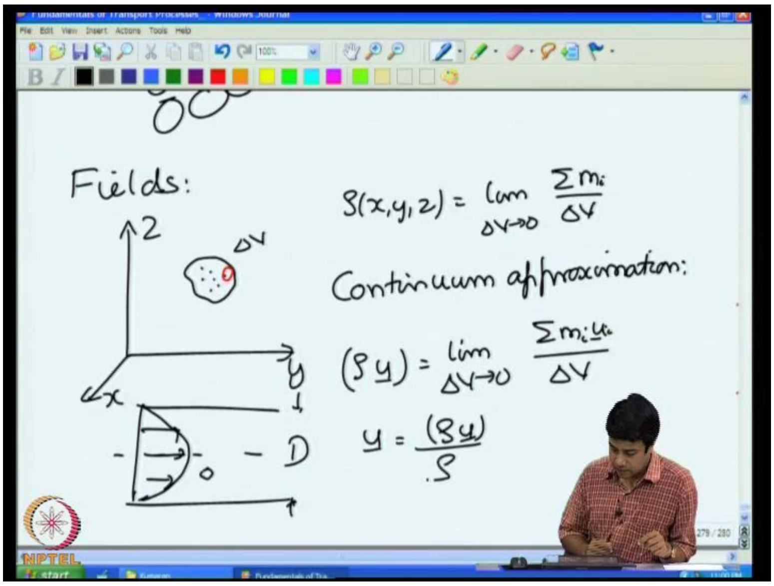
Fields:

$\rho(x,y,z) = \lim_{\Delta V \to 0} \frac{\sum m_i}{\Delta V}$

Continuum approximation:

$(\rho u) = \lim_{\Delta V \to 0} \frac{\sum m_i u_i}{\Delta V}$

$u = \frac{(\rho u)}{\rho}$

Of course, this volume has to be small compared to the scale so for example, if I am considering the flow in a pipe. The differential volume that I consider has to be small compared to the radius or the diameter of the pipe because if it is not small compared to the diameter of the pipe I cannot write down a unique value for the velocity there. The velocity is a varying function it is it is varying as a function of position in order to be able to define a unique value for the velocity I require that this differential volume has to be small compared to the radius of the pipe, but still large compared to the molecular size.

Therefore, in this intermediate regime where the differential volume that I consider is small compared to the macroscopic scale size of the pipe, size of a particle in the case of diffusion from a particle, but it is still large compared to the molecular size in this intermediate regime I can define fields which have continuously varying functions of position defined at every point within the space for the density temperature velocity. Physically what these mean is that I have taken a small differential volume around that particular location the size of the differential volume is large compared to molecular size, but still small compared to the macroscopic scale and for that differential volume I find out the total mass within the differential volume of all the molecules divide by the volume.

As I take delta V going to 0 in comparison to the macroscopic scale, but still large enough that it contains a large number of molecules I define, I can get a definition for the density the sum of the mass of all molecules divide by velocity divided by the volume. One can get a similar definition for momentum rho times u the total momentum is equal to limit as delta V goes to 0 of summation of m I times u I note that u is a vector u is a vector because the velocity is a vector. So, I have to take the vector sum of the velocities of all the molecules and add them all up and divide by the total volume to get the momentum density to get the momentum density momentum per unit volume and the velocity is just equal to the momentum density divided by the momentum itself that is the mean velocity.

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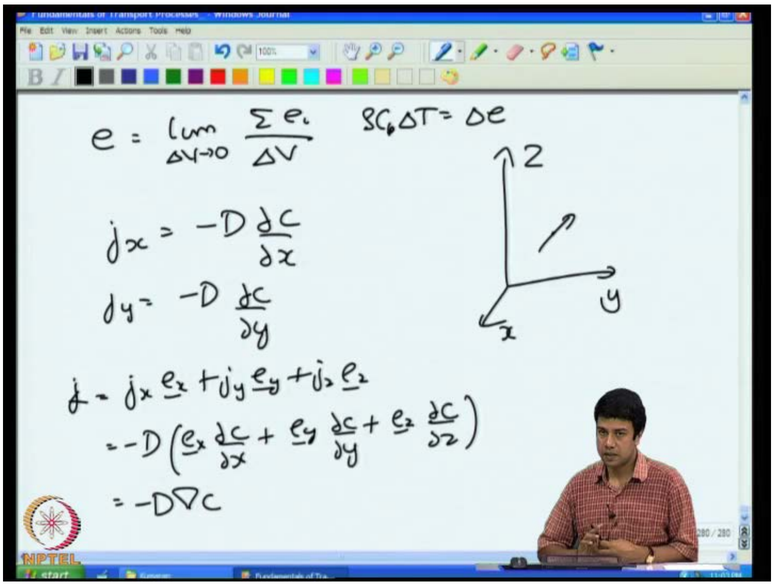

$$e = \lim_{\Delta V \to 0} \frac{\Sigma e_i}{\Delta V} \qquad \rho C_p \Delta T = \Delta e$$

$$j_x = -D \frac{\partial c}{\partial x}$$

$$j_y = -D \frac{\partial c}{\partial y}$$

$$\underline{j} = j_x \underline{e}_x + j_y \underline{e}_y + j_z \underline{e}_z$$

$$= -D \left(\underline{e}_x \frac{\partial c}{\partial x} + \underline{e}_y \frac{\partial c}{\partial y} + \underline{e}_z \frac{\partial c}{\partial z} \right)$$

$$= -D \nabla c$$

Similarly, I can define the specific energy as limit delta V goes to 0 of sigma of the internal energy of all the molecules this includes the internal as well as the kinetic energy and divide by delta V. The temperature field is determined from the internal energy by the consideration that rho C p T is equal to is equal to delta e any change in energy is equal to the change in temperature times the specific heat at constant pressure or constant volume depending upon the system that you are considering.

So, once we define these fields we can then define fluxes in terms of variations of these fields. So, for example, the mass flux if I have some coordinate system and I want to find out what is the mass flux at a given point the mass flux in the x direction is equal to minus d times partial C by partial x. j y is equal to minus d times partial C by partial y and similarly, for the Z coordinate at a given position from the variations of the concentration field I can find out what are the fluxes. The flux is actually a vector j vector is equal to j x e x plus j y e y plus j Z e Z minus D into e x into d c by d c plus e y into d c by d y plus e x into d c by d z is equal to minus D grad C.

So, the flux itself is a vector object at a given point it has both magnitude and a direction and it is related to the way that the concentration varies around that position. Note that when I write partial c by partial x it is implicit that I am keeping y and z a constant similarly, when I write partial c by partial y I keep x and z a constant and similarly, when I write partial c by partial z, x and y are a constant . So, these fluxes are also varying with position the temperature concentration fields are also varying with position and we obtain equations for these fields by doing what are called shell balances.

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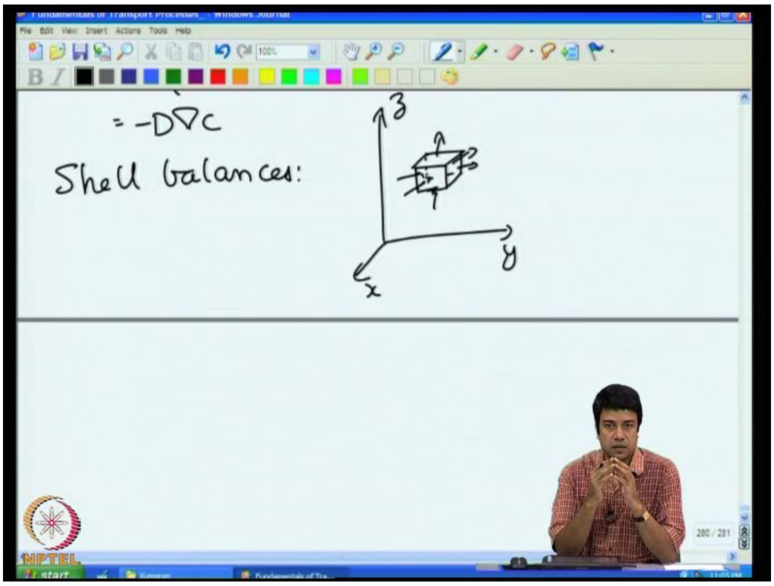

$= -D\nabla c$

Shell balances:

Now in shell balances we basically use mass momentum or energy conservation conditions and depending upon the coordinate system that we take let us say we take a Cartesian coordinate system. We take a differential volume this differential volume is chosen this differential volume for the shell balance is chosen. In such a way that the surfaces of the differential volume are surfaces of constant coordinate. So, the differential volume is chosen in such a way that the surfaces of the differential volume are surfaces of constant coordinate and then we write a balance equation which basically balances the masses coming in and going out of this differential volume.

So, if you are writing a mass balance equation we write for this entire differential volume mass coming in on all the surfaces mass going out on all the surfaces and the accumulation within the differential volume. We did this first for problems of unidirectional flow where you had variation on only one direction unsteady problems in general, but problems of unidirectional flow then we looked at problems with variation in two directions and then we wrote general balance equation for these differential volumes.

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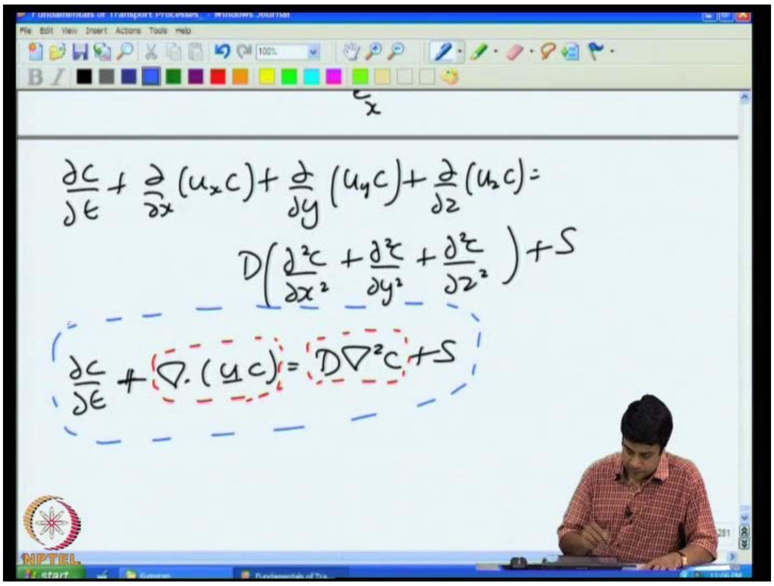

$$\frac{\partial c}{\partial t} + \frac{\partial}{\partial x}(u_x c) + \frac{\partial}{\partial y}(u_y c) + \frac{\partial}{\partial z}(u_z c) = D\left(\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2} + \frac{\partial^2 c}{\partial z^2}\right) + S$$

$$\boxed{\frac{\partial c}{\partial t} + \boxed{\nabla\cdot(\underline{u}\,c)} = \boxed{D\nabla^2 c} + S}$$

In Cartesian coordinates this differential volume equation has the form d c by d t plus d by d x of u x c..

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plus d by d y of u y c plus d by d z of u z c is equal to d squared c by d x squared plus d squared c by d y squared plus d squared c by d z squared plus S any sources due to reaction that are present, but I showed you that a simpler way to write this is as d c by d t plus the divergence of u times C is equal to D del square C plus S. Where this is the divergence of the velocity times a concentration and this is the laplacian of the concentration field this form of the equation is general.

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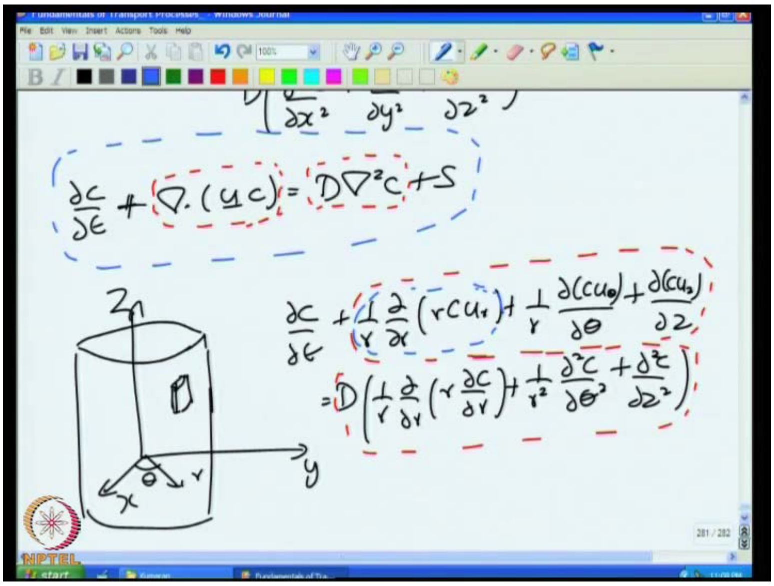

$$\frac{\partial c}{\partial t} + \nabla \cdot (\underline{u} c) = D \nabla^2 c + S$$

$$\frac{\partial c}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r c u_r) + \frac{1}{r}\frac{\partial (c u_\theta)}{\partial \theta} + \frac{\partial (c u_z)}{\partial z}$$

$$= D\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial c}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 c}{\partial \theta^2} + \frac{\partial^2 c}{\partial z^2}\right)$$

I showed you that you can get this even for a cylindrical coordinate system where r is the distance along the X Y direction and theta is the angle from the x axis. So, if I have a system configuration with cylindrical geometry I can use an r theta Z coordinate system and I write a balance over a differential volume in the r theta and Z directions and if I do this write balance over this differential volume I will get an equation of the form partial.

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So, this will be the differential equation where this here is the divergence of velocity times u and this whole thing is D del square of the velocity in the cylindrical coordinate system the more complicated forms that we get over here for these terms in comparison to the Cartesian coordinate system are because of the variation in the surface area because the surfaces are curved.

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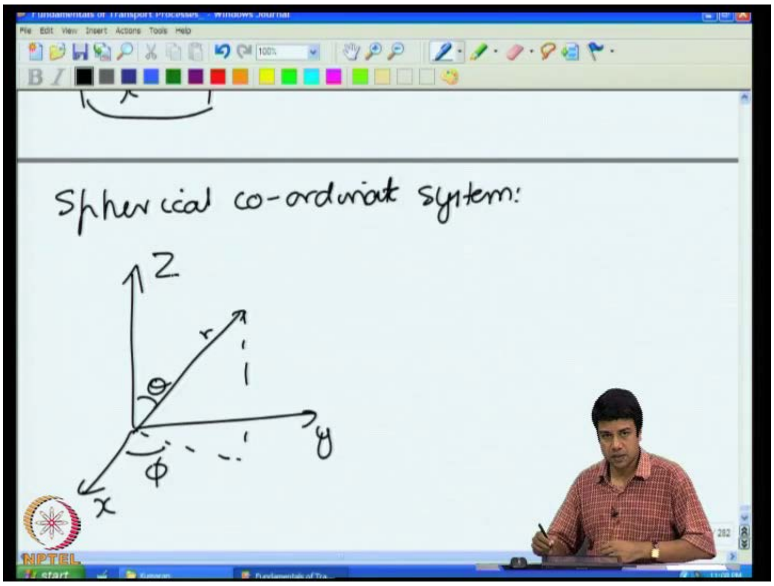

Spherical co-ordinate system:

There is a variation in surface area as you go outwards because they are parallel surfaces, but they are curved surfaces and the surface area changes as the coordinate changes. Similarly for a spherical coordinate system we had used the r theta phi coordinates. We had used the r theta phi coordinate system in a spherical coordinate system.

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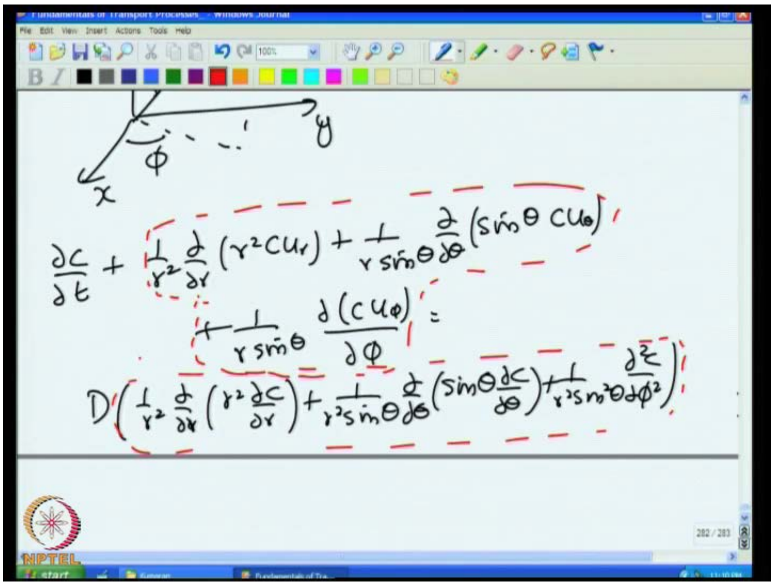

$$\frac{\partial c}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 c u_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\, c u_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial (c u_\phi)}{\partial \phi} = D\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial c}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial c}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 c}{\partial\phi^2}\right)$$

In that case the differential equation was of the form of d c by d t plus 1 by r squared into d by d r of r squared C u plus 1 by r squared r sin theta into d by d theta into sin theta C u plus 1 by r sin theta into d of C u theta by d theta is equal to D of 1 by r squared into d by d r of r squared d c by d r plus 1 by r squared sin theta into d by d theta of sin theta d c by d theta plus 1 by r squared sin squared theta into d squared C by d theta squared.

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So, this is the form of the differential equation where this entire term is divergence of u times C and this D del square C. We did solve problems specifically for variations only in one and two directions, but those were special cases of this mode general equation those were special cases where the variation in one or two of the coordinates was not no longer present.

So, this is the general equation for transport in all three dimensions and transport in one and two dimensions are general cases of this for those general cases. We had devised various techniques to solve the equations. The first one if you recall was a similarity solution for the transport from due to the instantaneous motion of a flat plate for momentum transfer or the instantaneous heating of flat plate for heat transfer. In that case we had transport from a flat plate into an infinite fluid and in that case we had no length or time scales in the problem and we were able to reduce the problem just based upon dimensional analysis to just one similarity variable and that similarity variable was Z by square root of alpha times T or Z by square root of nu times T.

This is the only possibility if you have transport into an infinite fluid and we use that technique in order to reduce the dependence from two independent variables time and Z to just one independent variable. So, the similarity solution procedure was one of the procedures that we had devised for cases where just based upon dimensional analysis you can reduce the dependence. The second one was separation of variables for example, transport into a finite channel in that case you have dependence both on time and on the Z axis on the Z coordinate. So, it separated out the concentration into two functions one which depended only upon time the other which depended only upon Z the differential equation was first order in time and second order in Z if you recall and therefore, we had to reduce we obtained a second order differential equation for the Z dependence of the concentration and a first order for the time.

We now had to choose a homogeneous direction in this particular case we ensured that both boundary conditions in the Z coordinate were homogeneous. We subtracted out the steady solution from the actual solution once you subtract out the steady solution from the actual solution you get homogeneous boundary conditions in the Z coordinate the solution we got were in the form of sine and cosine functions with discrete eigen values and pi for the flat plate case and on that basis we were able to construct a solution using orthogonality relations, because these sine functions form basis functions which are complete and orthogonal any solution can be expressed as a some of the sine or cosine functions which are defined with to be to have to whose value is defined to be 0 on both bounding surfaces and on this basis we were able to reduce the dependence into a function of times function of position.

We had an infinite series the functions decreased exponentially in time that is generally true because the solution has to go to the homogeneous solution in the long time limit and we had sine and cosine dependence upon space and based upon that we had constructed a solution as an infinite series expansion. This is a general feature for all separation of variables problems when you separate variables you would like to have homogeneous boundary condition in one of the coordinates such that you get discrete eigen values and is a set of solutions in the form of basic functions which are orthogonal for the particular operator that you are considering.

So, that was one method separation of variables solution and we also saw how we can do an expansion how we can study oscillatory flows using complex variables. Whenever there is an oscillatory flow rather than writing down the forcing form S cos of omega T or sin of omega T. We can write that as e power I omega T so, if the equation is linear the solution for the temperature field also has to have the same frequency there can be a phase shift, but the frequency has to be exactly the same. So, the final solution that I get has to be exponential in cos and sine functions with the same frequency as the forcing and from that we managed to get some perfect solutions. So, those were solutions for specific configurations for specific cases.

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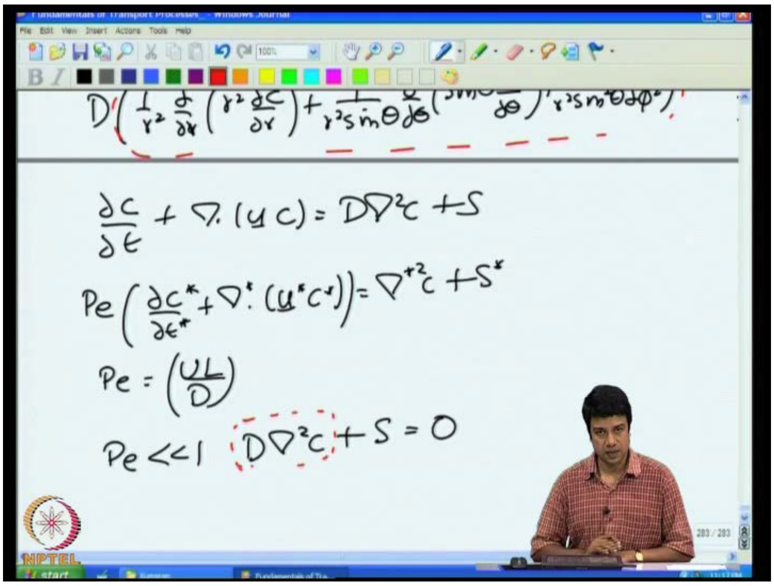

$$D\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial c}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\right.$$

$$\frac{\partial c}{\partial t}+\nabla\cdot(\underline{u}\,c)=D\nabla^2 c+S$$

$$Pe\left(\frac{\partial c^*}{\partial t^*}+\nabla^*\cdot(\underline{u}^*c^*)\right)=\nabla^{*2}c+S^*$$

$$Pe=\left(\frac{UL}{D}\right)$$

$$Pe<<1 \quad D\nabla^2 c+S=0$$

We then derived the word general equation that I just showed you of the form d c by d t plus divergence of u C is equal to D del square C plus any source or sinks. Now if you scale this equation by characteristic length and a characteristic velocity you will get an equation of the form the peclet number times d c by d t thus del dot u C is equal to del square C plus a scaled source or sink this peclet number is equal to u L by D where u is the characteristic velocity L is the characteristic length and D is the diffusion coefficient. In the limit where peclet number is small the diffusion is dominant compared to convection so, if the peclet number is small compared to 1 I can solve an equation of the form del square D del square C plus any sources or sinks is equal to 0. So, this basically reduces to a strategy for solving D del square C is equal to 0.

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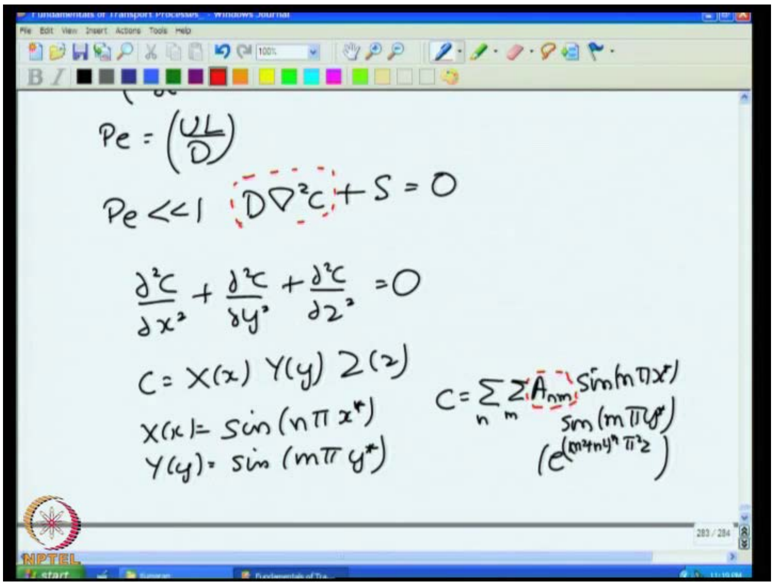

$$Pe = \left(\frac{UL}{D}\right)$$

$$Pe << 1 \quad D\nabla^2 c + S = 0$$

$$\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2} + \frac{\partial^2 c}{\partial z^2} = 0$$

$$C = X(x)\, Y(y)\, Z(z)$$

$$X(x) = \sin(n\pi x^*)$$

$$Y(y) = \sin(m\pi y^*)$$

$$C = \sum_n \sum_m A_{nm} \sin(n\pi x^*) \sin(m\pi y^*) \left(e^{(m^2+n^2)^{1/2}\pi^2 z}\right)$$

In general so, in Cartesian coordinates this has the form d square c by d x square plus d square c by d y square plus d square c by d z square is equal to 0 and I showed you how you can obtain solutions of the form C is equal to some function of x some function of y some function of Z and use separation of variables to get the individual functions we solved a couple of problems on this heat transfer from a cube for example, in that case you can separate out the variables into different functions in the homogeneous directions you will get solutions of the form x is equal to sine n pi x where x is a scaled variable and y is equal to sin of n pi y star.

In the homogeneous directions in the inhomogeneous direction you will get exponential solutions if Z is the inhomogeneous direction the solutions in that direction will be exponential functions and therefore, the final solution that you will get will be of the form C is equal to summation over n summation over m of a n m sin n pi x sin m pi y times some function exponential e power square power half pi square Z. We will get exponential functions in the Z direction and these coefficients will be determined using orthogonality relations, but this solution is very specific it works only for the cartesian coordinate system and does not give us very much physical insight into the diffusion process.

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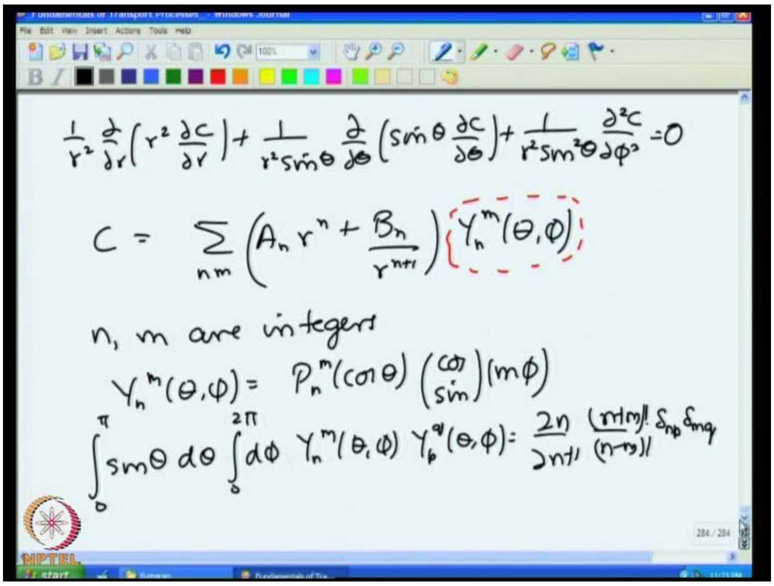

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial c}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial c}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 c}{\partial\phi^2}=0$$

$$C=\sum_{nm}\left(A_n r^n+\frac{B_n}{r^{n+1}}\right)Y_n^m(\theta,\phi)$$

n, m are integers

$$Y_n^m(\theta,\phi)=P_n^m(\cos\theta)\binom{\cos}{\sin}(m\phi)$$

$$\int_0^{\pi}\sin\theta\,d\theta\int_0^{2\pi}d\phi\;Y_n^m(\theta,\phi)\,Y_p^q(\theta,\phi)=\frac{2n}{2n+1}\frac{(n+m)!}{(n-m)!}\delta_{np}\delta_{mq}$$

For that reason we solved the diffusion equation by separation of variables in a spherical coordinate system and this had the form sigma n m A n times r power n plus B n by r power n plus 1 Y n m of theta phi, where once again n m are integers Y n m of theta phi is equal to P n m of cos theta times cos or sin of m phi y n m are the spherical harmonic solutions in the theta and phi directions and the r dependence is in the form of power loss either r power plus n or r power minus n and these functions are once again orthogonal functions.

So, if I have sin theta d theta Y n m of theta phi y P Q is equal to 2 n by 2 n plus 1 so, this is non 0 only when n is equal to P and m is equal to Q. So, this is the orthogonality relation which can be used to determine the concentration of the temperature fields in a spherical coordinate system and you actually saw how to determine the conduction from a sphere which has a source of heat as well as the effective conductivity of a composite material using these basis functions.

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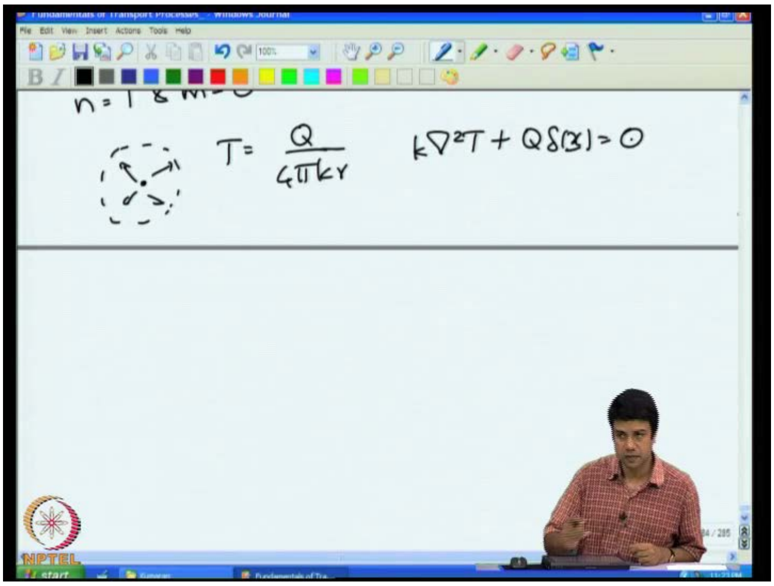

$n = 1$ & $m = 0$

$$T = \frac{Q}{4\pi k r} \qquad k\nabla^2 T + Q\delta(x) = 0$$

The spherical coordinate system the solutions that are obtained using separation of variables also has another physical interpretation for n is equal to 1 and m is equal to 0 this corresponds to a point source of heat Q is equal T is equal to Q by 4 pi k r in that case the temperature decreases radically outwards temperature goes to infinity at the point itself, but Q is finite the total amount of heat coming out of this of this point source is Q in the limit as the radius of this point source goes to 0 if the total amount of heat coming out is Q then this temperature field goes 1 over r the heat flux goes 1 over r square and the total heat out of any surface is constant because there is no source anywhere except at the origin.

I gave you one further interpretation of this is also the solution to the equation k del square T plus Q times delta of x is equal to 0 where delta of x is the 3 dimensional delta function for a point source.

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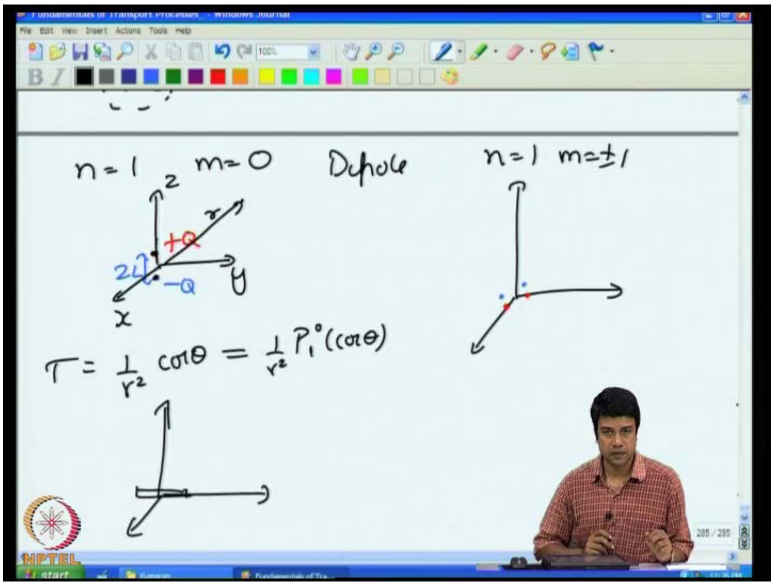

Then I showed you that for n is equal to 2 and m is equal to 0 this corresponds to a source and a sink plus Q and minus Q separated by a small distance 2 L. If the distance from the origin r is large compared to the separation between the source and the sink the solution that I get for the temperature field is going to be equal to 1 by r square cos theta T is equal to which is equivalent to 1 by r square P 2 0 of cos theta. So, this is one..

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and this will be 0 and n is equal to 1 and m is equal to plus or minus 1 corresponds to situations where you have either a source and a sink separated along the x axis or along the y axis. So, if you go sufficiently far away you will get 1 by r square P 1 one of cos theta e power cos phi and P 1 minus 1 times sin phi.

So, the spherical harmonic solutions obtained by separation of variables can also be interpreted as distributions of sources combination of source and sink which is basically a dipole. So, n is equal to 1 corresponds to one source and one sink separated by a small distance which is a dipole n is equal to 2 corresponds to two sources and two sinks distributed in such a way that the net source is 0 net dipole is 0 the temperature field will go as 1 over r cubed and plus 1 and so on. So, this is the other interpretation and I also showed you how these delta functions can be used to obtain the temperature fields due to distributed sources we had in particular scene the temperature field due to line source.

In one of our previous lectures we can integrate we can consider the line sources as a distribution of point sources determine the temperature field due to each one of them and add them up to get the total temperature field. So, these were for the solutions of the diffusion equation by various methods one is by delta functions the other is by the separation of variables.

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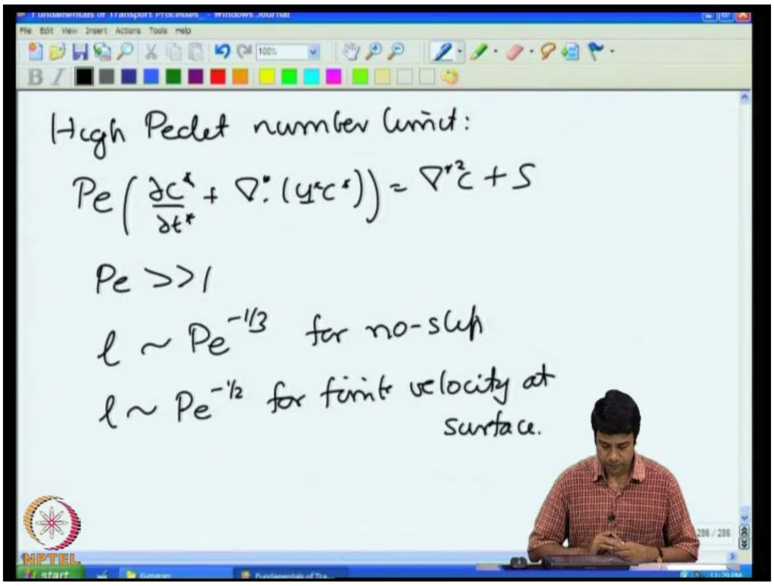

High Peclet number limit:

$$Pe\left(\frac{\partial c^*}{\partial t^*} + \nabla^* \cdot (u^* c^*)\right) = \nabla^{*2} c + S$$

$Pe >> 1$

$l \sim Pe^{-1/3}$ for no-slip

$l \sim Pe^{-1/2}$ for finite velocity at surface.

The next was the high peclet number (No audio from 52:57 to 53:11) P e times partial C by partial T plus del dot u C is equal to del square C plus sources or sinks and for P e large compared to one, you would expect that you can neglect the diffusion terms in comparison to the convection terms however that reduces the differential equation form a second order differential equation to a first order differential equation and you cannot satisfy boundary conditions the physical reason for the inconsistency is that convective transport only takes place along the flow diffusion takes place in all directions equally at a surface there is no flow perpendicular to the surface. So, one cannot have convection perpendicular to the surface itself, at the surface itself ultimately transport has to take place due to diffusion due to the fluctuating motion of the molecules.

If the velocity is large the convective effect is large there is still diffusion take place from the surface, but it will penetrate not very far because it is being swept backwards by the convection. So, it is going to be a smaller length scale over which there is going to be a balance between convection and diffusion that length scale is obtained by scaling the cross stream coordinate by that length scale putting it into the balance equation and then determining how that length scale varies as a function of peclet number, for the case where we have a surface with a no slip condition the peclet number, the distance, the lengh scale smaller length scale goes as goes as peclet number to the minus 1 by 3.

In the limit as peclet number becomes large for no slip and L goes as P e power minus half for finite velocity and in order to solve the problem in order get the temperature fields we need to make one additional assumption that the temperature at a given location depends only upon the distance from the start of the heating point. If we make that assumption then the total length no longer matters only the length from the start of the heating section matters and then we get a similarity solution similar to the similarity solution that we had obtained for the impulsive start of plate earlier and I showed you how to get those similarity solutions both for a flat plate as well as for a spherical particle.

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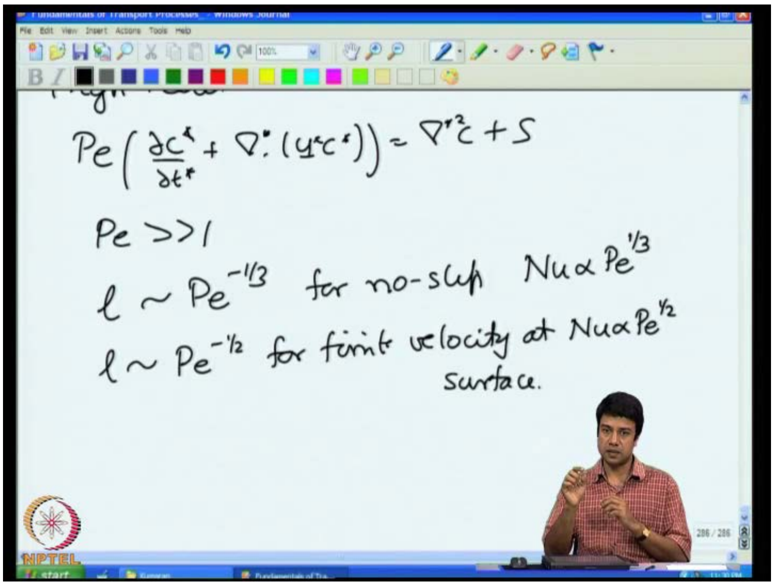

$$Pe\left(\frac{\partial c^*}{\partial t^*} + \nabla^* \cdot (\underline{u}^* c^*)\right) = \nabla^{*2} c + S$$

$$Pe >> 1$$

$\ell \sim Pe^{-1/3}$ for no-slip $Nu \propto Pe^{1/3}$

$\ell \sim Pe^{-1/2}$ for finite velocity at surface. $Nu \propto Pe^{1/2}$

From that we get the nusselt number verses peclet number relation and u proportional to P e power 1 by 3 as P e becomes large the length scale goes as P e power minus one third the gradient goes as P e power plus one third. So, the heat flux is proportional to the gradient which goes as P e power plus one third. When there is slip at the surface you will get n u proportional to P e power plus half the constants in these relations do depend upon the configuration, but the scaling itself does not depend upon the configuration. So, two different solution techniques one for the diffusion equation when the peclet number is small I showed you different techniques separation of variables the boundary integral technique the delta function sources.

Boundary layer theory in the limit of high peclet number in this case transport is restricted to a thin region near the surface and we use a scaling analysis within that region to find out how the thickness of that region varies with peclet number and then use similarity solutions to get the exact form. So, that is a brief summary of everything that we did in this course central message is we write down equations for fields temperature, concentration, momentum fields these equations are obtained by balances over differential volumes in different coordinate systems they end up having the same form in vector notation, but for each coordinate system the operators are different.

Then we looked at how to analyse this in two different limiting cases the low and the high peclet number limits and in both of these cases you have different forms which give you different types of correlations between the dimensionless fluxes and the driving forces the peclet number, the Reynolds number and the dimensionless fluxes. So, that briefly is a summary of everything that we did in this course I hope you had an enjoyable time and thank you for attending the course.