Fundamentals of Transport Processes
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Lecture No. # 35
Diffusion Equation Greens Function Formulation

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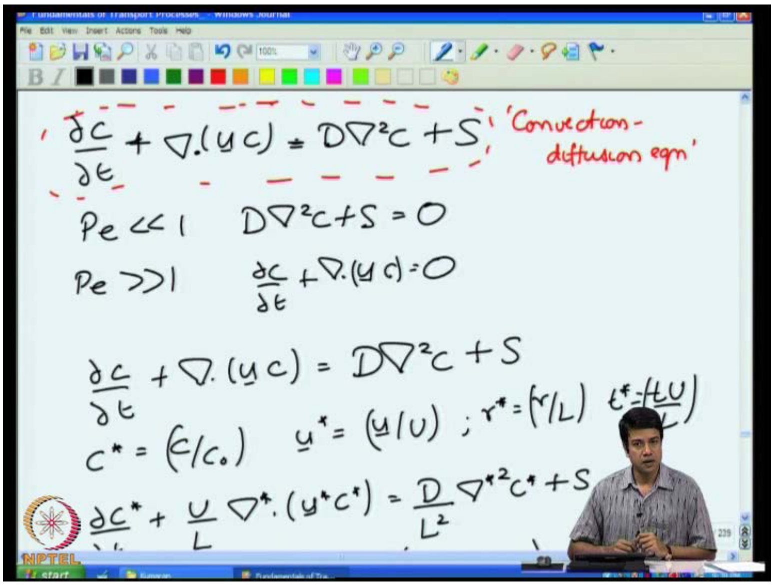

Welcome to lecture number thirty five in our course on fundamentals of transport processes where I will just briefly take a little bit of time to recollect where we were in this course. We had derived a general convection diffusion equation for mass and heat transfer shown in the red over there. For mass transfer this is of the form partial c by partial t plus divergence of the velocity times C is equal to the diffusion coefficient times the Laplacian of c plus any sources or sinks that are there. These operators are general. So, this equation is valid for any coordinate system specifically for the, but, the the operators the divergence in the Laplacian assume different values in different forms in different coordinate systems. And we had derived explicitly the forms of these operators in different coordinate systems by doing shell balances.

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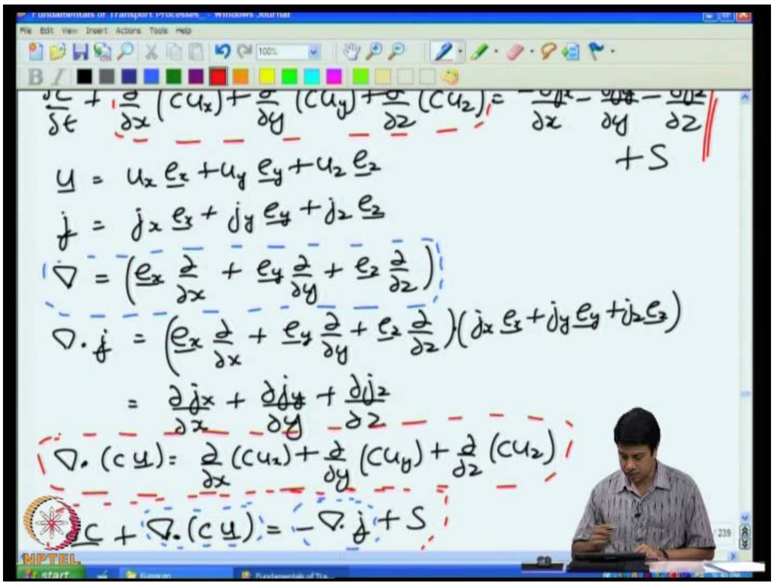

Now, if you recall we first did a shell balance for a Cartesian coordinate system and we got this form for the divergence operator and the Laplacian. So, divergence of concentration times velocity was given by this particular expression in a Cartesian coordinate system.

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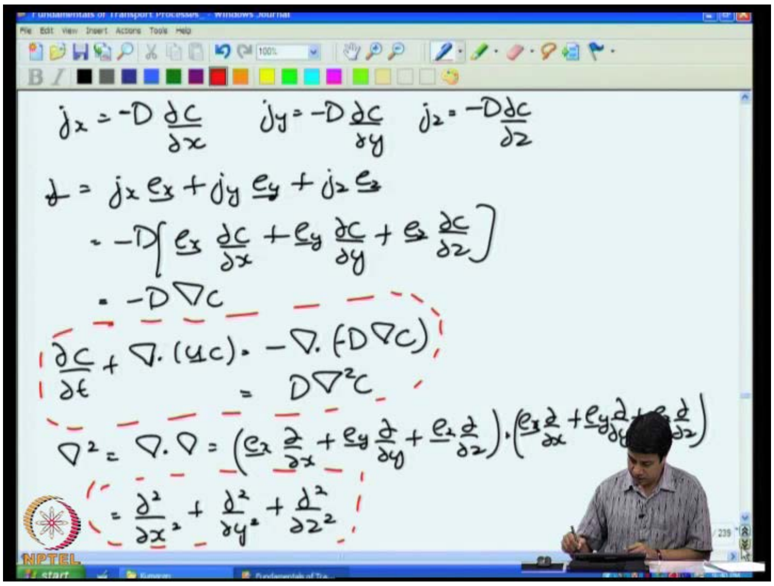

And the Laplacian operator was given by this. This was done by doing a shell balance in a Cartesian coordinate system where the surfaces of the differential volume that we consider are surfaces of constant coordinate. Two surfaces with constant x, two with constant y and two with constant z.

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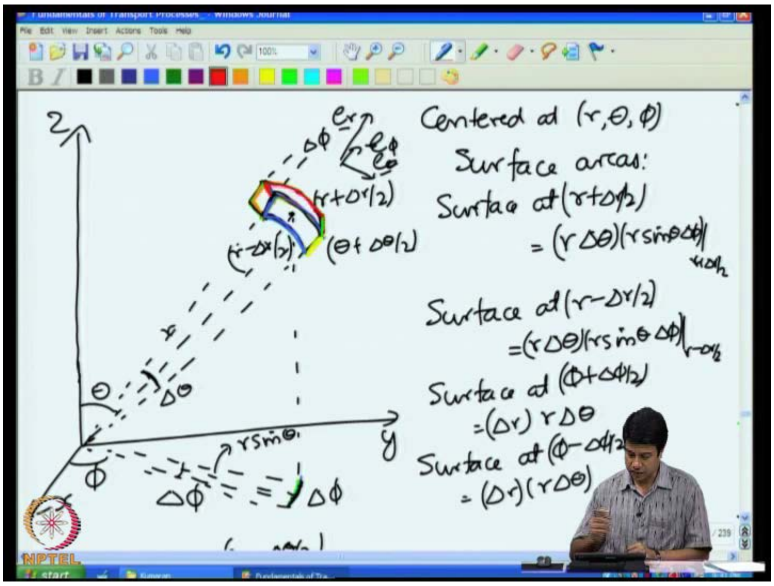

And then we had gone ahead and done the same in a spherical coordinate system using a shell of this kind. Two surfaces with constant r, two surfaces at constant theta and two surfaces at constant phi where r is the distance from the origin theta is the Azimuthal angle the angle made by the radius vector with the z axis and phi is the meridian of the angle, the angle along the x y plane from the x axis of the projection of this radius vector onto the x y plane.

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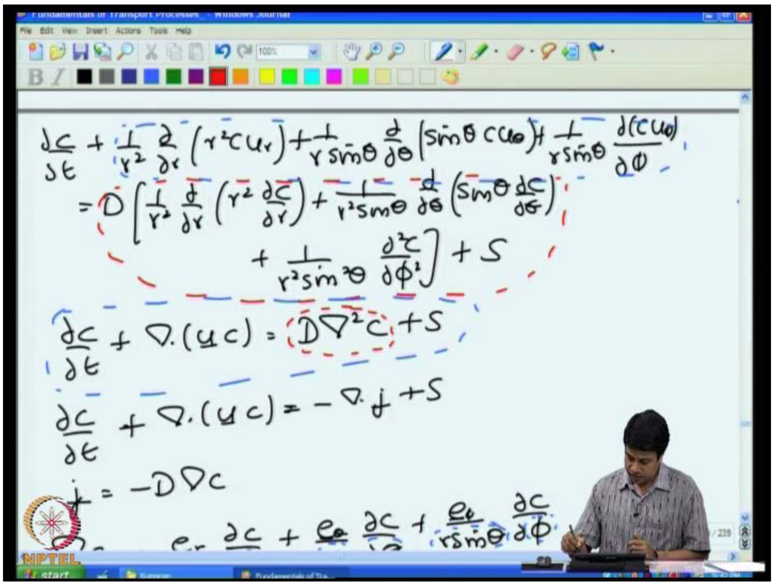

And there by doing the shell balances we had got a different form for the the the divergence in the Laplacian operators.

The terms on the left side in the blue on the equation on top or the divergence of c times u and they have a more complicated form 1 over r square d by dr of r square times c u r plus 1 by r sin theta d by d theta of sine theta c u theta plus 1 by r sin theta partial of c u phi by partial phi where u r u theta and u phi are the components of the velocity in the spherical coordinate system. And the Laplacian operator is shown on the right 1 by r square d by dr of r square partial c by partial r plus 1by r square sine theta d by d theta of sin theta partial c partial theta plus 1 by r square sin square theta partial square c by partial phi square.

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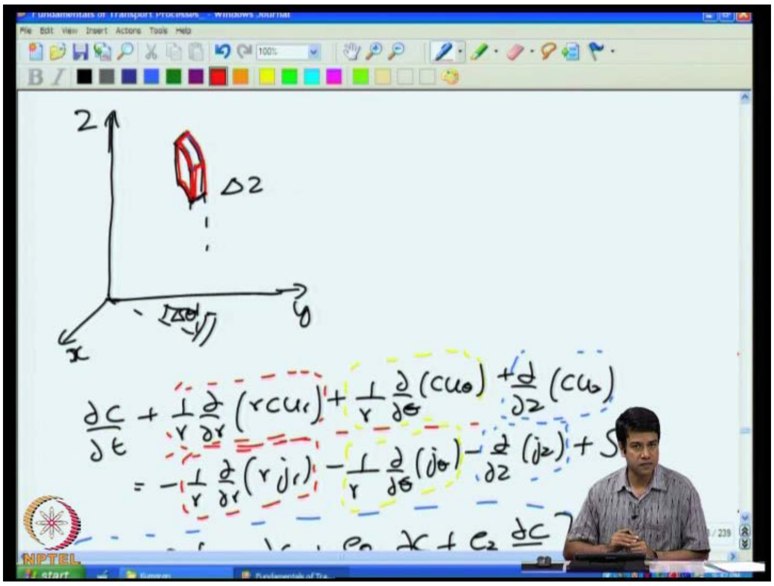

And I had briefly shown you how to get equivalent operators in cylindrical coordinate system by using a cylindrical shell with surfaces at constant r, constant z and constant theta where theta is the angle from the x axis of the projection of the radius vector on to the x y plane in a cylindrical coordinate system. And so, we were left with this general form of the fusion equation which contains the the convection diffusion equation which contains the time derivative, the unsteady term.

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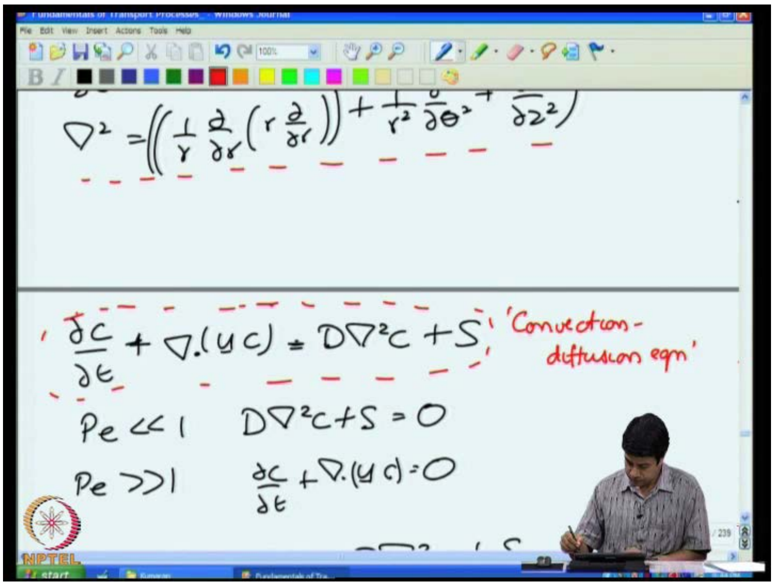

the unsteady term and the convective term on the left hand side. The unsteady term partial c by partial t the convective term divergence of the velocity times concentration. On the right side are the diffusion term d del square c, the Laplacian of the concentration plus whatever sources or sinks that are there within the field. And I told you that we can consider 2 limits 1 is when pecklet number is small. In that case we neglect the unsteady and the convection term all together and the equation reduces to d del square c plus s is equal to 0. The other is when the pecklet number is large compared to 1. In this case you would think that we can completely neglect the diffusion term, turns out it is not so. And we will see in the next few lectures how to deal with the high pecklet number limit of this convection diffusion equation.

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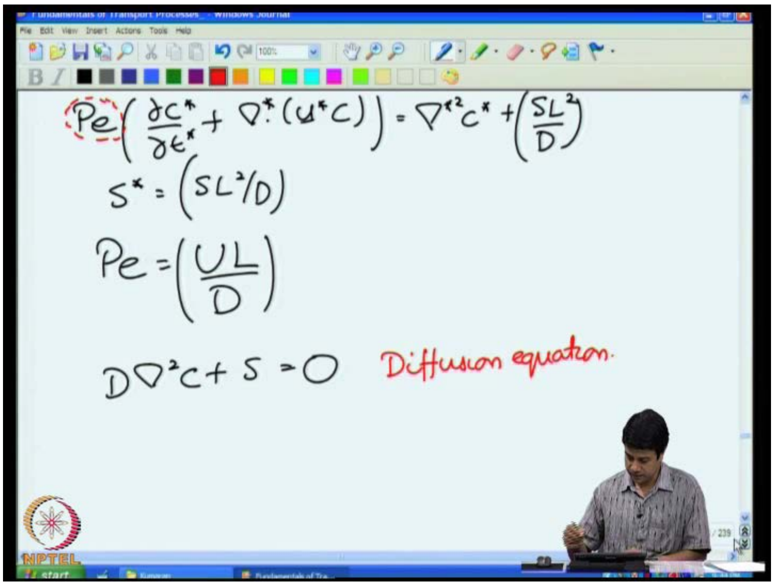

So, first we started of solving d del square c plus s is equal to 0 which is just the diffusion equation. If there are no sources or sinks in the field then this is just the Laplace equation for the concentration field.

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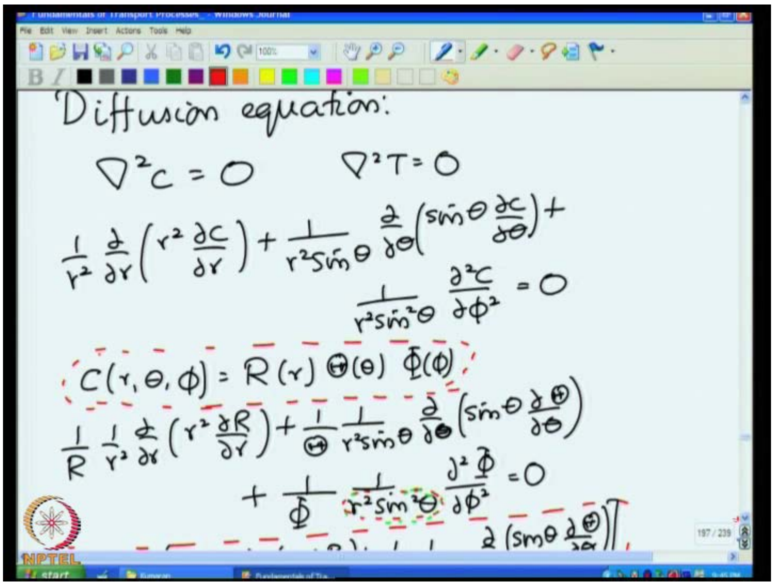

And first I had solved it for you in a spherical coordinate system as I explained spherical coordinate system can be used only for objects of spherical symmetry. But, also a spherical coordinate system gives us some general forms or solution which can be used for any problem and first we solve this using a procedure of separation of variables where we write the concentration equation as the product of three terms. One of which is the function only of r, one is only a function of theta and the third one is only a function of phi.

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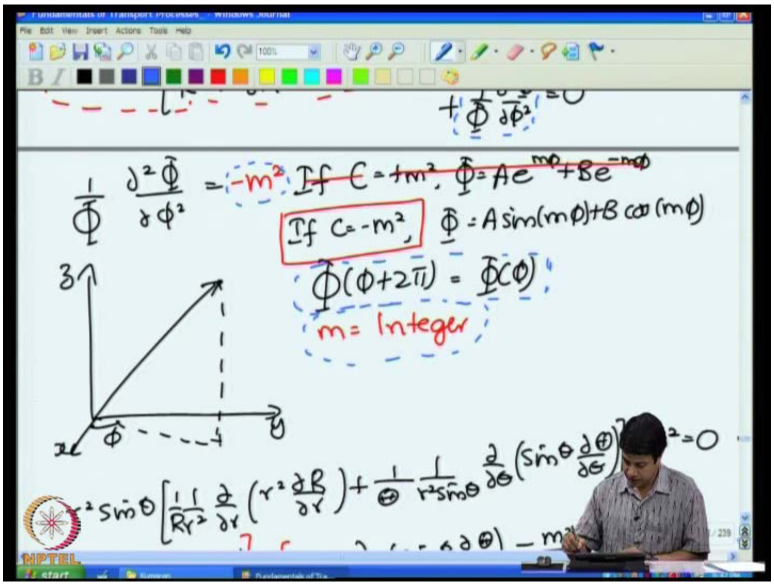

And we separate out the dependences on all of these three. The equation for phi just reduces to 1 over capital phi t square of capital phi by d phi square is equal to a constant

So, this is the usual second order differential equation identical to what we are encountered in Cartesian coordinates. However, in Cartesian coordinates we got discrete Eigen values because we had homogeneous boundary conditions. In one particular direction the value of of a the the concentration at 0 and at some particular length. One had to be 0 and on that basis we got discrete Eigen values. In this particular case, we get discrete Eigen values from a symmetric consideration. That is that if I am at any particular angle phi and I go all the way round and come back to that same angle, in other words if I increase phi by a by 2 pi; I should come back to the same location and the concentration has to be exactly the same because it is the same location physical location in space even though the phi coordinate has been incremented by 2 pi.

So, symmetry itself requires that capital phi of phi plus 2 pi is equal to capital phi of phi itself. That means, that this coefficient here the constant in the separation of variables has got to be minus m square where m is an integer because only if I have functions of the kind sin and cosine of m phi and m is an integer, I will get solutions which have that periodicity of 2 pi.

So, this straight away gave us sin and cosine functions in the phi direction and then we went on and solved for the theta direction.

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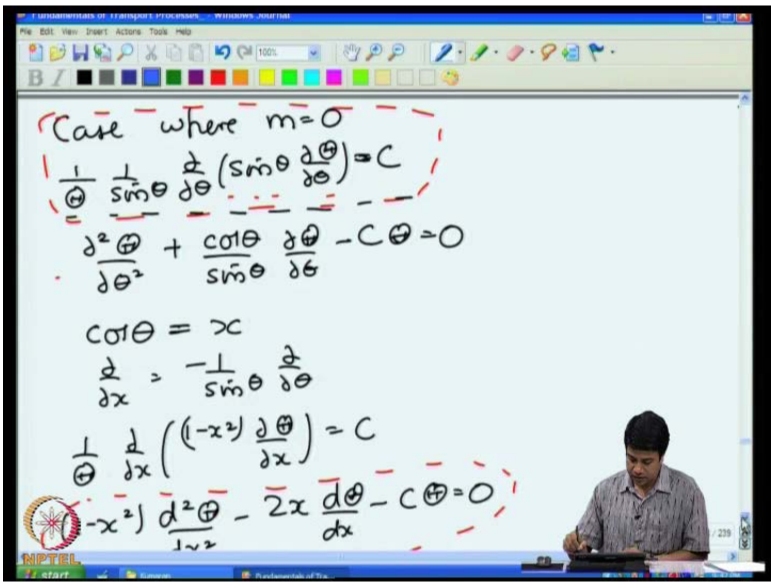

And I showed you that in the theta direction you get another constant c. This constant has to be minus of n into m plus 1 where n is an integer.

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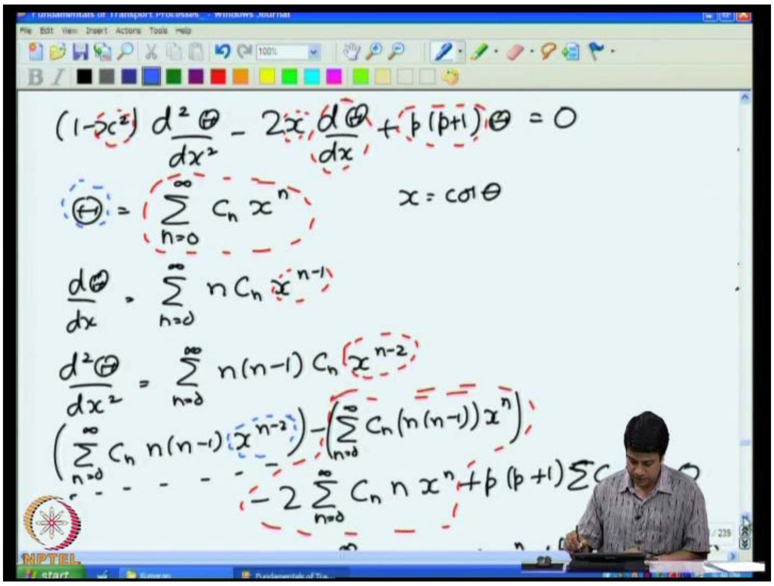

I showed you that I showed that to you by expressing this function capital theta as an expansion in x where is equal to cos theta and this series expansion is convergent at x is equal to plus and minus 1 only if this constant is equal to n into n plus 1.

So, that gives discrete Eigen values in the theta direction.

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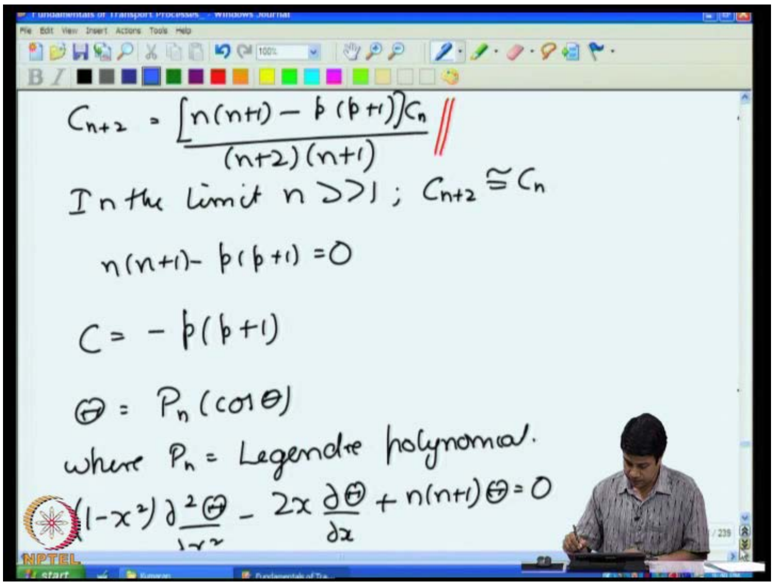

And finally, we have to solve along the radial direction for the diffusion equation. This solution this solution along the radial direction just turns out to be a power of n of the form a n into r power n plus b n into r power minus n minus 1.

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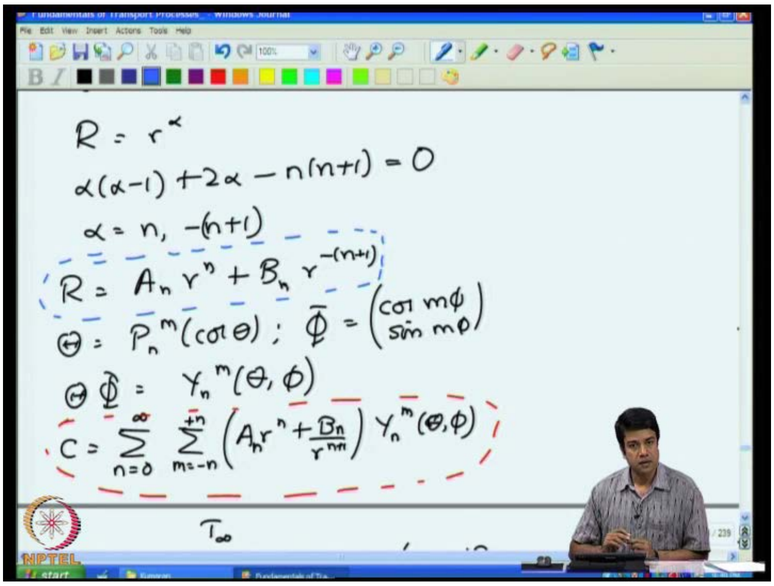

So, finally, I get these spherical harmonic expansions concentration is equal to A n times r power n plus B n by r power n plus 1 y n m of theta phi where Y n m is product of the Legendre polynomials p n m of cos theta as well as cos and sin functions in the phi coordinate and these spherical harmonic functions Y n m satisfy orthogonality relations. That is if I take one particular, the orthogonality relations which are shown in the blue on top.

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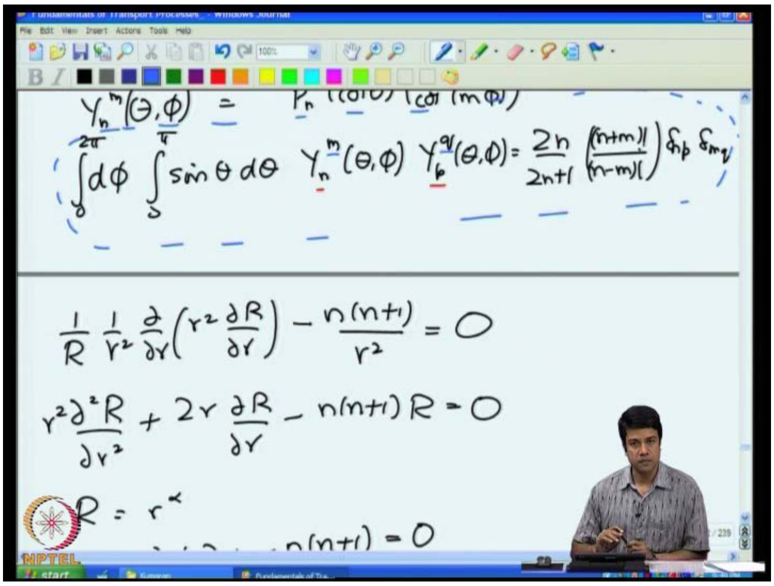

If I take two different functions and multiply them the answer is 0. If the functions are different, it is non zero only when the two functions are identical to each other. So, using these orthogonality relations we actually solved the problem of heat conduction in a composite which has spherical inclusions.

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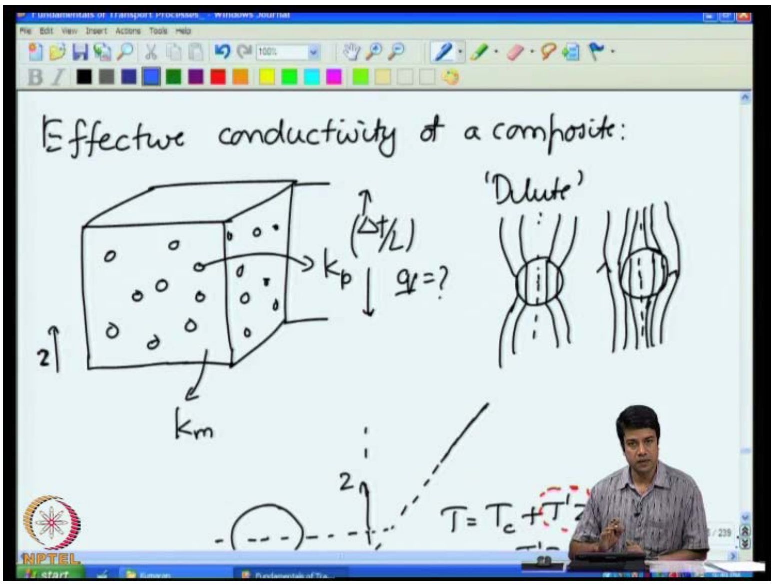

If you recall, we did this problem on spherical inclusions.

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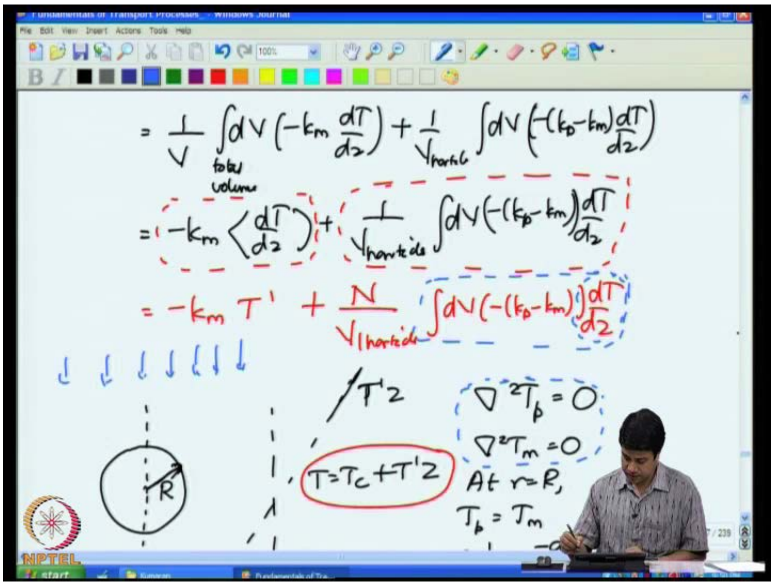

And found out a solution for the temperature field around a particle placed in a linear temperature gradient far away using spherical harmonic expansions.

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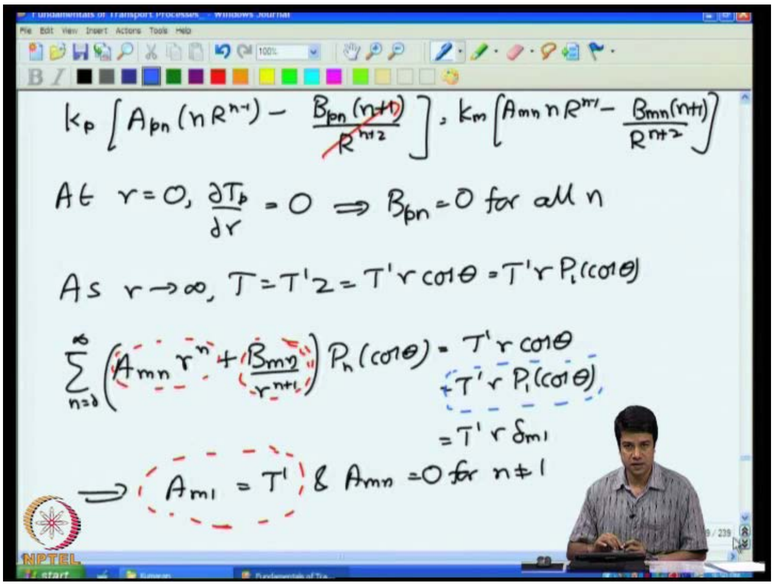

And since the the the forcing temperature field or the temperature field far away had the symmetry of 1 0 that is it is proportionate to p 1 0 of cos theta which is same as y 1 0. Therefore, the the the the actual temperature everywhere also has exactly that same symmetry.

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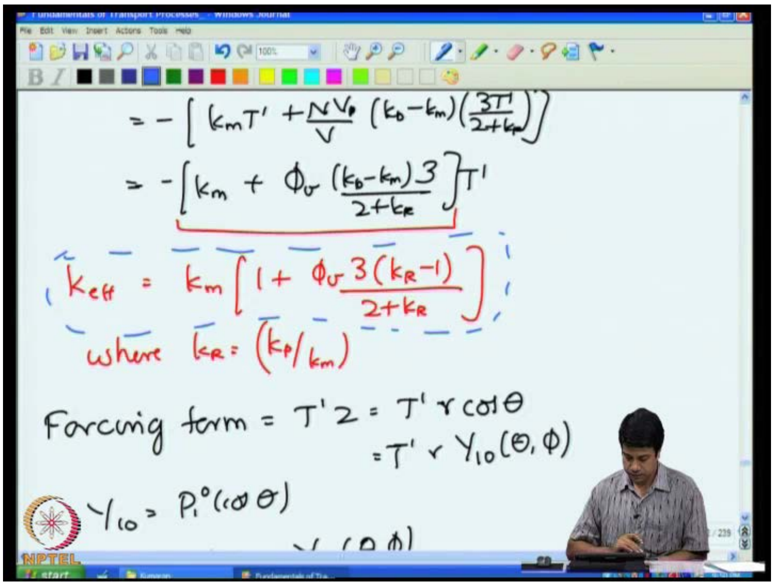

And because of that we were able to solve the problem and get the effective conductivity of the the the composite in the dilute limit where the temperature field around one particle does not affect the temperature around another particle.

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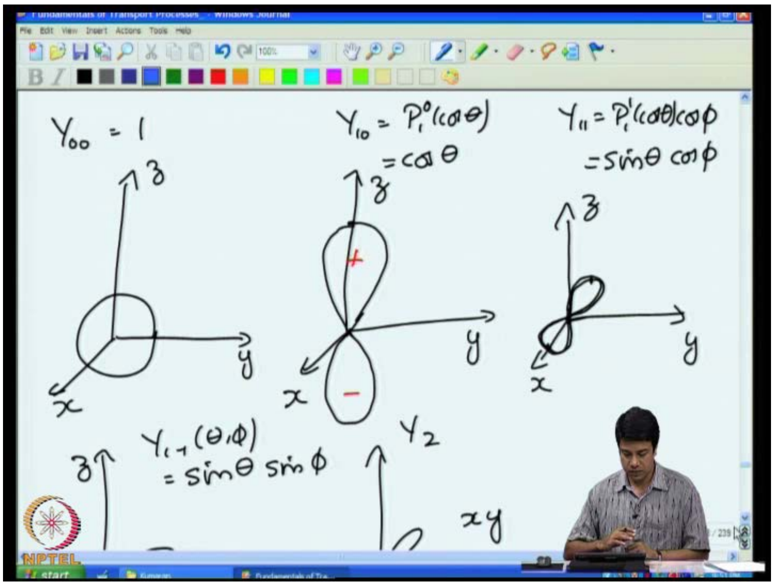

And I had told you about the symmetries of these spherical harmonics briefly in terms of how they vary spatially Y 0 0 spatially its is spherically uniform.

So, surface is of constant Y 0 0 will just look like a spheres and where as Y 1 0 Y 1 1 and Y 1 minus 1 have two reasons of maximum and minimum.

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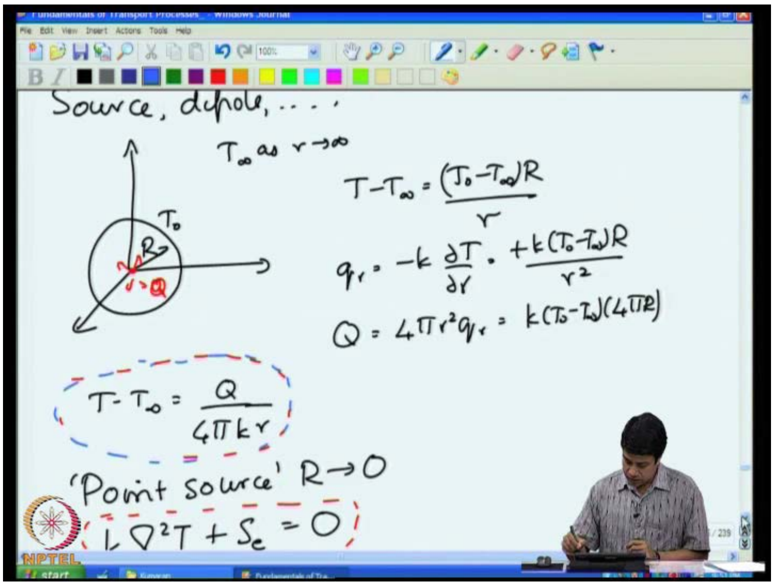

And then we looked at a different way of getting the same solution in terms of sources dipoles etc.

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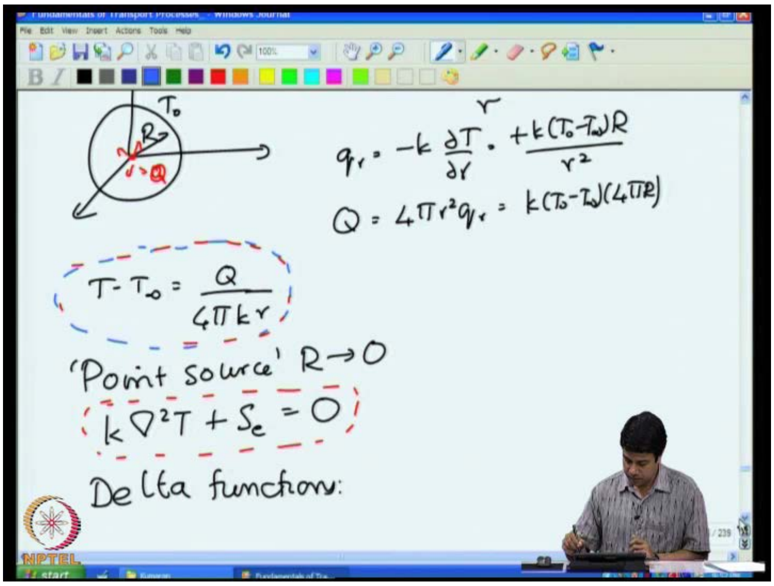

So, first I solved for you the equation. the The solution for the temperature field in a point source point source radius goes to 0, but, heat emitted per unit time is a constant ok.

So, that is a point source. The temperature field for that point source is Q by four pi K r where Q is the energy coming out per unit time and K is the thermal conductivity and r is the distance from the center. This point source is of infinite decimal radius it is a point. However, the amount of heat coming out of this point source is still finite in the limit as the radius goes to 0. So, the heat flux has to go to infinity as you go very close to a point source.

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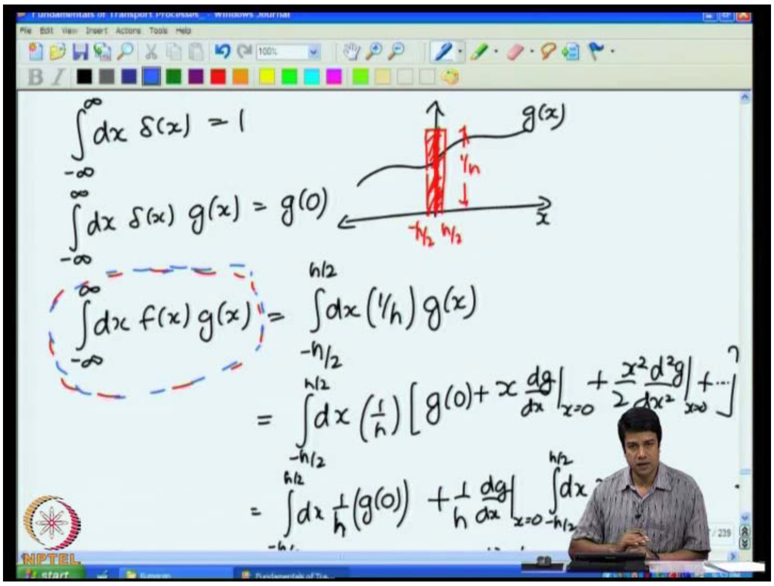

I had solved this point source problem for you using by defining delta functions ok. So, to briefly recap in one dimension; delta function is 0 whenever x is not equal to 0 it is non zero only at x is equal to 0 and the area under the function is equal to 1 and if I multiplied the delta function by any other function and integrate overall space I get the value of that function at x is equal to 0.

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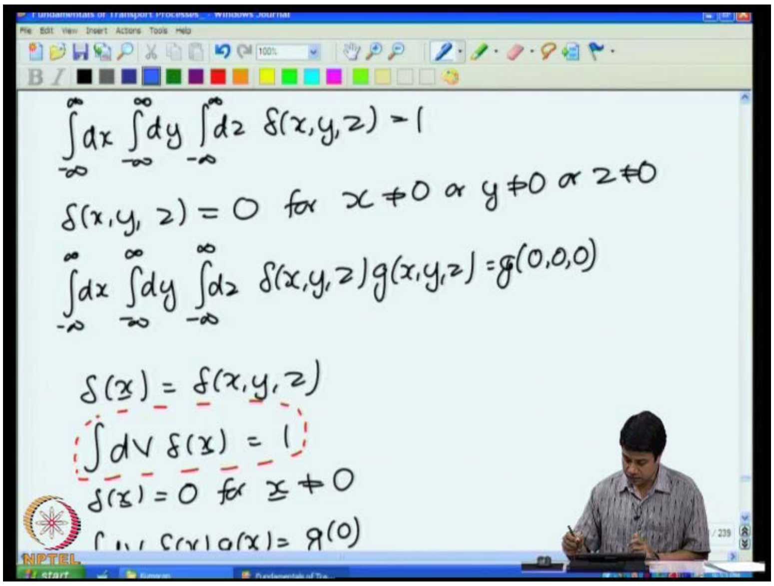

Similarly, one can extend this to two and three dimensions. In the case of three dimensions this delta function is 0 for x vector, not equal to 0 that is when simultaneously either x is not equal to 0 or y is not equal to 0 or z is not equal to 0 is non 0 only at the origin at x is equal to 0 y is equal to 0 and z is equal to 0. The integral over all space of delta of the delta function is equal to 1, an integral over all space of delta of x y z times some any function g of x y z basically picks out the value of the delta function at the origin as shown at the bottom there ok.

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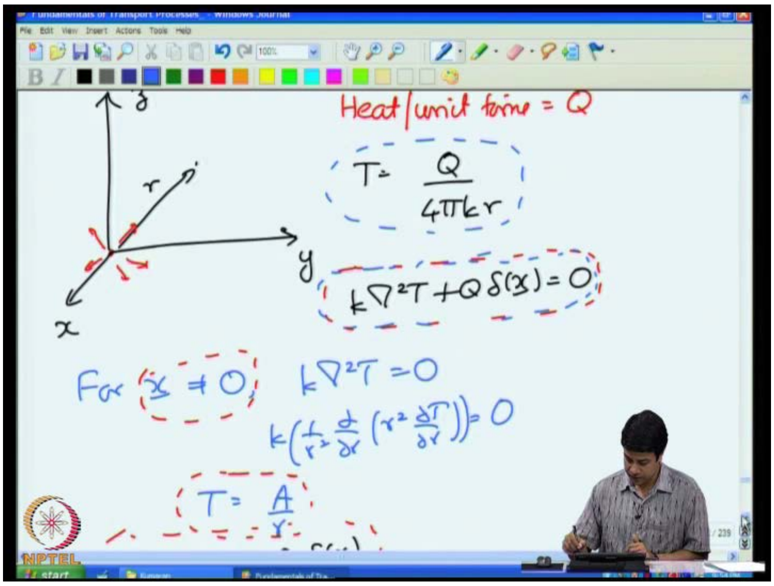

And finally, we saw that this the the equation the solution for the point source is a solution of the equation K del square t plus Q times delta of x is equal to 0.

So, if I have a source of the form of a delta function times Q, note that the delta function has dimensions of one over volume q is heat emitted per unit time. So, times delta of x has the correct dimension of heat emitted per unit volume, per unit time for a distributed source. For this particular form of the delta function, the solution is the same as a solution for a point source in the limit as r goes to, the radius of the source goes to 0.

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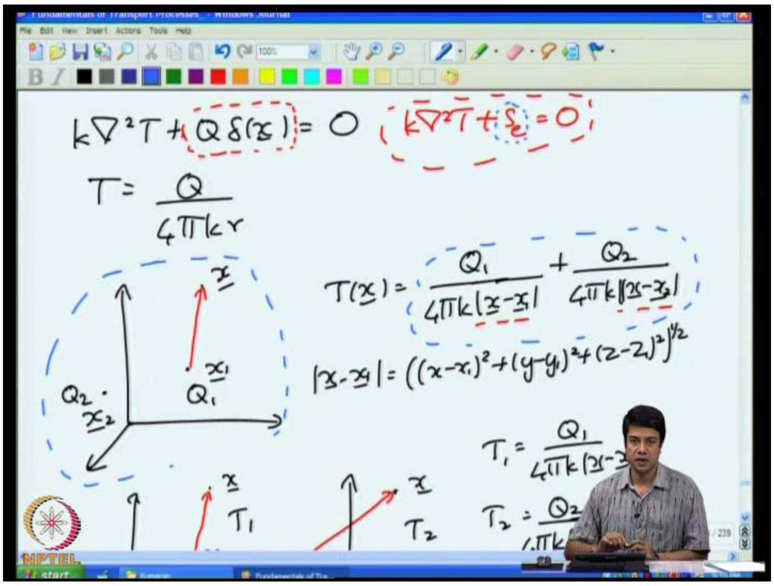

And then I showed you that you can use linear superposition for point sources. We have two different point sources; the temperature field into an observation point due to two sources it is the sum of the temperature field due to each individual source.

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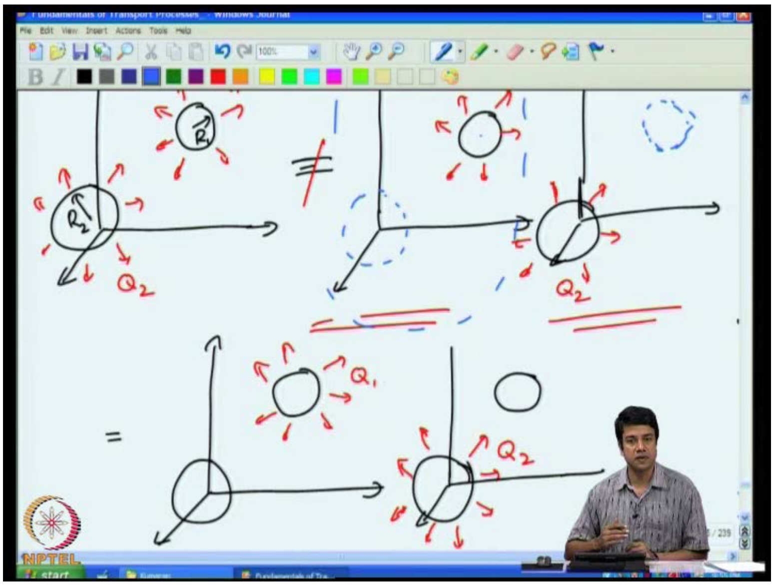

It is important to caution that the same superposition principle does not apply for two different particles of finite size. If I have two particles of finite size, I cannot just consider the temperature due to one in the absence of the second and the temperature due to the second in the absence of the first. I have to consider both particles being present. Simultaneously, when I solve the problem because the boundaries for the two problems, the two sub problems have to be exactly the same. However, since a point source has no volume, in this case I can either have or not have the point source. It makes no difference because it does not create an additional boundary in the problem.

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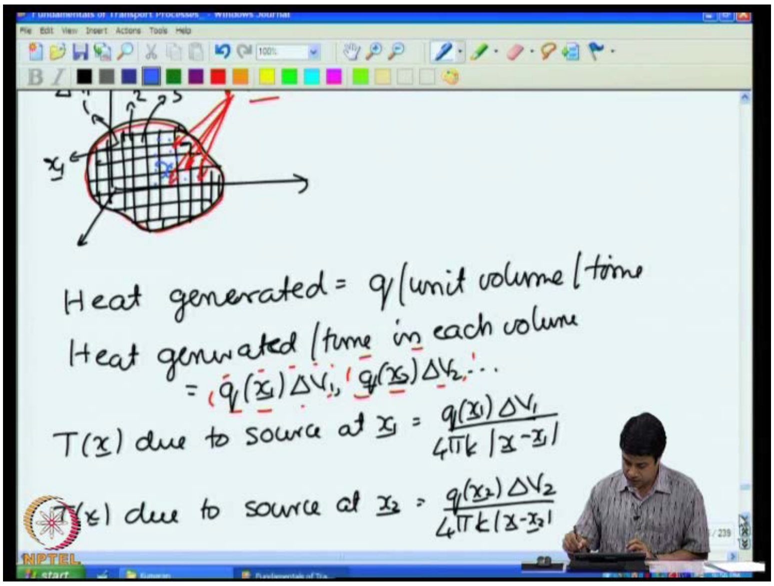

And then I showed you how to express the temperature field due to a distributed source in terms of the temperature field due to a point source.

I divide the distributed source into discrete volumes and assume that the total heat coming out of each small volume is located at the center of that volume. So, that the total heat coming out of each small volume is a point source and I just add up the total temperature field due to each of these point sources at an observation point in the limit as the volume goes to 0. This summation can be converted into an integral.

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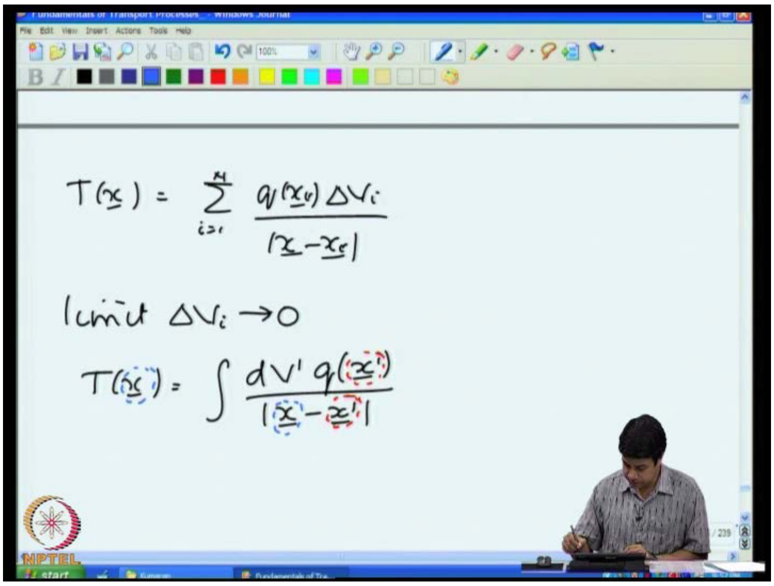

So, I get an equation of the form t at the location x is equal to integral d v prime of Q of x prime divided by x minus x prime where x minus x prime is the distance between the source and the observation point. Note that, x is the observation point, the point at which you are taking the temperature. X prime is the source point. the The location of the point source and in this case for a distributed source you are integrating over all locations of the source point.

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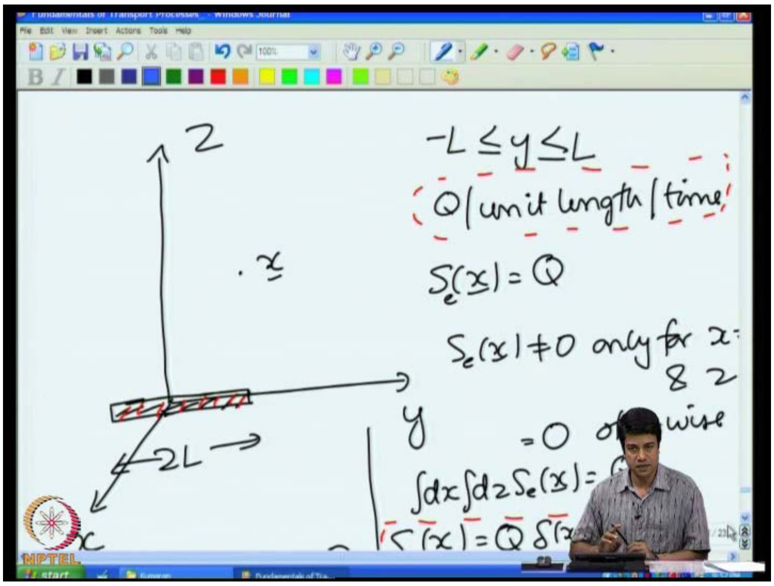

And using this I had solved for you the temperature field due to a wire along the x axis.

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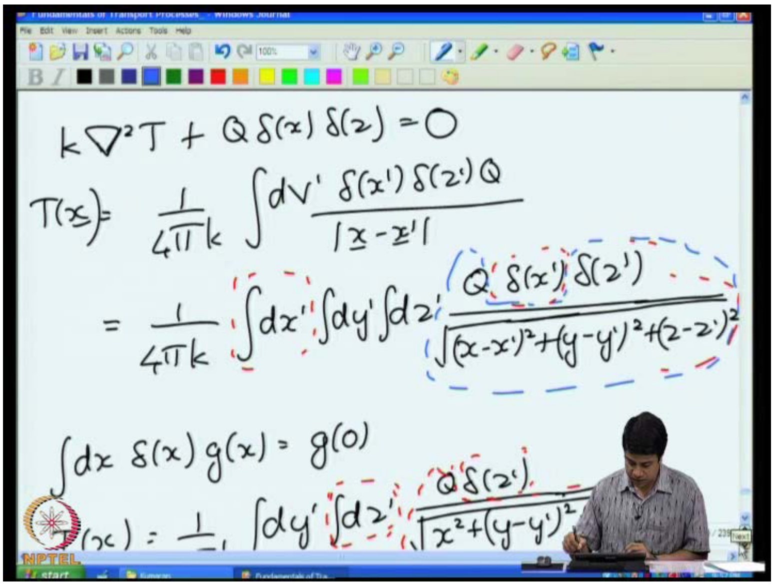

And we had explicitly calculated the temperature field by carrying out the integral as you can see in the black outline box.

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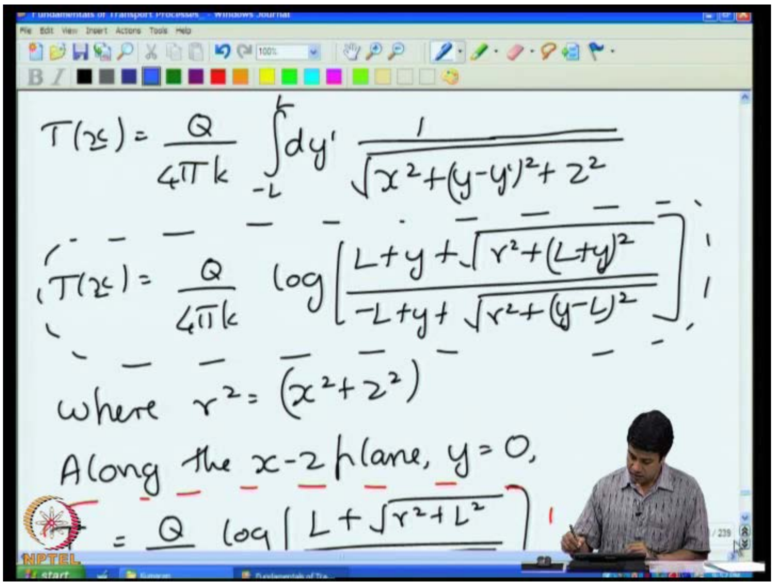

And I had shown you that in the limit as the distance from the wire to the observation point.

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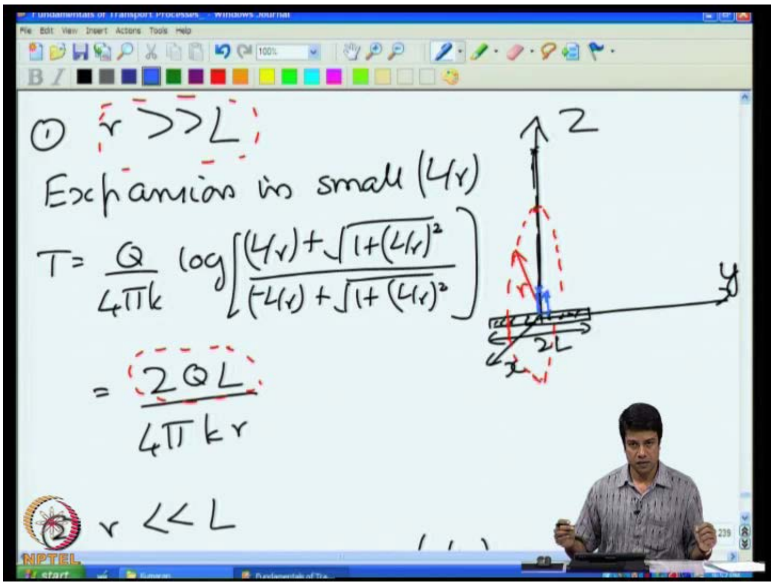

If it is large compared to the length of the wire; the result that I get looks like a point source because I do not see the details of that wire and r is small. It looks like conduction from an infinitely long wire.

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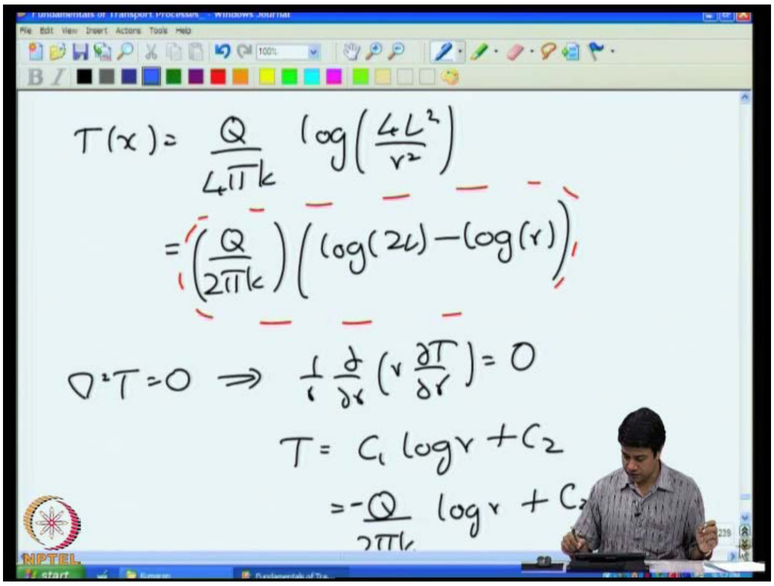

For conduction from an infinitely long wire, I get a logarithmic function for the temperature variation.

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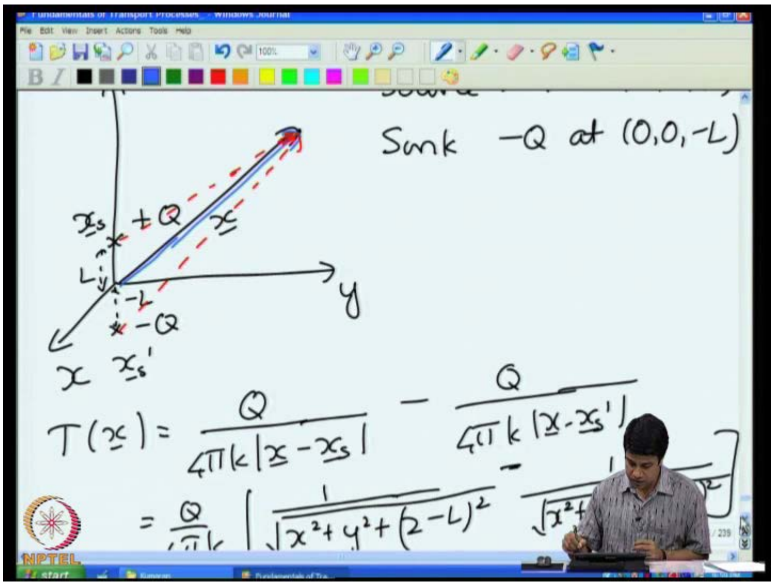

And finally, I had made a connection between the spherical harmonic expansions and the point source formulation.

The the the solution due to a point source, the decaying harmonic solution for n is equal to 0. The spherically symmetric decaying harmonic solution is identical to the to the temperature field due to a point source. Both of them go as 1 over r and they do not depend upon theta. So, that is the slowest decaying term in the limit as r goes to infinity it goes as 1 over r the next higher term n is equal to 2. i am sorry n is equal to 1 for n is equal to 1 in this spherical harmonic expansion n goes from minus 1 to 0 to plus 1. So, there are three solutions.

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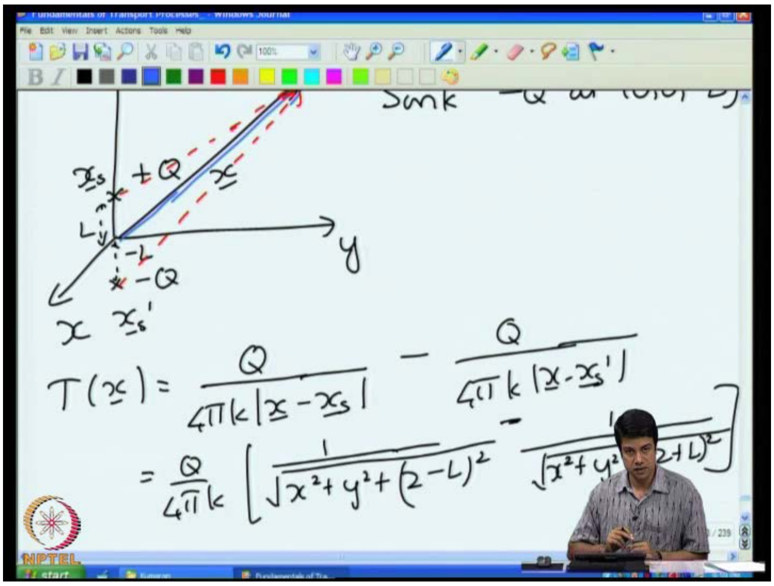

I showed you that these solutions are identical to the solutions that you would get due to the super position of the temperature fields due to two point sources. 1 plus Q at a location one above the z axis, above the x y plane, the other at the location minus l l below with source strength minus Q.

So, I have source Q on the plus z axis, a sink minus Q on the minus z axis separated by distance 2 l if I go sufficiently far away. That the distance r is large compared to l.

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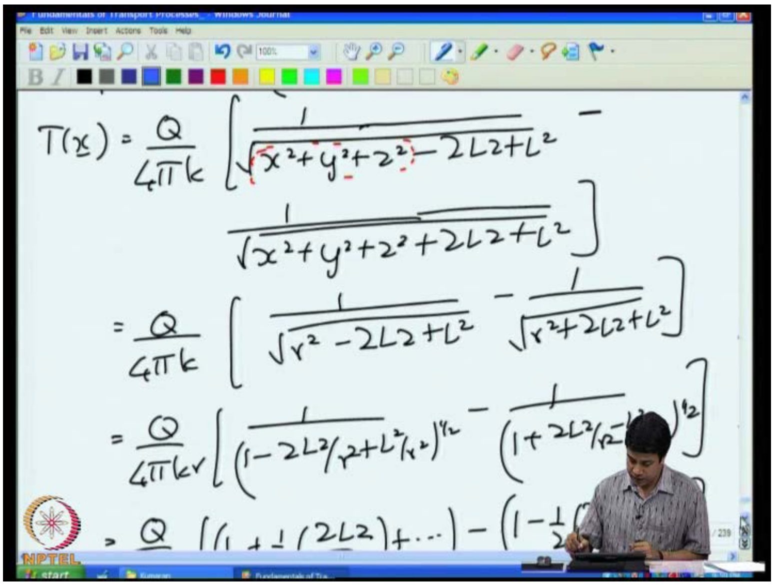

The leading order solution that I get by expanding in this small parameter l by r is identical to the solution.

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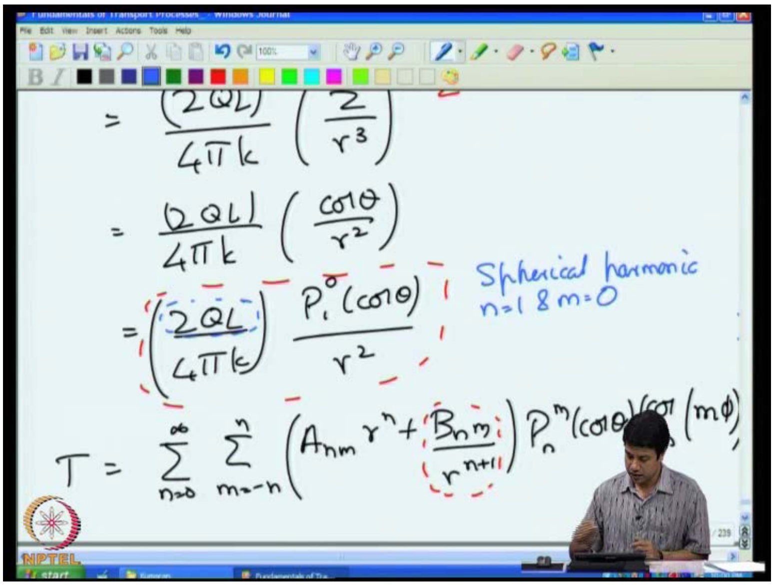

Spherical harmonic solution solution for n is equal to 1 and m is equal to 0. So, that was for a source and the sink separated by distance l along the z axis. If they are separated by a distance l along the x axis I get n is equal to 1 m is equal to plus 1 and if they are separated along the y axis, I get n is equal to 1 and m is equal to minus 1.

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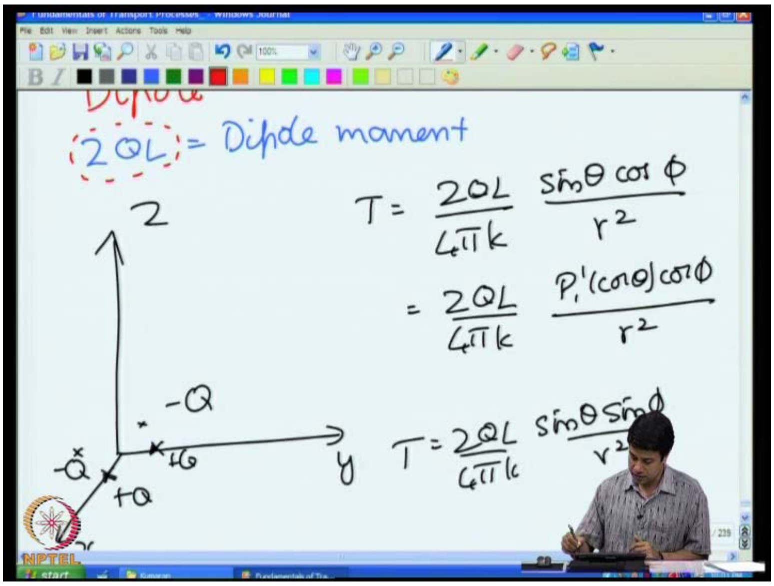

So, this is what is called a dipole and 2 Q times l is the dipole movement there is no net source because there is a source which is generating q pi unit time. There is a sink which is absorbing the same amount of heat. So, there is no net source, but, there is a temperature field due to this because of the small separation between the source and the dipole in the limit as the distance l goes to 0 while simultaneously maintaining 2 Q l as a constant you will get this n is equal to 1 m is equal to 0 or plus or minus 1 temperature field. So, Q has to increase and l has to decrease such that the dipole movement remains finite in the limit as l goes to 0. In that case you get the dipole solution.

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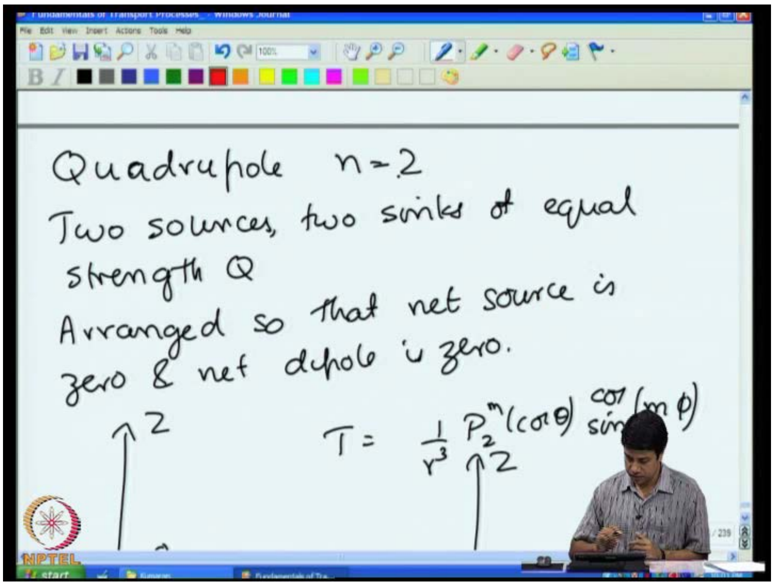

And then I showed you how n is equal to 2 can be obtained by superposing two sinks and two sources of equal strength arranged. So, that the next source is 0 and the net dipole movement is 0 and we looked at the symmetries of these and I showed you that they are identical to the symmetries of n is equal to 2 and m is equal to minus 2 to plus 2 in the spherical expansion.

So, let us continue our discussion of delta function sources and sinks.

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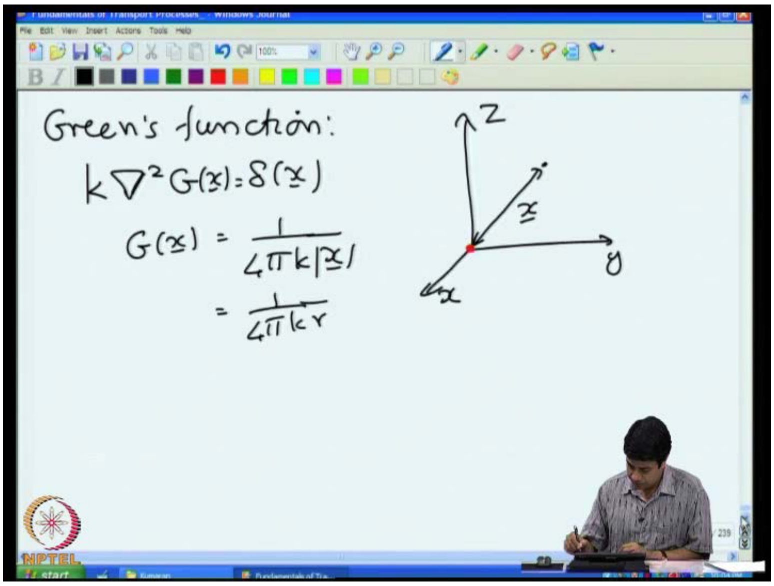

Now, the greens function can be its defined in various ways, but, for the present heat conduction problem; let me define it formally its defined as k del square g is equal to delta of x. So, this is the formal definition of a greens function. So, k del square g at the location x is equal to delta of x.

Now, the solution of this as we all know is given by g of x is equal to 1 by four pi k into the distance x. That is if I have this this is the solution for a point source located at the origin. So, this is the distance x vector, this is x vector and I have a point source at the origin. So, del square g k del square g is non 0 only when x is equal to 0 it is equal to 0 everywhere else.

So, in in that case solution for this equation is basically 1 by four pi k into r r is the modulus of x, the distance, the absolute value of the x factor, the distance from the origin of the location at which we are measuring the temperature. So, this is the green's function for a point source in an infinite domain. I can define green's function in finite domains based upon symmetries.

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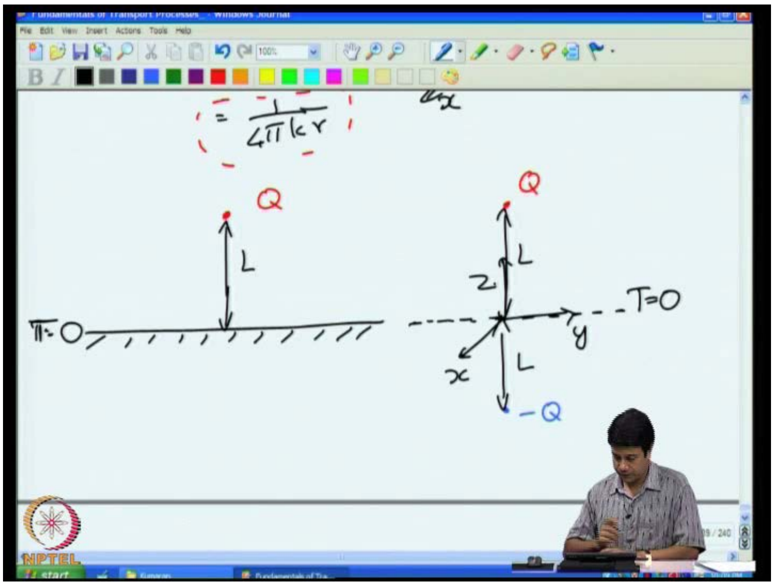

Consider the following problem; I have a wall here the wall is insulating. So, or let us let us just take I have a wall here at horizontal plane wall and I have a point source source with source strength Q q per unit time. This point source is not located in an infinite domain. It is located near a wall and on this wall I am setting the temperature t is equal to 0.

So, this is like, if I have for example, in winter if there is the ground is at a particular temperature and I have a point source near that surface and I would like to know what is the temperature field due to the presence of the point source; the temperature t is equal to 0 at the surface. If I have a temperature which was non zero; I could define a rescale temperature such that I subtract out the temperature of the wall from the total temperature. Now, the question is what is the temperature field due to this point source? Obviously, the temperature field is not going to be equal to the temperature field due to a point source in an infinite medium. In that case we have already got the Green's function here.

However, to get the temperature field in a finite domain I can use symmetries. So, t is equal to 0 here. Therefore, I if I have a point source, if I have another configuration let us say that let us say I have another system where I do not have a wall. So, this wall is not there. So, in this case let us say that I have source q at a distance l from the surface, I consider an alternate configuration where there is no wall there is a source of strength q at a distance l from the surface. However, I also have a sink minus q also at a distance l, but, below the surface I have a sink minus q at a distance l. But, below the surface the perpendicular distance of the source is l above the surface perpendicular distance of the sink is l. Below the surface in this case you can see that this plane here that I had drawn mid way between the source and sink is a plane of symmetry. There is heating from the top there is cooling below and simply because of symmetry the temperature has to be equal to 0 on the surface.

Because there is heating on top, there is cooling below and because of that exactly at the center between the two, the temperature has to be exactly equal to 0. So, the boundary condition at the plane of symmetry in this second configuration with 1 source and 1 sink is identical to the boundary condition at the wall. In the first configuration which has only a source the boundary conditions are exactly the same. You solving the the equation del square t is equal to source or sink in both cases and the boundary conditions are exactly the same. Therefore, the temperature field in the second configuration has to be exactly the same as the temperature field in the first configuration.

So, the temperature field in the second configuration is relatively easy to get. I need to put in a coordinate system here x y and this vertical coordinate. I will call it as z.

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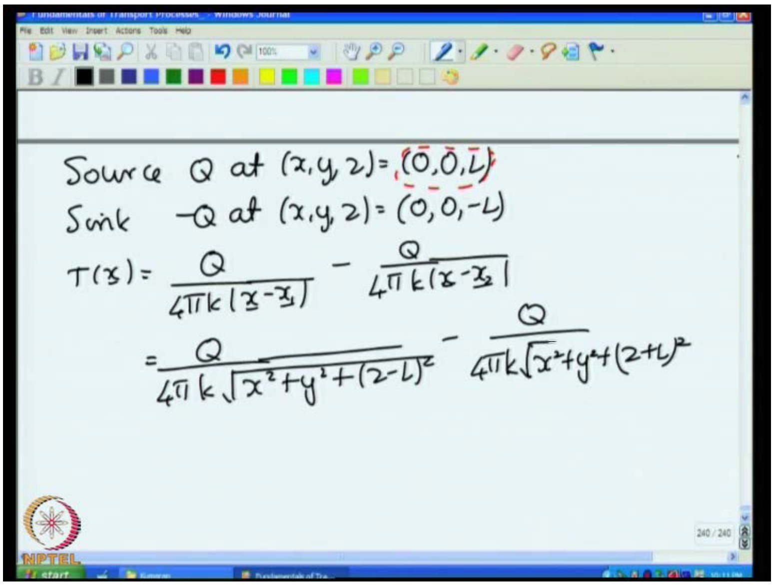

Therefore I have source q at x y z is equal to 0 0 l and I have a sink minus q at x y z is equal to 0 0 minus l.

So, that is in the second configuration, I have source at the location 0 0 l I have a sink at the location 0 0 minus l. So, therefore, the temperature at any location x, in the second problem is just given by the temperature at the given location x is equal to Q by four pi k into x minus x 1 where x 1 is the location of the source and there is an additional contribution due to the sink which is minus q by four pi k into x minus x 2.

So, where x 2 is the location of the sink. So, this is the temperature field in the second configuration. But, as I said the temperature field in the second configuration has to be identical to the temperature field in the first configuration because the equations that I am solving are the same. The boundary conditions act the plane of symmetry in the second configuration is exactly the same as the condition at the wall in the first configuration. So, because of that the temperature field in the first configuration is also given by this.

So, I can expand this out this is q by four pi k the source is at the location 0 0 l. So, I will have the source as square root of x square plus y square plus z minus l the whole square minus q by four pi k square root of x square plus y square plus z plus l whole square.

So, this is the final solution for the temperature field. In the second configuration, that I just showed you and I told you that the temperature field in this second configuration has to be exactly the same as the temperature field due to a distributor, a point source near a wall. This essentially is the Green's function for the solution when you have a wall present with temperature 0 at the surface.

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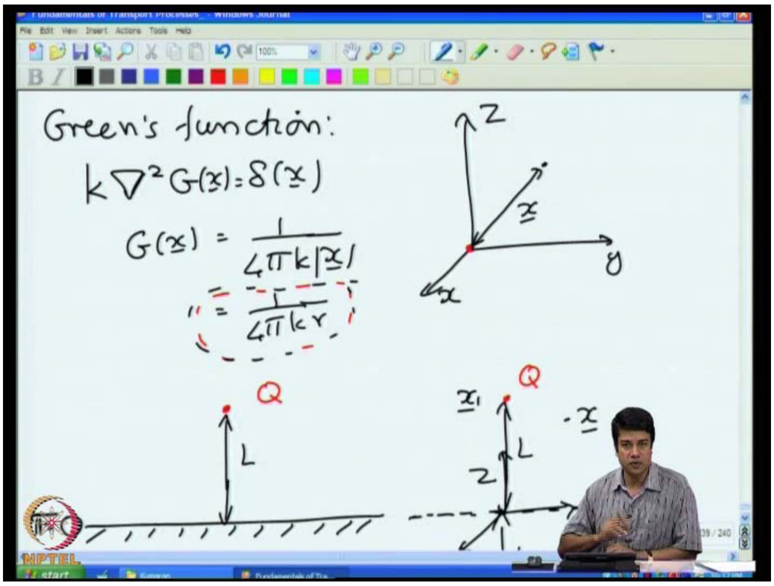

This was the original Green's function which I had in an unbounded meeting, but, when I have a wall present the Green's function is different. The Green's function due to a point source near a wall has one contribution due to the source point, source itself and a second contribution due to the image sink that was located at minus l. So, that you get the exact the exactly correct boundary conditions at the wall itself.

So, this was with temperature equal to 0 at the surface. In many practical applications 1 will have a heat flux equal to 0 at the surface.

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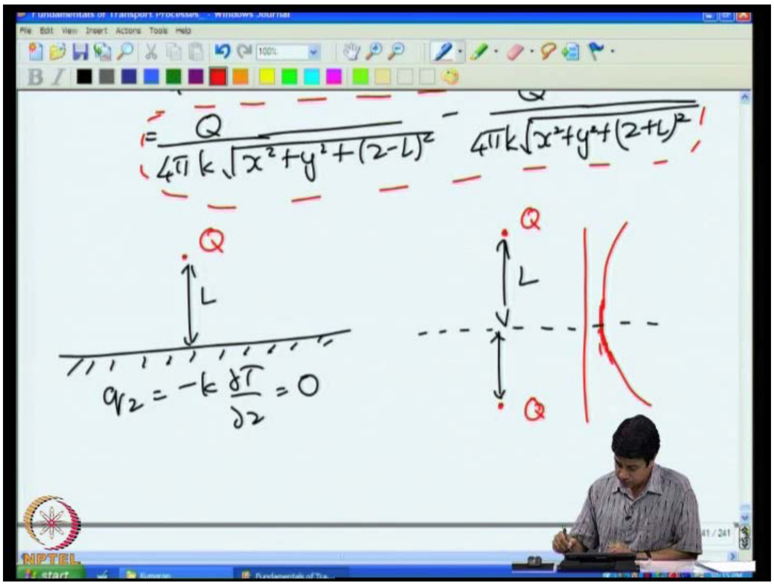

So, this is the Green's function for 0 temperature at the surface or for a fixed temperature at the surface because I can always define a rescale temperature in such a way that the temperature at that surface is equal to 0. One can consider an alternate situation where I have the heat flux q z is equal to minus k d t by d z is equal to 0 at the surface and then I have a source point source with a source strength q.

So, at this flat surface I have a source of source strength Q. But, the heat flux at the wall is identically equal to 0. This I can consider an alternate configuration where I have let us say this is at a distance l and at that same distance l, I have a source plus q. However, at the distance minus l I have another source of both source strength plus Q in this case. Since, I have plus q above and plus q below this on this plane of symmetry, I will have the derivative of of t with respect to z being identically equal to 0 because it is a maximum. The the source strength is here is a same as a source strength here. Therefore, as I come, as I approach the surface from above and from below I should get the exact same temperature from below as I get from above ok.

So, if I am equal distance above and equal distance below; the temperature should be the same. So, I require that the temperature if I plot the temperature on this axis the temperature has to be the same both above and below. So, that exactly at the mid point where they meet if the temperatures are the same from above and below of the slope the, at this midpoint has to be equal to 0. Therefore, the temperature field satisfies 0 derivative or 0 flux conditions at this surface. Now, the solution for this is easy to write as well I have a source plus q a source plus q below.

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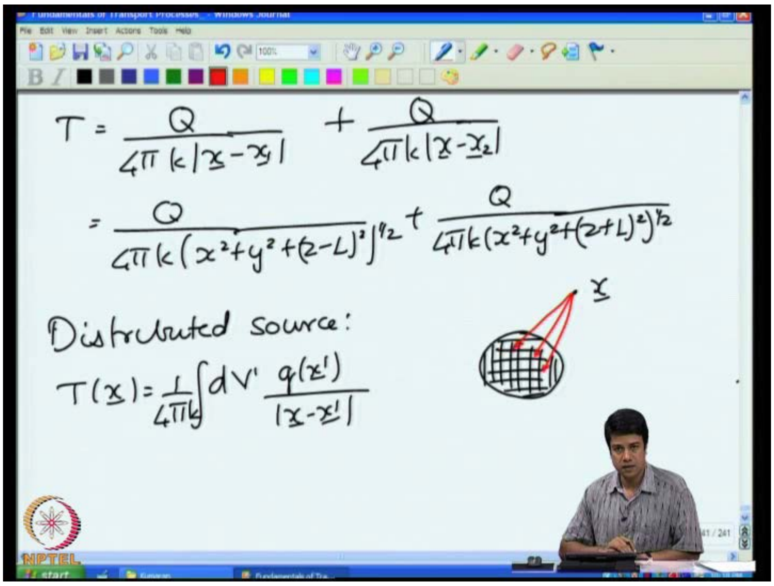

And therefore, the temperature field will be written as q by four pi k into x minus plus q by four pi k into x minus x 2 or if I have to expand it out for this case q by four pi k into x square plus y square plus z minus l whole square power half plus q by four pi k the distance between the source and the sink.

So, we got one particular solution if I had a point source close to wall with 0 temperature and another solution if I had a point source close to a wall with 0 flux for an unbounded domain. I showed you that the temperature field due to a distributed source can be written as the integral of the temperature field due to a point source. So, I can write due to distributed source t of x is equal to integral dv prime q of x prime 1 by four pi k by x minus x prime where x prime is the source point and x is the observation point.

So, pictorially if I had some distributed source, I divide that into small domains and at a given location x, I have to sum up the contribution to the temperature due to each of these small 1's. So, that gives me the temperature at a given location x due to a distributed source of heat and then I showed you if I have a wall with temperature 0 near a point source I can get the solution by taking an image of that point source at the wall. It is called the method of images.

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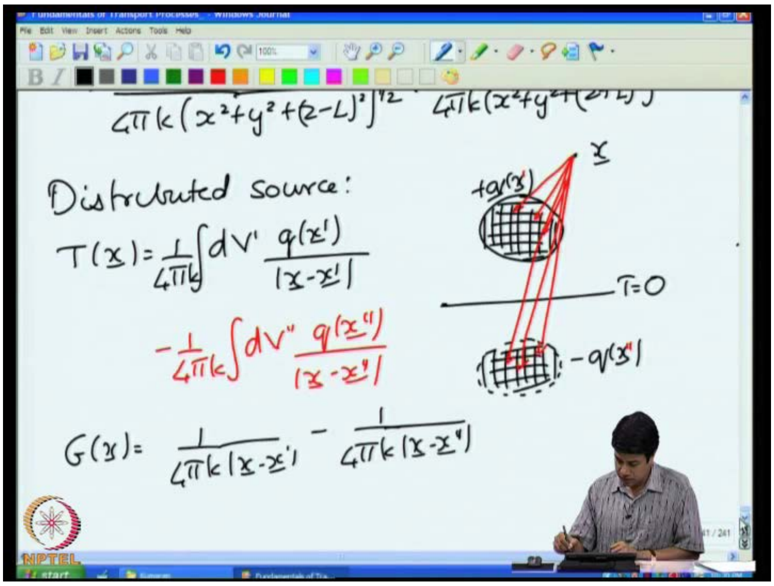

So, if I have a wall with temperature t is equal to 0 and I have a point source at a given location then I just need to integrate over both the point. I just need to add up the contributions due to the point source as well as due to its image. If I have a distributed source, I do exactly the same thing. If temperature is equal to 0 rather than trying to enforce the temperature boundary condition on the wall itself, I just consider an image source as well.

So, in this case there is going to be a contribution both due to the source itself as well as due to the image and the total temperature field is going to be equal to this. Note that the image if the source is plus Q of x the image has to be minus Q of x if the temperature is 0 at that surface. So, the additional contribution that I get will be equal to minus 1 by four pi k integral d v double time q at x double prime by x minus x double prime where x doubled prime is the location of these image points and x prime is the location of the source point.

So, if I do not have a source in an unbounded volume, but, I had a source in a finite volume with a wall which was enforcing 0 temperature boundary conditions at the surface I could still use the method of images in this particular case. The green’s function for this particular case g of x is equal to 1 by four pi k into x minus x prime plus minus into x minus x doubled prime where x double prime is the location of the image the reflection of the source point on the wall.

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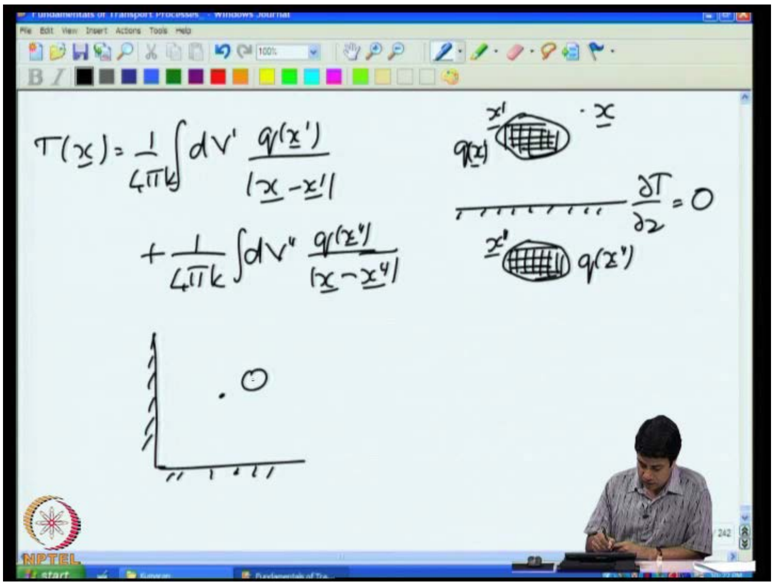

So, this is for 0 temperature boundary conditions at the surface on the other hand if I had an insulating wall on which dt by dz is equal to 0 the normal derivative of the temperature is equal to 0 and I have a source here. This is the observation point x. This is the source point x prime. Then I need to consider an image of the source below this is x double prime. And in this case if I have a source q of x prime here since the flux is 0 at the wall you have the same q at x double prime. Here the same source both above and below in such a way that that derivative of the temperature is equal to 0 at the wall and this case it is quiet easy to write the temperature field is equal to 1 by four pi k integral d v prime q at x prime by x minus x prime. In this case, I have plus because I need 0 flux condition at the wall ok.

So, using the concept of sources and sinks, I can use that to enforce boundary conditions as well I can enforce 0 temperature boundary conditions as well as 0 flux boundary conditions or mixed boundary conditions where the temperature and the flux are related by using different combination of sources and sinks. So, this works when I have just 1 wall. If I have two walls with a corner, if I have two walls with a corner and I have a source Q here if these two walls have 0 temperature.

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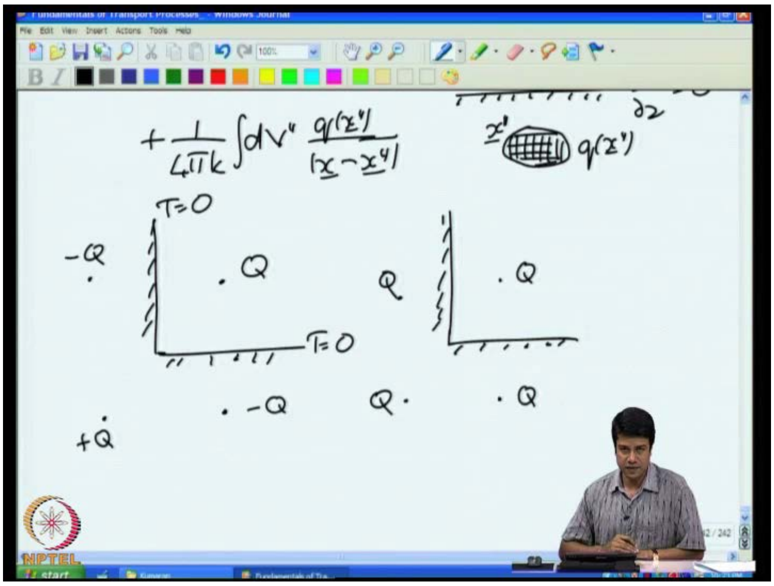

Then by symmetry, I need to put minus q and minus q and plus q here. This will ensure that t is equal to 0 on the surface and t is equal to 0 on this surface. If I had 0 flux boundary conditions then I could ensure that by putting plus Q here and plus Q here.

In in order to get 0 flux boundary conditions.

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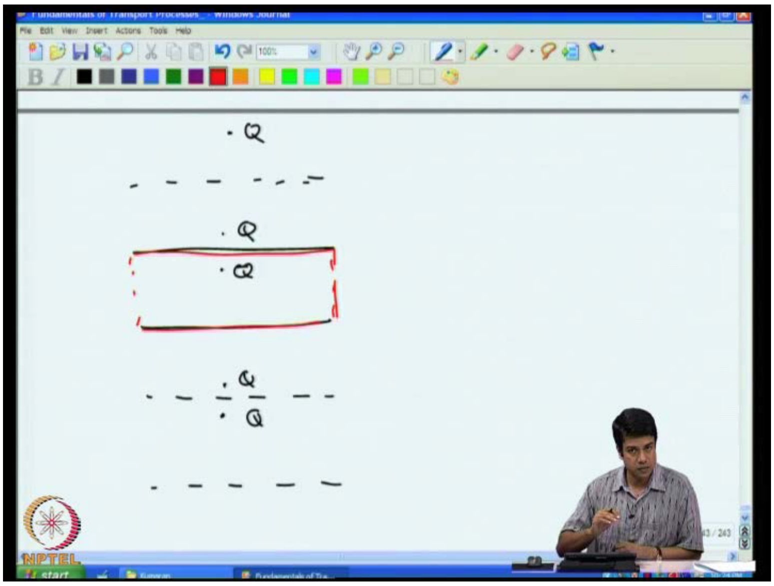

And combinations of these if I had two walls and a source in between then I have to consider many images both above and below. Just two images will not be sufficient and to consider an infinite series of images both above and below. However, the advantage of this is that the temperature field that I need to calculate is only within this domain. The temperature field that I need to calculate is only within this domain and the effect of sources and sinks far away decreases 1 over distance as you go further and further away.

So, because the the the the the the contribution due to the source and sinks which are far away decreases 1 over distance. If I want a good numerical approximation; I can truncate this these images at some finite number in such a way that I get a good numerical approximation within the domain of interest. Now, formally the Green's function formulation works as follows. So, I will just explain to you the formal way in which the Green's function formulation is written down.

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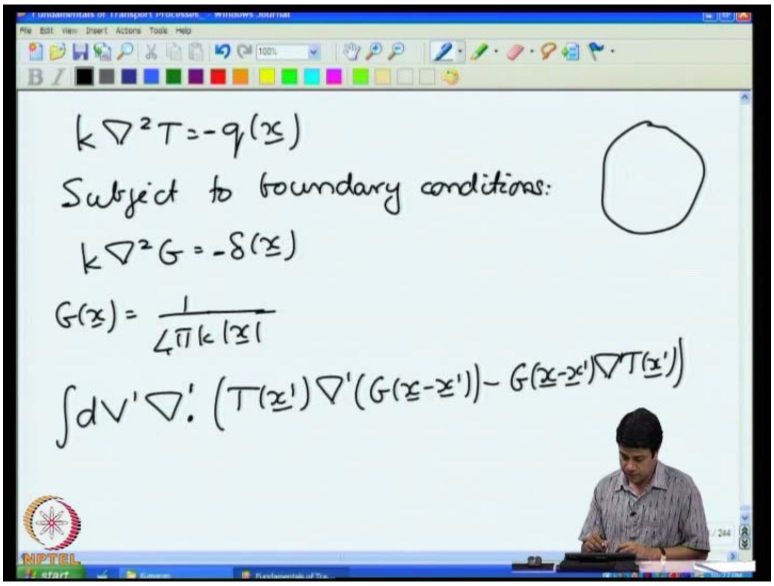

So, let us assume that we have some a problem to solve k del square t is equal to some q or can we write this is as q of x subject to boundary condition subject to some boundary conditions. On some domain that I have I have some domain and I am trying to solve this equation subject to boundary conditions on this domain, I define an alternate problem that is k del square g is equal to delta of x. I should have a negative sign here because both of them have negative signs k del square g is equal to minus delta of x a delta function a point source at some location x.

I know the solution for the second problem g is equal to 1 by four pi k k times x the distance between the two locations. Now, consider the integral over the entire volume integral with the entire volume of del dot of can we be a little more precise here ((no audio 47:59 to 48:36)).

So, I consider this volume integral, the divergence of t of x prime times the gradient g of x minus x prime minus g of x prime times a gradient of t of x minus x prime should be a little careful here in this case. x prime is the source point and x is the observation point.

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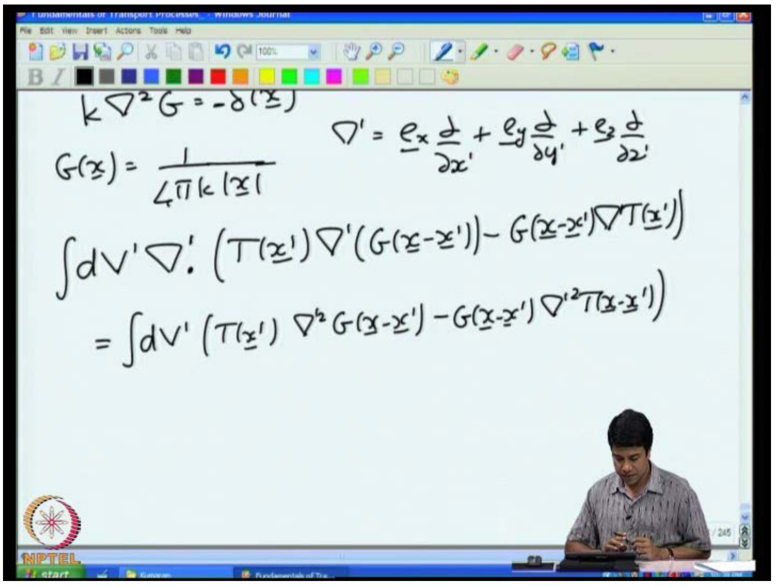

So, the operator del prime that I defined here is equal to e x d by d x prime plus e y d by d y prime plus e z d by d z prime. So, that is the divergence operator with respect to the source point. This now, I can expand out this I can expand out I have divergence of two terms here and two terms on the right hand side which I can expand out using product rule.

So, this thing is equal to integral d v prime of t of x prime del square g of x minus x prime minus g of x minus x prime del prime square t of x minus x prime you can see that the first gradient of t gradient of g will cancel out because I have taken the difference between these two and now here for del square I have k del square g is equal to minus delta of x. So, k del square g will be equal to minus delta k del square del prime square g of x minus x prime will be equal to minus delta of x minus x prime.

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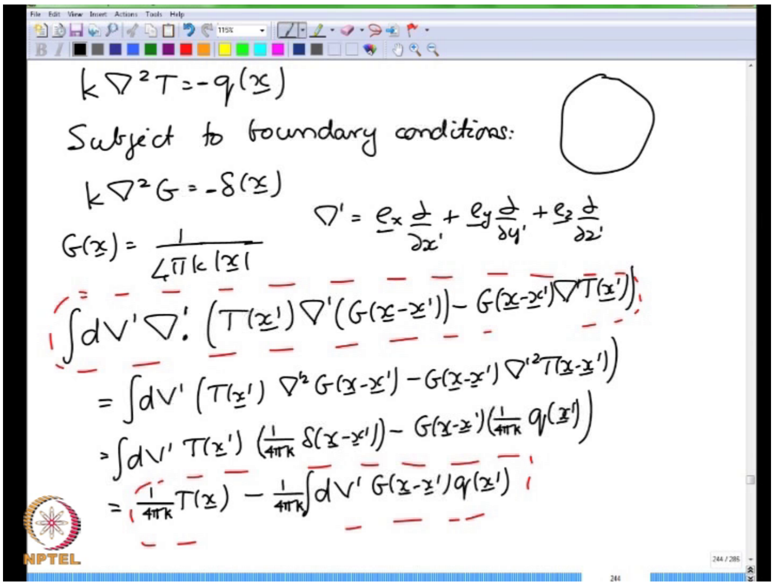

So, this is equal to integral d v prime of t of x prime into 1 over k delta of x minus x prime minus g of x minus x prime into 1 over k del square t is equal to the source del square t is equal to the source minus q del square g is equal to minus delta and del square t is equal to minus q.

So, this will just be equal to q of x prime and integral of x over x prime of delta of x minus x prime is just going to be equal to 1 over k times t of x minus integral d v prime g of x minus x prime 1 over k q of x prime. So, that is what I get from this particular integral. I could also simplify this integral using the divergence theorem. I could also simplify this integral using the divergence theorem. If I use the divergence theorem then this integral integral d v prime del prime dot. If I use the divergence theorem this becomes integral over the surface of the unit normal dotted with ((no audio 52:17 to 53:00)). So, this gives me the expression.

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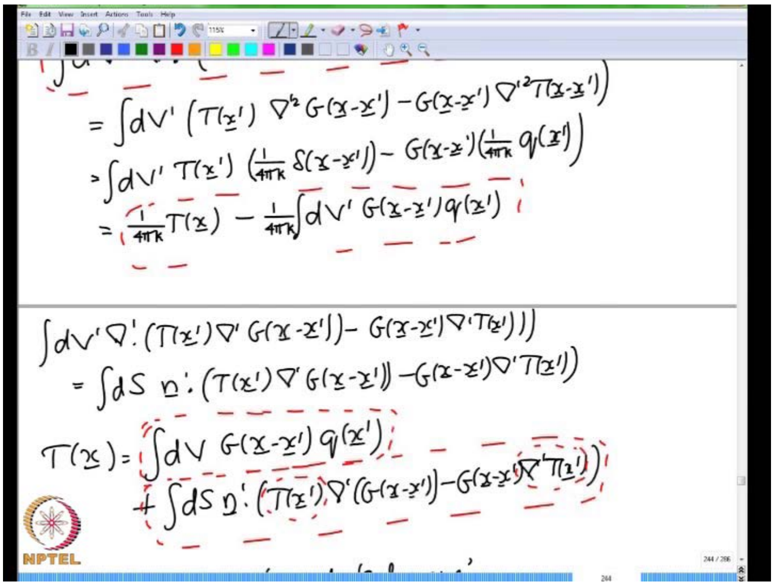

So, I get 1 over k times t of x minus this sources or sinks over the entire domain is equal to this surface integral. So, putting all these together, I will get t at the location x is equal to integral dv prime g of x minus x prime times Q of x prime plus integral over all the surfaces found in this volume of n prime dot t of x prime gradient of g of x minus x prime minus g of x minus x prime del prime t of x prime. So, this now is the final solution for the temperature field.

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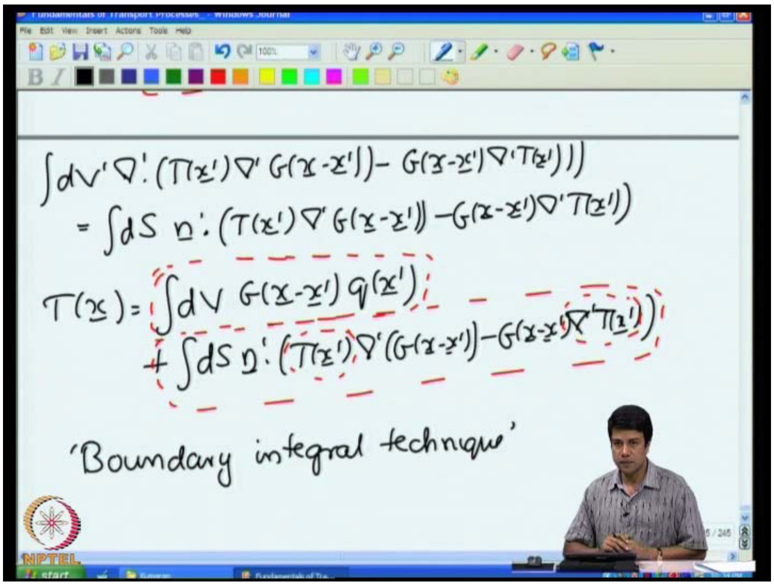

This is precisely the temperature field that we had got due to a distributed source. If you recall if g is 1 by four pi k into x minus x prime, this is exactly the temperature field that I got due to the distributed source. This additional term here is the temperature field that you get due to the boundary conditions that are applied.

Note that, g is a known function g is equal to 1 over four pi k into x minus x prime. So, g is a known function. If I know the temperature or its derivative on the bonding surfaces if either t is equal to 0 or the derivative of the temperature is equal to 0 on the bonding surfaces then I can choose my Green's function in such a way that it satisfies that particular boundary condition. And once I have done that, I can now get the temperature field everywhere from a knowledge of the temperature field on the bonding surface. Consider the special case where Q of x prime is equal to 0. No sources or sinks within the the domain. In that case, the first term on the right is 0. The second term is a surface integral. This surface integral can be evaluated if I know the temperature on the surface. Therefore, the temperature at every point within the volume can be evaluated exactly if I know temperature on all of the bonding surfaces.

So, in the case where there are no sources or sinks within the volume; if the temperature field supplies, satisfies the Laplace equation and I know the temperature field on the surface; then I know the temperature field automatically everywhere within the domain. So, this principle basically forms the basis of what is called the boundary integral technique. If I want to solve Laplace equation for the temperature field, the diffusion equation in some domain and there is no source or sink within the domain. If I know the temperature field on the surface then I know the temperature everywhere within the domain because I can reduce the temperature on the surface to an integral and i am i am sorry i can reduce the temperature everywhere within the domain to an integral over the bonding surfaces of the Green's functions times the temperature on that bonding surface.

So, I do not know, I do not need to know the temperature everywhere in the volume provided I have the temperature specified at the surface. I know precisely what the temperature is everywhere. So, this Green's function formulation basically as solved del square g is equal to some delta function. So, the Green's function is basically the inverse of the Laplacian acting on the delta function and once I know what the Green's function is; I can then get the distribution, the temperature in terms of that Green's function for source in a infinite medium. That Green's function is 1 by four pi k times x minus x prime and I have a planes of symmetry, I can find it using images and for the most general case I can find it in this manner by writing, expressing the temperature field in terms of the temperature on the bounding surfaces as well as any sources or sinks that are there within the volume.

So, with this we conclude our discussion on solutions of the diffusion equation. I showed you by two methods; one by separation of variables, the symmetries that you get there and the other by means of point sources and and and sinks and there there distributions and how these two are related.

So, and this illustrates for you what a useful concept, the delta function concept is. So, this will complete our discussion on the diffusion equation. Next lecture we will look at the limit of high peclet number where we would expect convection to be dominant in comparison to diffusion. How do we solve problems in that limit? So, that will be the discussion for the next few lectures. We will see you in the next lecture.