Fundamentals of Transport Processes
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Lecture No. # 34
Diffusion Equation Multipole Expansions

Welcome to lecture number thirty four of our course on fundamentals of transport processes. We were looking at the solution of the diffusion equation as I had told you in the last class. We derived conservation equations for mass and energy. All of these had a common form. The time derivative of concentration plus divergence of the mean velocity times concentration is equal to d times the Laplacian of concentration where d is the diffusion coefficient plus any sources or sinks of mass. Similar equation is obtained in the case of energy except that you substitute temperature instead of concentration and thermal diffusivity instead of mass diffusivity. The peclet number is the ratio of convection and diffusion in these problems. yeah

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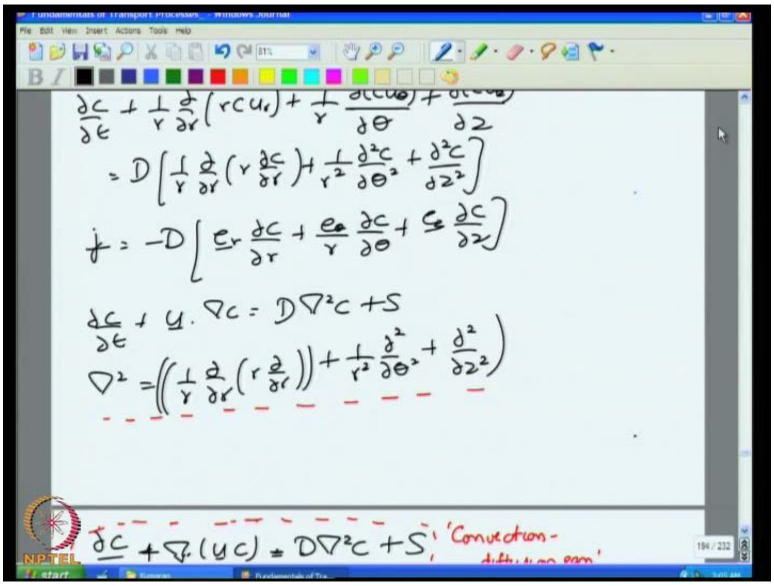

If I scale all lengths by characteristic length and all velocities by characteristic velocity;

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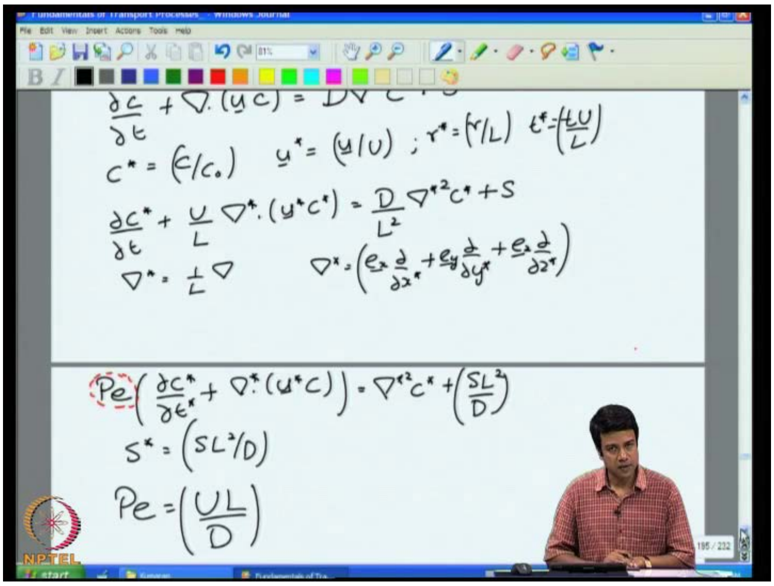

Then I get a dimensionless equation of the form the peclet number times the time derivative plus the convective terms is equal to the diffusion term plus any source or discretion of energy.

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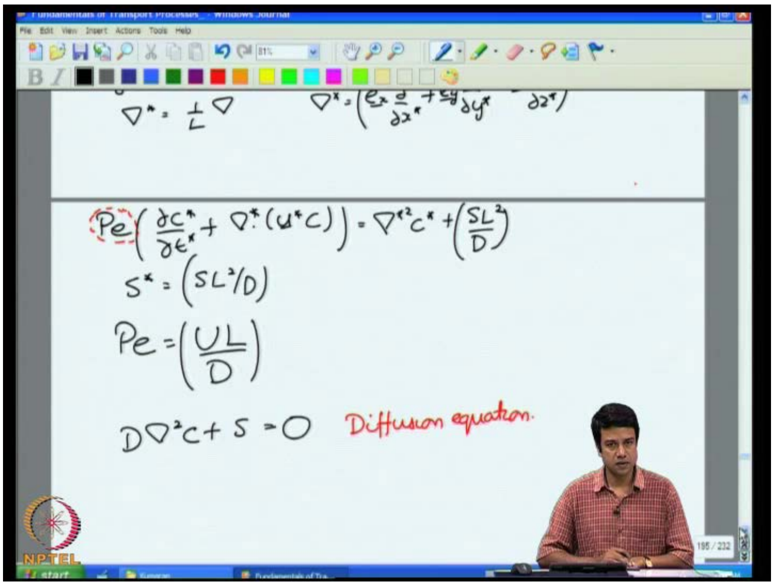

And the peclet number is small in the ratio of convection to diffusion is small and in the leading approximation, we neglect the convection of energy all together and we are left with solving the diffusion equation. D del square c plus any source of energy is equal to

0. So, this is the solution of the diffusion equation for the case where diffusion is large compared to convection.

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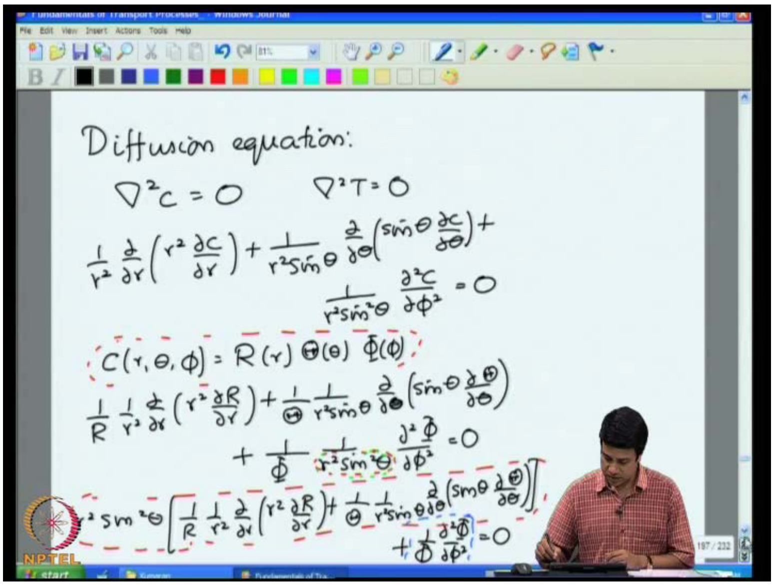

And first we had looked at ways of solving it by separation of variables in spherical coordinate system. The three coordinates are R, the distance from the origin; theta, the azimuthal angle made by the radius vector with respect to the z coordinate and phi, which is the angle along the x y plane.

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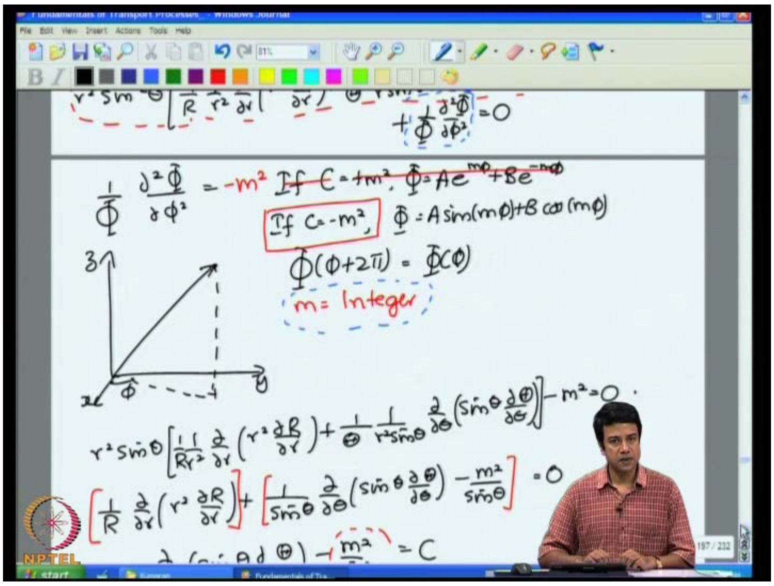

We use separation of variables and from symmetric conditions we got discrete eigen values in the phi direction. The requirement that when you go around by an angle 2 pi, you come to the same location means that the solution for the concentration should be identical when you go around by an angle of 2 pi. That means that this constant, the constant minus m square in the equation for the phi coordinate m has to be an integer. So, this gives us the set of discrete Eigen values in the phi direction and the associated Eigen functions or basis functions or sine and cosine functions in that direction. In a similar manner we solved the equation for the theta coordinate.

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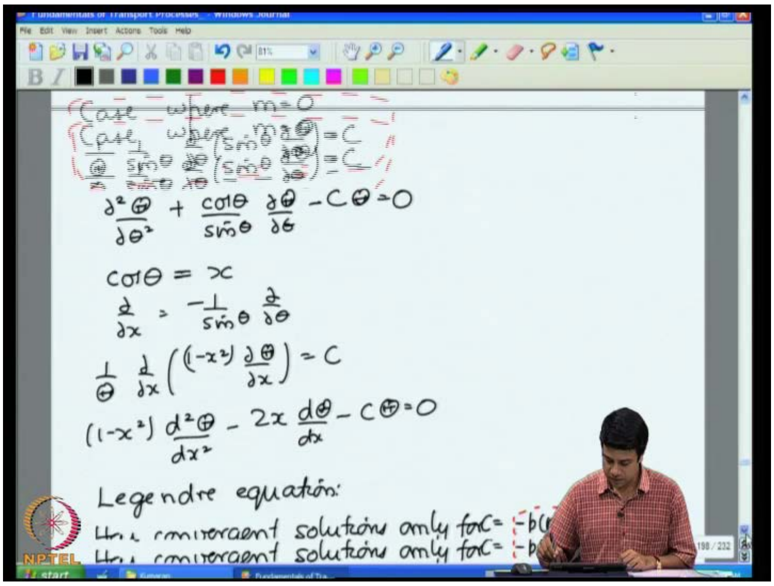

And in this case we got solution in terms of the Legendre polynomials, the equation that we ended up with in the theta direction was the Legendre equation and the solutions for this were in terms of the Legendre polynomials in the theta direction.

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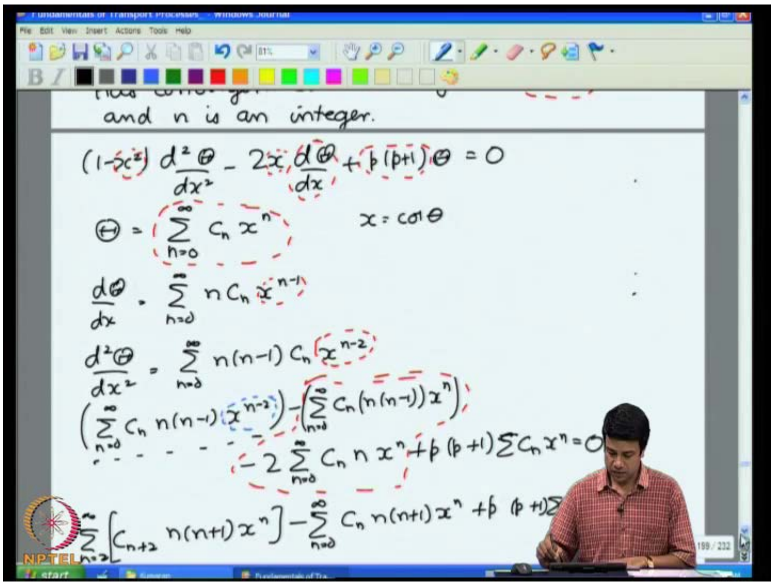

And I showed you by means of the series solution that if the polynomials are to be finite both at theta is equal to 0 and theta is equal to pi or at x is equal to minus 1 and x is equal to plus 1. The constant in the equation has to be of the form n into n plus 1.

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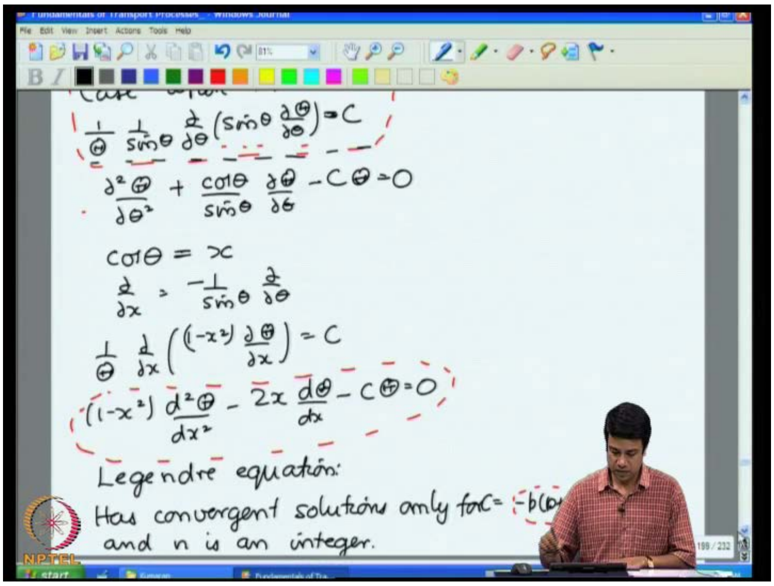

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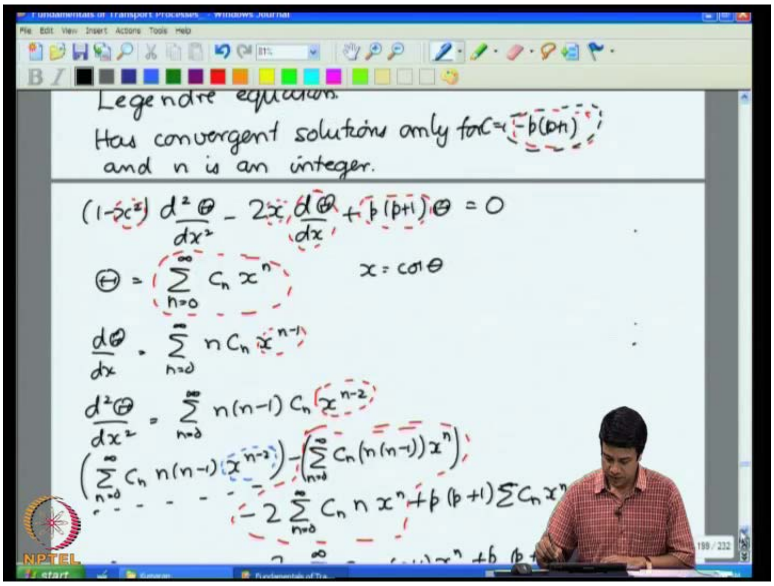

So, the convergent solutions for this equation can be obtained only for discrete Eigen values of this problem from the requirement that the solution has to be finite both at x is equal to plus and at x is equal to minus 1.

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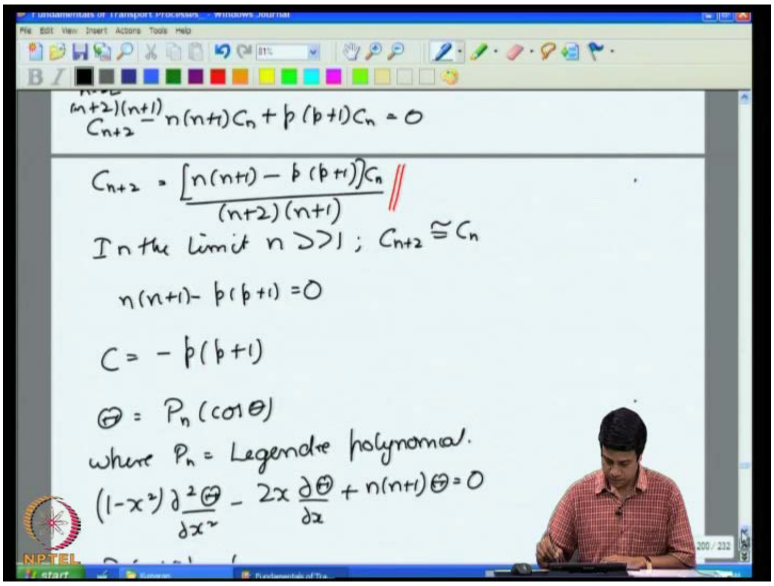

And so, we got this Legendre polynomial expansions.

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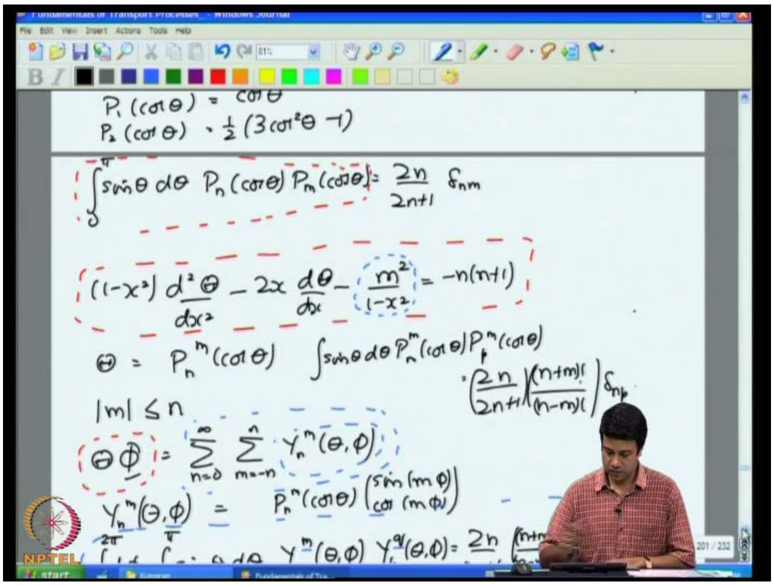

And I had written down the first few polynomials for you and the product of theta and phi gives you functions of theta and phi which are called spherical harmonics p n m of cos theta times sin and cos of n phi. These are orthogonal to each other when the inner product is defined as integral sin theta d theta from 0 to pi and integral over phi from 0 to 2 pi. So, these form an orthogonal set of polynomials. If you multiply one polynomial with another one and integrate you get 0. You get non-zero result only if you take the inner product of two identical polynomials.

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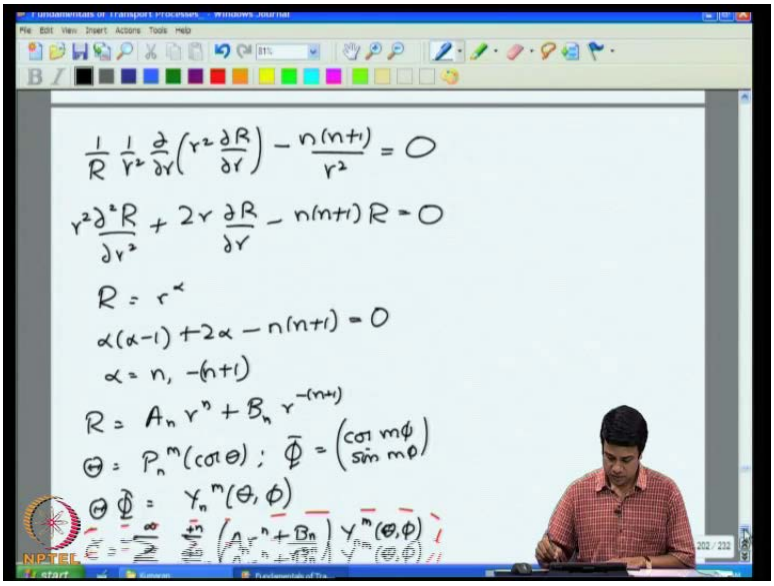

And then we solve for the r the the radial coordinate and we got equations. We got solutions which are in terms of power loss with respect to the radius r power plus n and r power minus of n plus 1. So, that give us the final solution for the spherical harmonics, for the solution of the diffusion equation in a spherical coordinate system. This is a series summation n is equal to 0 to infinity and n goes from minus n to plus n. So, for every value of n, you have m going from minus n to plus n. That means that there will be 2n plus 1, such solutions. So, if n is equal to 3 and you go from minus 3 to plus 3; you have minus three minus 2 minus 1 0 1 2 and 3. So, seven solution for each value of n.

So, this is the series solution. The the separation of variable solution for the general case for any coordinate system, the coefficients in this are evaluated ultimately from the orthogonality conditions.

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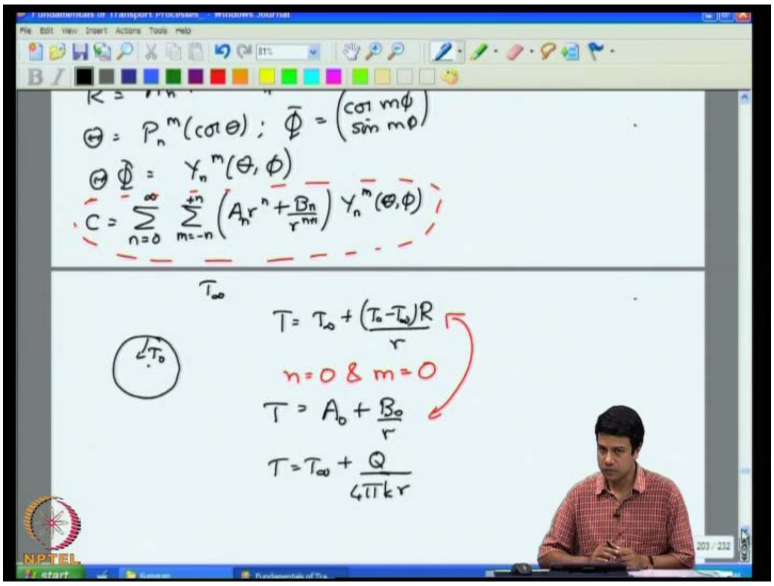

I showed you that the particular case of the heat conduction from a sphere which we had solved some time back in one dimension. It is just the solution for n is equal to 0 and m is equal to 0. This is spherically symmetric. So, y 0 0 is just equal to 1. It is independent of theta and phi.

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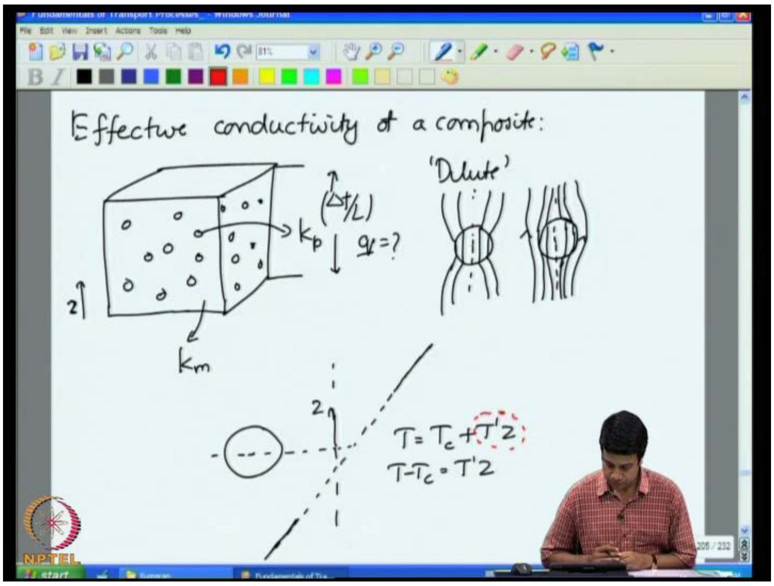

And then we did the example of the effective conductivity of a composite. In this case, there is a spherical particle of conductivity K p in a matrix with conductivity K n. Far away from the particle there is a linear temperature gradient, the temperature has the form t c plus t prime times z where t c is the temperature at the center of the particle. And in order to, if determine the effective conductivity one has to calculate the correction to the heat flux due to the presence of particles. So, one has to calculate what is the actual heat flux within a particle when it is placed in this linear temperature gradient. The temperature gradient has the form t prime. So, the temperature goes as t prime times z far away.

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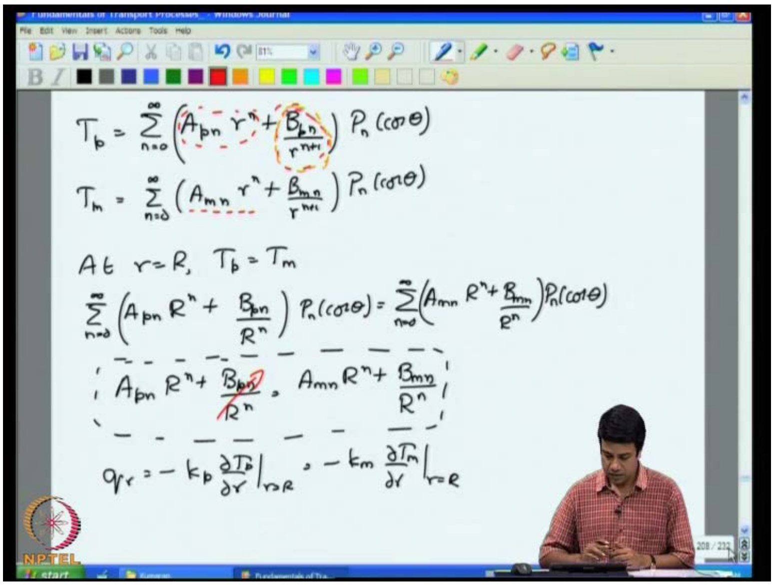

So, as if the temperature goes as t prime times z. That means, that t prime times z is equal to t prime r p 1 of cos theta and I had shown you that because the temperature far away has symmetry 1 0. That is, this p 1 0 of cos theta. Far away the temperature has this p 1 0 symmetry. There is no forcing with with respect to the temperature field. For any other symmetry form, the only forcing for the temperature field is due to the 1 0 symmetry. Therefore, the solutions will also have that same symmetry ok.

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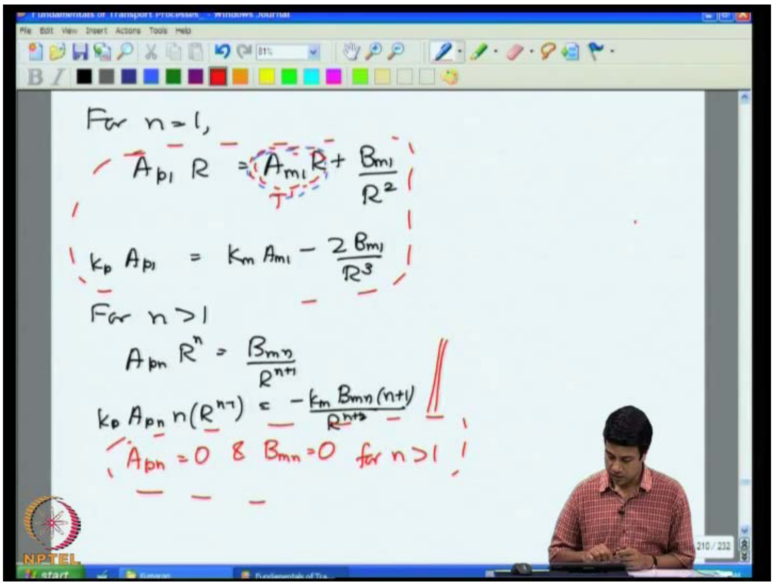

In this particular case therefore, I can just write down the solutions with n is equal to 1 and m is equal to 0 and evaluate those coefficients alone. As I showed you in the last lecture, all other coefficients turn out to be identically equal to 0 for n is greater than 1 or other coefficients turn out to be identically equal to 0. Only the coefficients 1 0 that is a p 1 and b m 1 turn out to be non 0.

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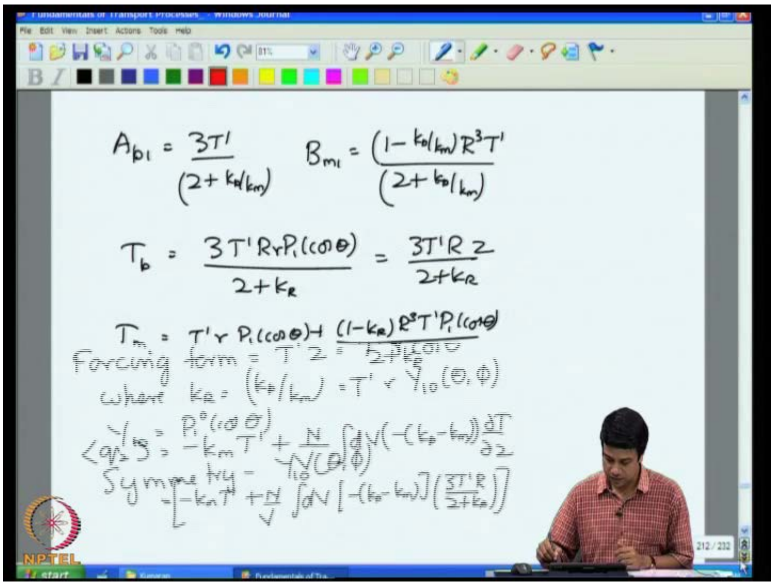

And from that, I can straight away evaluate what is correction to the temperature field due to the presence of this spherical inclusion.

And we had briefly discussed these spherical symmetries. We will come back to that a little later.

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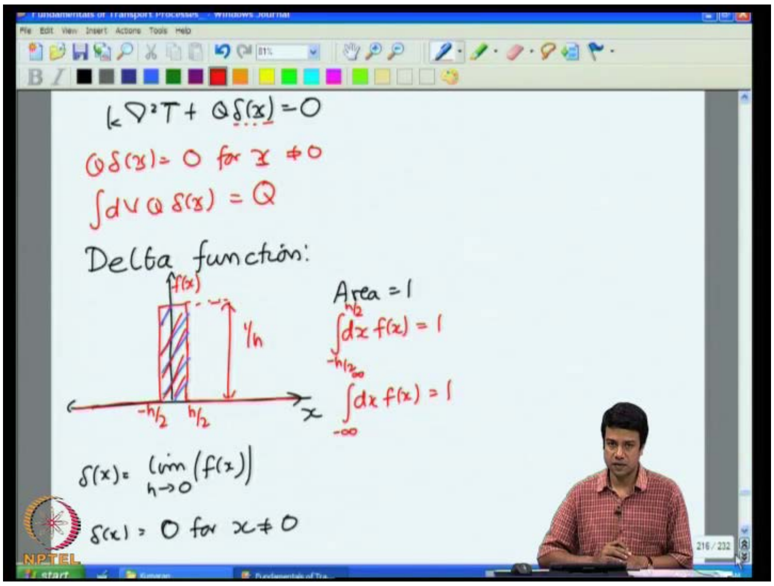

And then I had defined for you the delta function non zero only when x is equal to 0. It is 0 for all x not equal to 0 and the area under the delta function is equal to 1. It is given by the limiting form of this function which is 1 over h between minus h by 2 and plus h by 2 0 otherwise in the limit as h goes to 0. So, this function becomes taller and taller as h becomes small. It becomes thinner and thinner in such a way that the total area is a constant.

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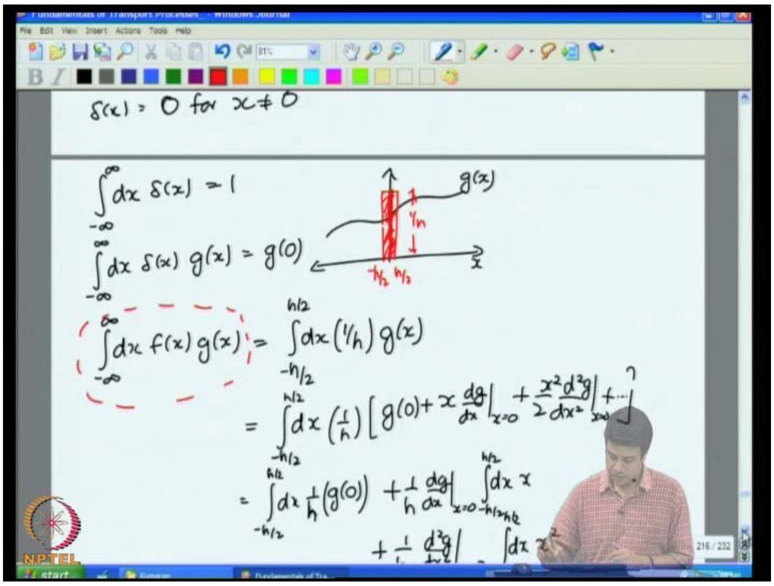

And if I multiply this function by any other function g of x and integrate, it just picks out the value of that g at x is equal to 0. So, briefly these are the properties of the delta function, this is a one dimensional delta function. It has dimensions of inverse length because integral d x times delta x is equal to 1. So, delta of x has dimensions of one over length can be extended quite easily to two and three dimensions.

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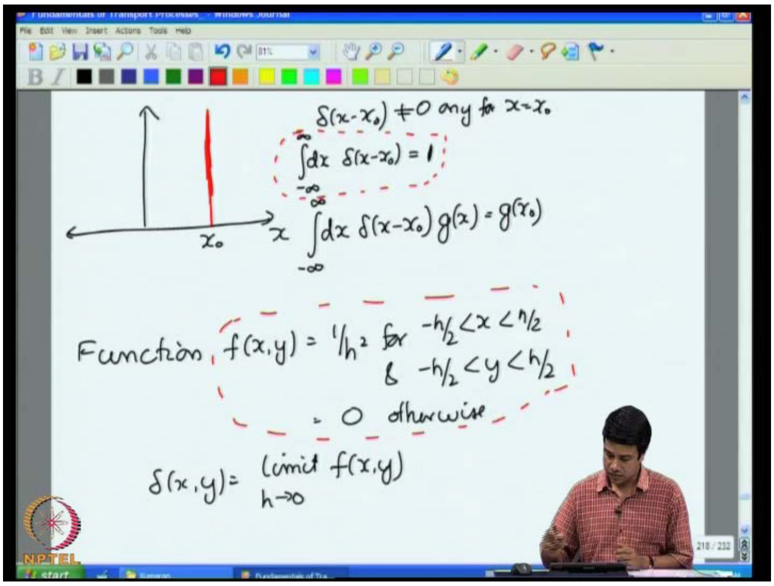

I just extend the definition. This function is equal to 1 over h square for minus h by 2 less than x less than h by 2 and minus h by 2 less than y less than h by 2 and its equal to 0. Otherwise, it is the delta function of x and y it is the limit as h goes to 0 of f of x and y.

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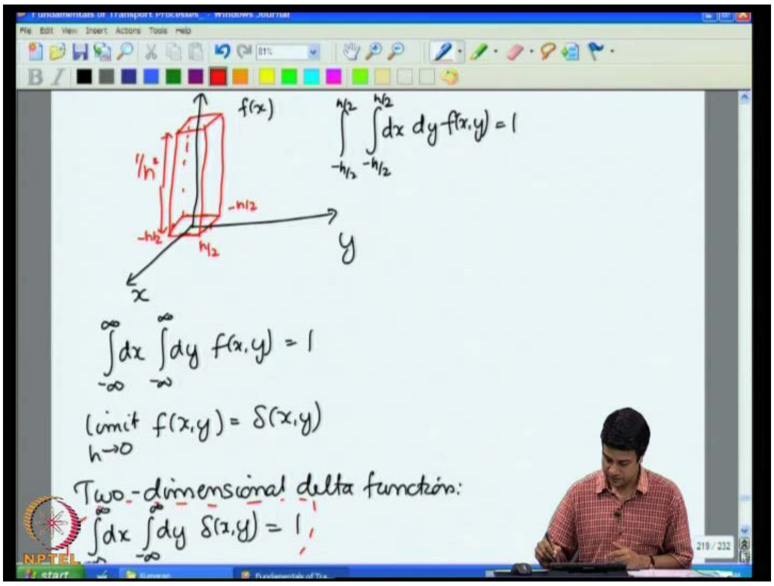

And I had plotted the form of this function for you. It looks like a cube whose height goes to infinity and whose area of cross section goes to 0 and this has dimensions of 1 over length square.

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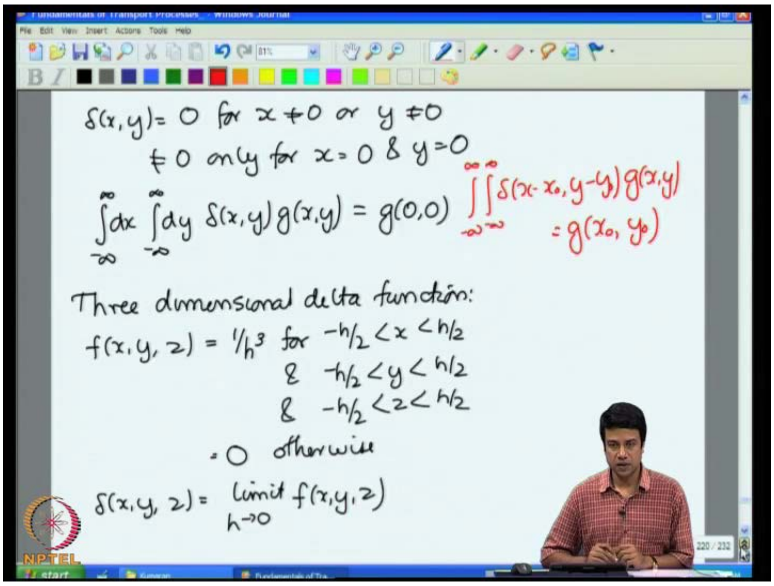

And finally, we came down to the three dimensional delta function. This is non zero only within a cube of of of side h. At the origin it is 0. Everywhere else, the total integral over the entire volume of delta of x times the volume is equal to 1.

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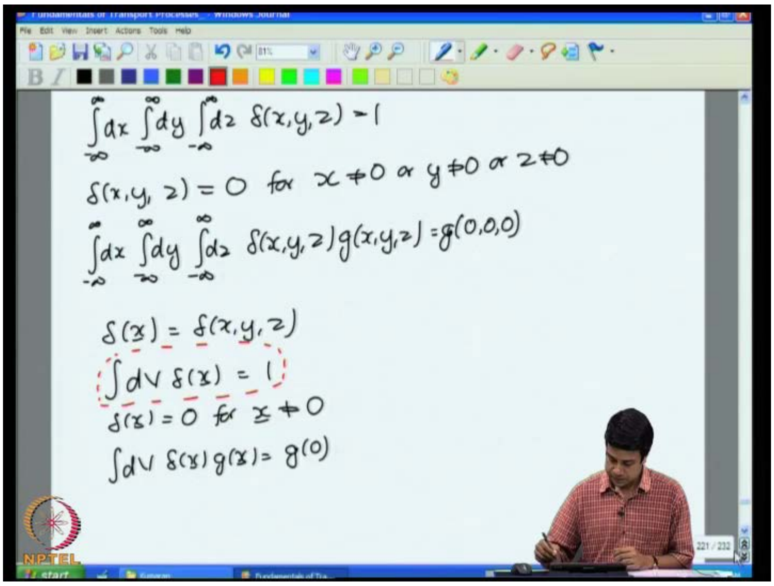

And once again it picks out the value of the function at the origin itself.

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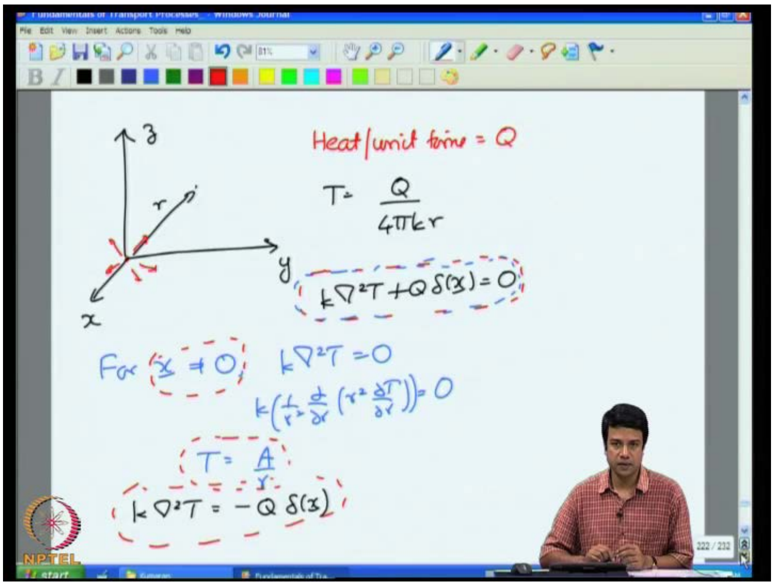

And I showed you that the solution for the heat conduction equation for a point source t is equal to q by 4 pi k r is also a solution for the differential equation k del square t plus q times delta of x is equal to 0. So, this the solution of this diffusion equation is identical to the solution for a point source at the origin where the heat coming out per unit time is equal to capital q and I showed you that by actually solving this for non zero value of x and then equating the total flux coming out to the heat generated per unit volume.

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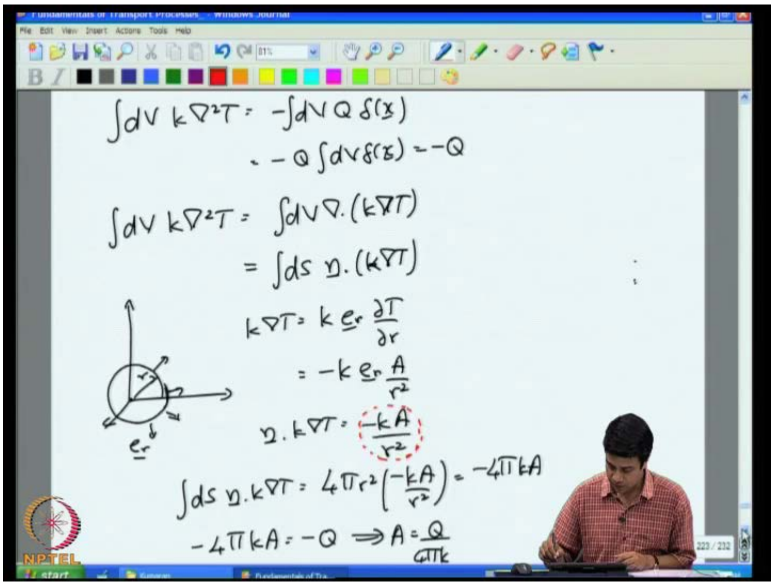

So, this is a point source. The solution for a point source is temperature is equal to q by 4 pi k r.

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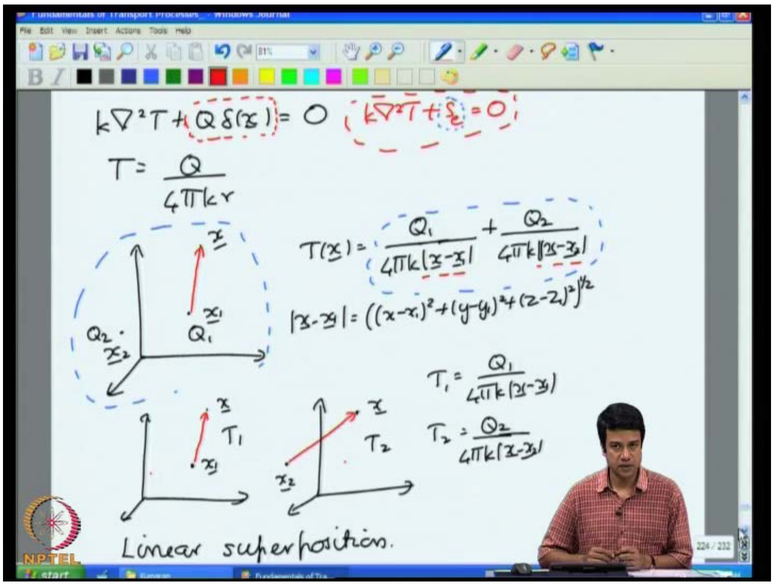

And I had told you that if you had multiple sources; 1 can use linear superposition. If you have multiple sources, one can use linear superposition to write down the temperature field due to each of these point sources. You cannot do that for finite objects. You can do it only when there is a point source and that is why the concept of a point source is such a powerful concept.

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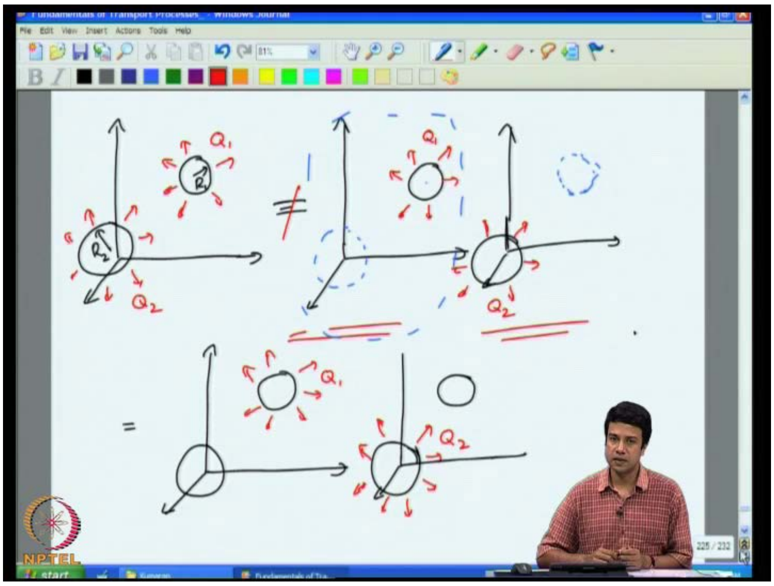

If you had finite objects, if you had tried to do linear superposition you have to make sure that all the boundaries remain the same. You cannot change the boundaries whereas, since the point source is an object of zero dimension anyway, it does not matter whether the boundaries exists or does not exist.

So, for that reason one can work with point sources and sinks quite easily in a manner that is not possible with finite state objects.

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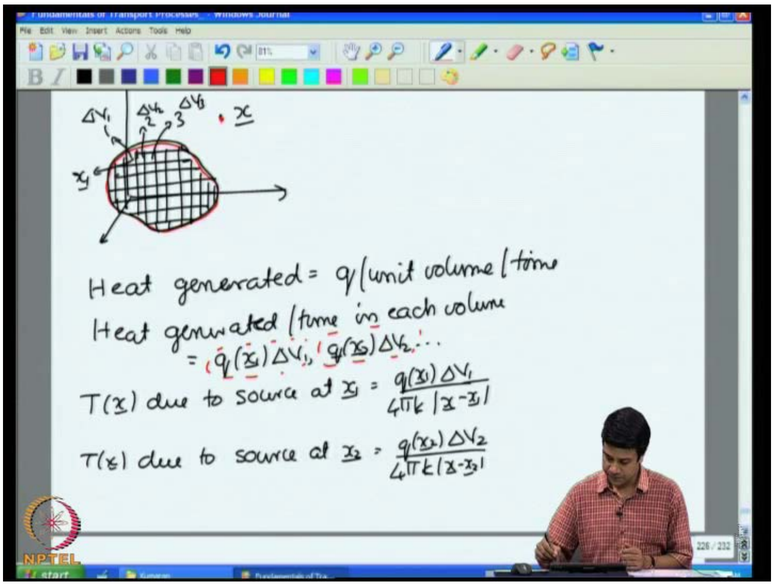

And 1 can write down the temperature field due to a distributed source in terms of temperature fields due to point sources. I had showed you how to do that. You divide the entire distributed source. That volume into small differential volumes each of which is emitting q times delta v. Q is the energy per unit volume per unit time. So, q at the position x times delta v q at the position x 1 times delta v 1, gives you the total energy coming out per unit time within the differential volume delta v 1. And similarly for q at the position x 2 times delta v 2 as you make the sizes of the volume smaller and smaller. This approximates a distributed source. This gets closer and closer to the distributed source.

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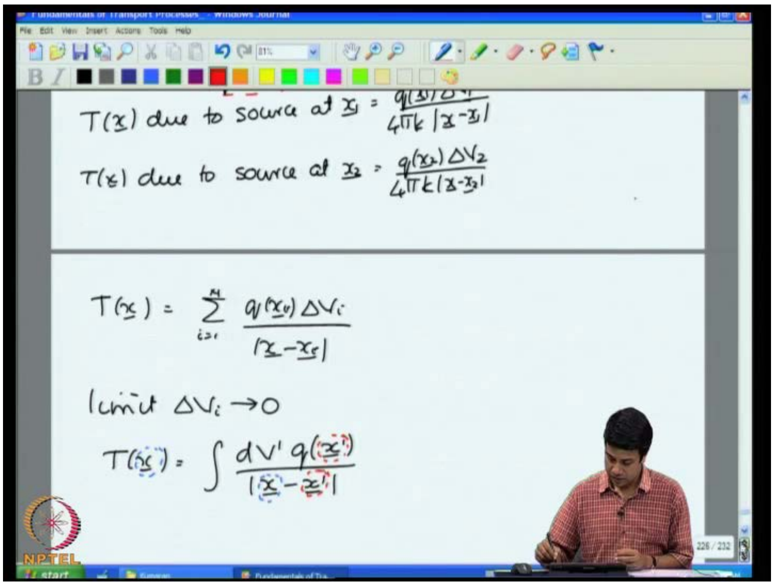

Therefore, you can add up the temperatures due to each of these individual sources and in the limit as delta v goes to 0; you can convert this summation into an integral. So, for this system t at the location x is equal to integral over x prime.

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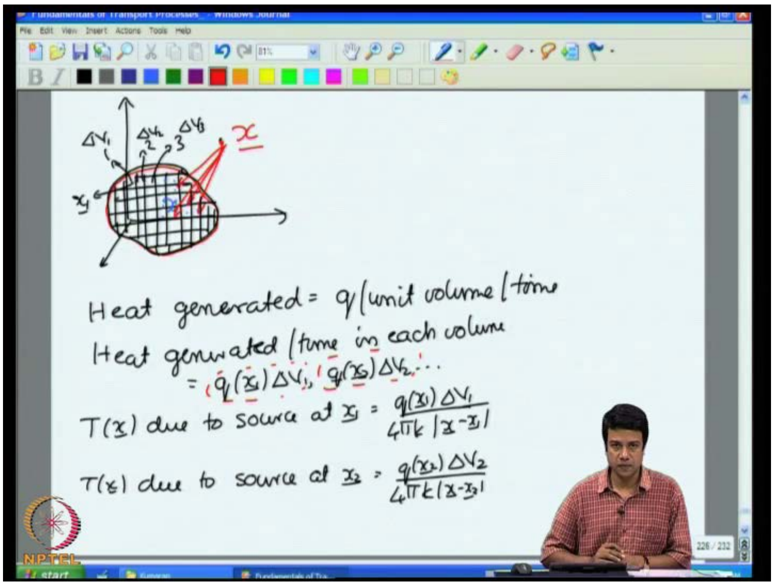

Note, that here x prime is the location of the source points. These are x prime and x is the location of the observation point. The position at which you are measuring the temperature field and basically what you are doing is you are adding up the temperature field at this location due to each of these individual small sources adding up the the source the temperature field due to each of these sources and integrating over the entire volume to get the final solution for the temperature field.

Once again using linear superposition because you have reduced each of these sources to just point sources, and in the last lecture we did an example of how 1 would use this for a concrete example of a wire emitting heat.

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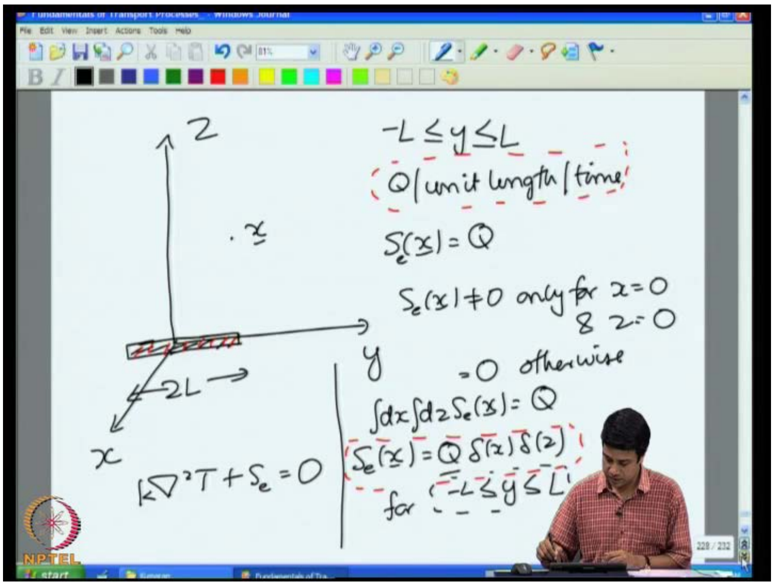

So, I had shown you that in this case. I can define the source as a delta function in the x and z coordinates.

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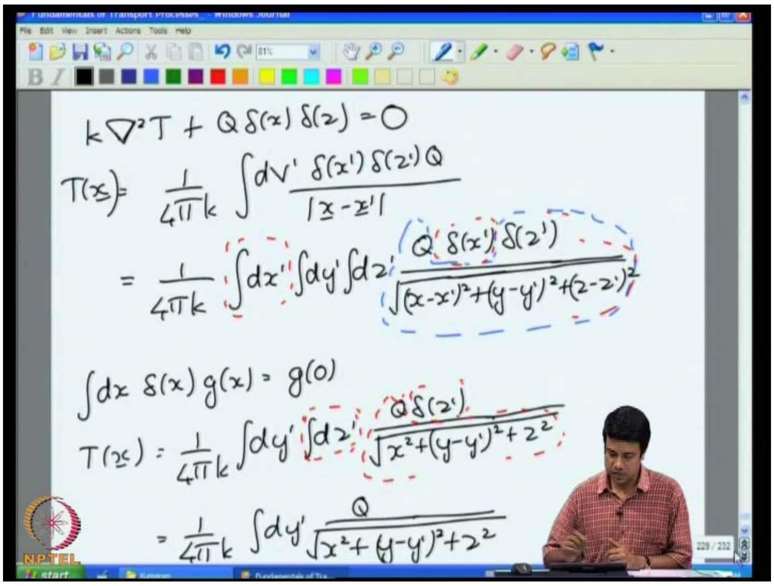

And once I do that, I can use the property of delta functions that it picks out the value of the function at x and z is equal to 0.

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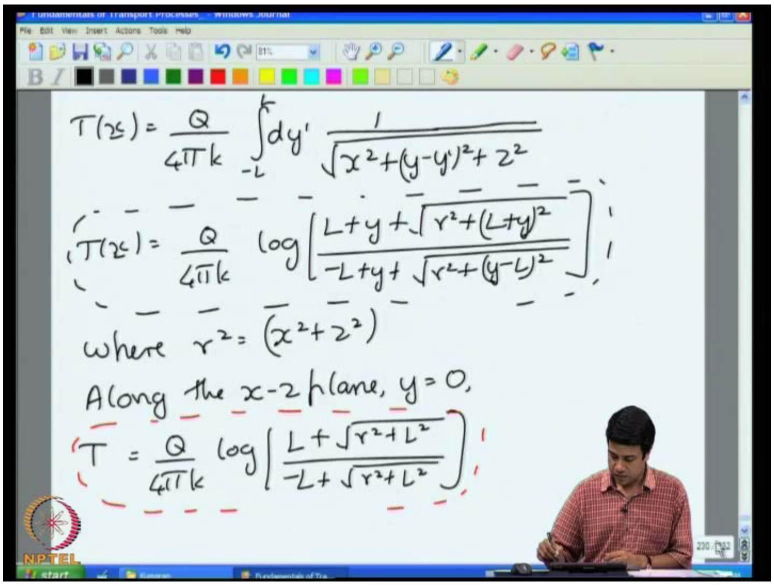

To actually get an analytical solution for the temperature field along the x z plane for y is equal to 0 this had a simplified form. However, 1 can actually plot out the temperature numerically using this more complicated form. So, this is the actual temperature field and for any value of x y and z; one can find out what is the temperature for the particular case along the x z plane. I had simplified this.

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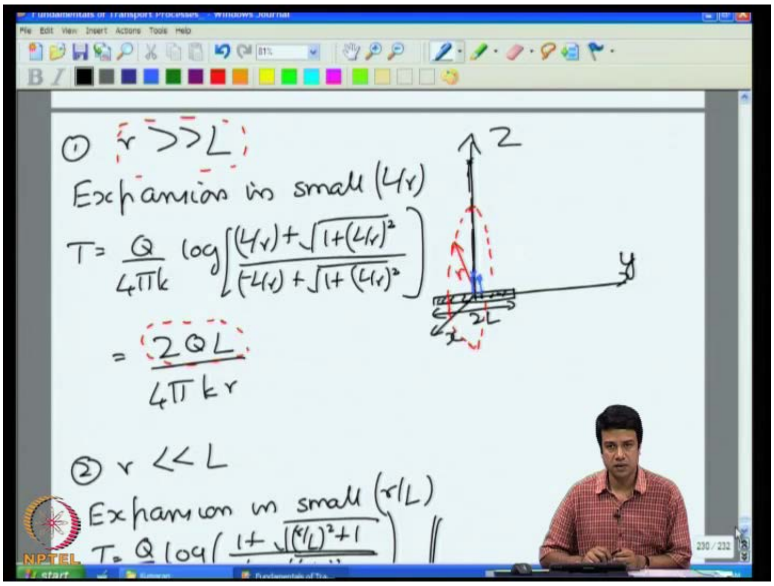

And then further considered two different limiting cases; one is where r the distance from the source is large compared to the length of the source itself. If I am sufficiently far away from this wire then it effectively looks like a point to me, a point and the only the details of the point are obscured because of the distance from the source. The only thing that determines the temperature far away is the total amount of heat generated by this point source. So, the solution that I get will depend only upon the total amount of heat generated by this point source. The source was q per unit length, per unit time. The length of this wire was 2 l. Therefore, the total heat generated per unit time is 2 q l. So, the temperature is 2 q l by 4 pi k r.

So, this happens when the distance from the source is large compared to the length of source itself.

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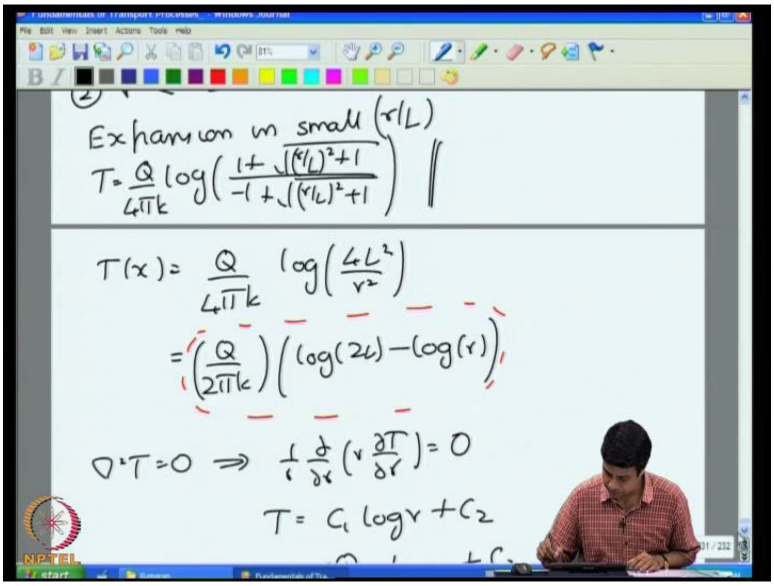

In the opposite limit where the distance is small compared to the length of the source; I get a logarithmic function. The solution is log of r. I told you that that is the solution of the equation in a two dimensions, in a cylindrical coordinate system for a point in two dimensions or for an infinite line in three dimensions. When I am sufficiently close to the source it looks effectively like a source of infinite length when the distance when the distance of the observation point from the source point is small compared to the length of the wire and it looks effectively like a point, like a infinite line infinite line in three dimensions or a point in two dimensions and for a point in two dimensions, the solution is a logarithmic function.

So, this gives us an example of how one would calculate the temperature filed due to a distributed source in this case distributed along a line in terms of the temperature field for a point source.

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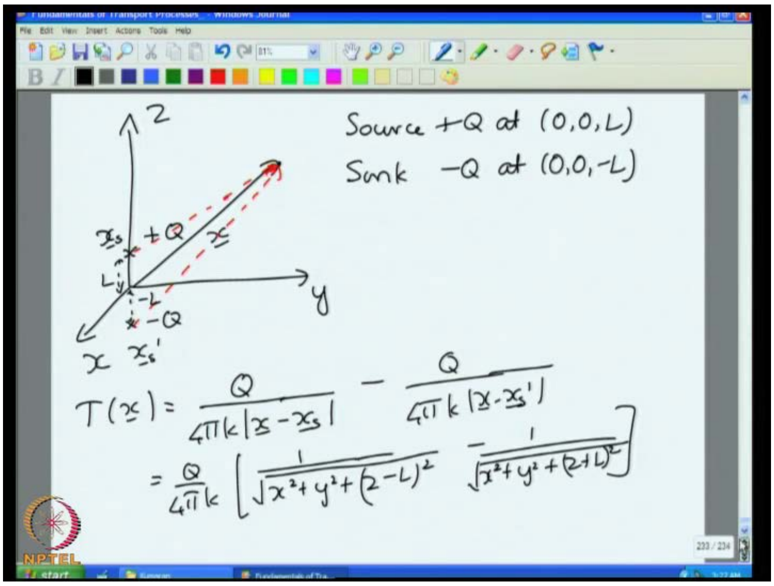

So, let us now look at a different kind of temperature field in a Cartesian coordinate system. If I have a source plus q at a location at a distance l from the origin along the z axis and I have minus q at a distance minus l. So, I have two sources; 1 plus q on the plus z axis, the other minus q on the minus z axis. Both sources of equal strength; one is positive. So, it is a source of heat the other is negative and so it is a sink of heat. Now, if you go sufficiently far from these if the distance x this is the x vector. So, if the distance r is sufficiently far from this, there is plus q generated by the source minus q. That is heat is absorbed by the sinks. So, there should be no net flow of heat sufficiently far from this source.

So, what is the temperature field due to this combination of one source and one sink? The temperature field x so, this is source at the location at 0 0 l. There is a sink minus q at 0 0 minus l and the combination of these two determines what the temperature is. So, t is equal to q by 4 pi k into x minus x s and then there is that is this is point x s of the source and this is the point x s prime of the sink. So, I have minus q by 4 pi k. So, there is one. This generation of heat due to the source and absorption of heat due to the sink and at this location, the distance is this. This is the distance from the source and this one is the distance of the observation point from the sink. So, the combination of these two is what gives me the temperature at this particular location.

So, I know the two locations of the source and the sink at 0 0 plus l x is equal to 0, y is equal to 0, z is equal to plus l as well at x, x is equal to 0, y is equal to 0 and z is equal to minus l. So, this becomes q by 4 pi k into 1 by square root of x square plus y square plus z minus l the whole square minus 1 over square root of x square plus y square plus z plus l of whole square. So, this is the total temperature field due to the combination of the source and the sink.

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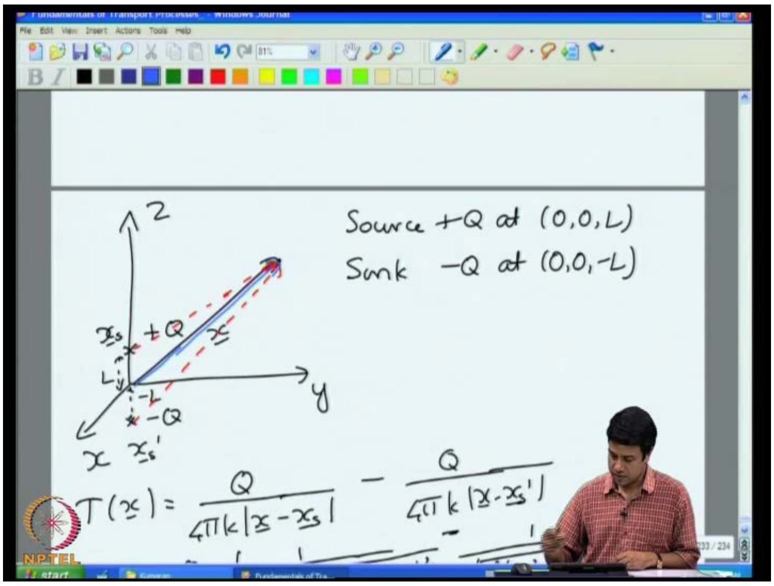

Now, let us take the case where the distance, let us take the case where the distance from the origin is large compared to the distance 2 l between the source and the sink. So, let us take the case where the distance from of the observation point is large compared to the separation between the source and the sink.

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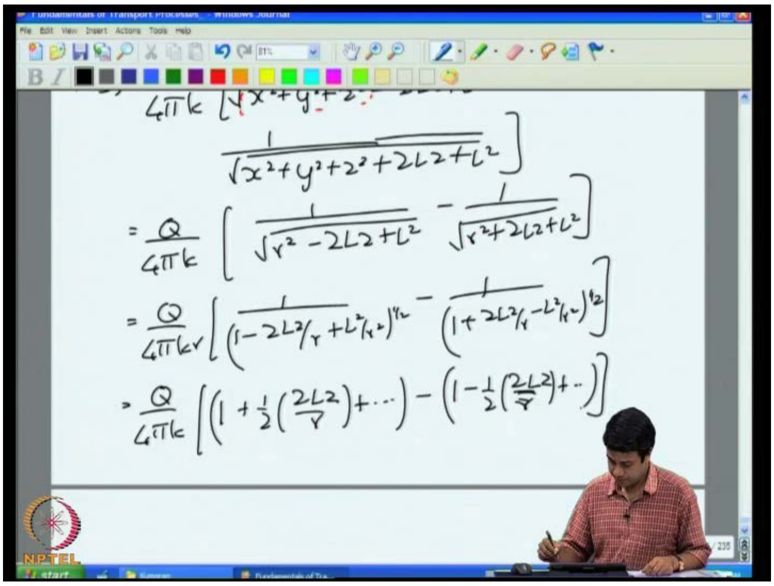

In that case I can expand in l over r. That is when the distance r from the origin is large compared to the distance between the source and the sink. So, this temperature is q by 4 pi k to 1 over square root of x square plus y square plus z square minus 2 l z plus l square minus. So, just expanding out the z minus l and the z plus l. Now, I know that this square root of x square plus y square plus z square is equal to r square in the spherical coordinate system r square is equal to x square plus y square plus z square.

So, this can be written as q by 4 pi k times 1 over square root of r square minus 2 l z plus l square minus 1 over square root of r square plus 2 l z plus l square. Now, this I can rewrite by taking out r as the common factor. I can take out r as the common factor to get q by 4 pi k r into 1 by 1 minus 2 l z by r plus l square by r square power half minus 1 over 1 plus 2 l z by r minus l square by r square and I can use a binomial expansion for this in the limit where where the distance is is in the limit where l is small compared to r. Recall that I want to expand in this small parameter l over r.

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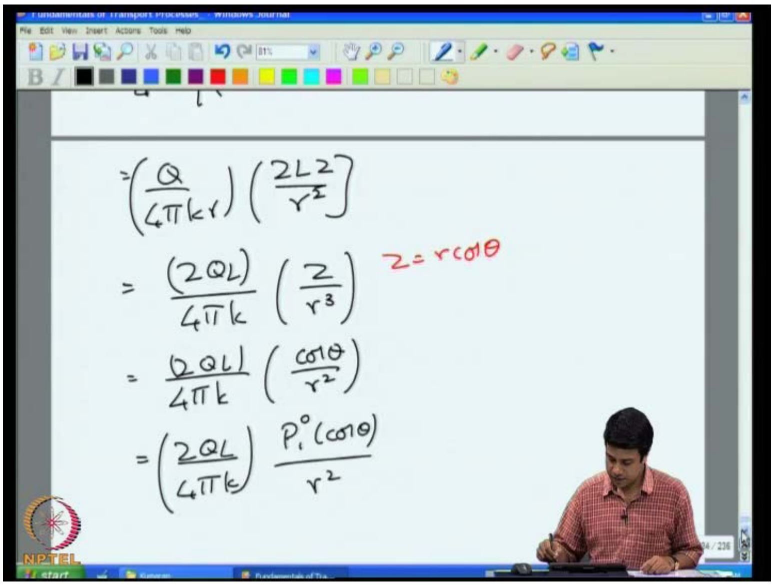

So, this becomes equal to q by 4 pi k into 1 plus half 2 l z by r plus plus terms of order l square by r square minus 1 minus half 2 l z by r plus terms of order of 1 over r square is equal to q by 4 pi k r into 2 l z by r i should take this is r square . So, this is the final solution that i get this is equal to i. I will just rewrite a little bit 2 q l by 4 pi k into z by r cubed or if I write this in a spherical coordinate system, I can use the substitution z is equal to r cos theta. So, this will become 2 q l by 4 pi k into cos theta by r square. Rewriting this once again 2 q l by 4 pi k into p 1 0 of cos theta by r square because p 1 0 of x is just equal to x. So, p 1 0 of cos theta is just equal to cos theta itself. Physically what does this mean?

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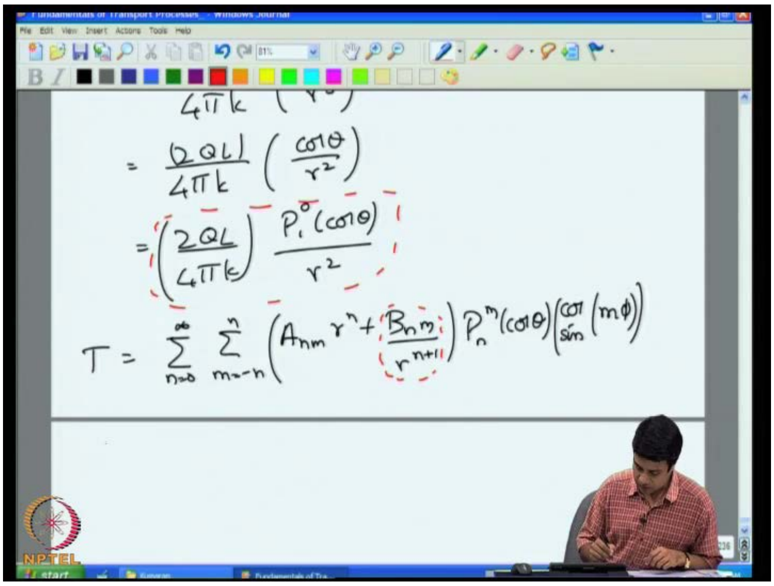

Let us go back and recall our earlier solution for the temperature field that we had obtained by separation of variables t is equal to summation a m n, summation n is equal to 0 to infinity, summation m is equal to minus n to plus n a m n a. This was the solution that we got from the spherical harmonic expansion. You can see that this solution is identical to this solution. For n is equal to 1, identical to this solution constant by r square. For n is equal to 1, you will get some constant divided by r square and m is equal to 0 you will get p 1 0 of cos theta.

So, the solution that we got from a source and a sink separated by a small distance is the same as the spherical harmonic expansion for n is equal to 1 and m is equal to 0 provided the two are separated along the z axis.

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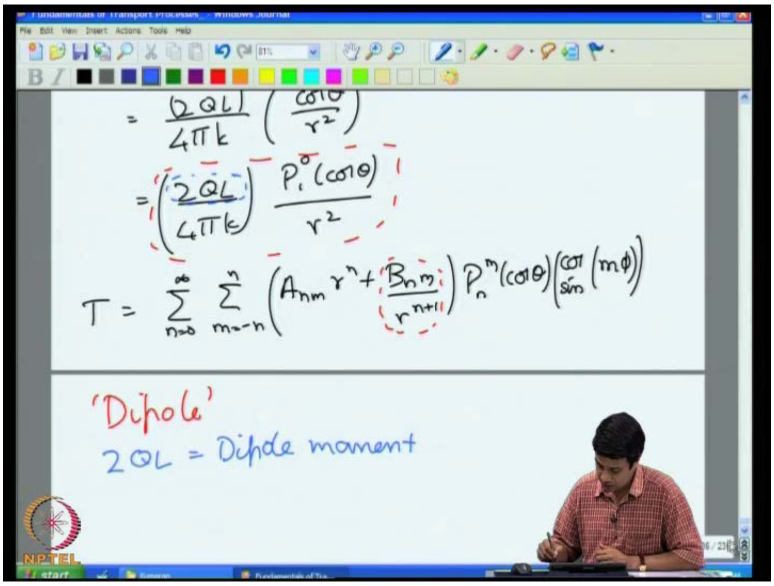

So, this source and sink combination is what is called a dipole. There is a source and a sink that are very close to each other. The distance of separation is small and because of that there is no net flow of energy. The sum of the the combination of the source and the sink results in no net energy coming out because q is generated by the source q is absorbed by the sink. If you go sufficiently far away, there will be no net energy coming out. No net energy coming out means that the temperature cannot decay as 1 over r because if you recall when the temperature decays as 1 over r the heat flux goes as 1 over r square surface area increases proportional to r square. So, therefore, there will be a flux that is a constant. However, if you have a source and a sink close to each other there can be no net flux. That means, the energy the temperature field cannot decay proportional to 1 over r. It can decay faster 1 over r to some higher power. In this particular case for this combination of source and sink the energy the the temperature decays as 1 over r square. This thing here previously for a source I got Q by 4 pi k r in this case I get 2 Q times l Q is the source strength 2 l is the distance between the source and sink and this 2 Q l is the dipole moment.

So, this is only for decaying harmonics. This this formulation cannot be used for growing harmonics. So, this is a physical interpretation of the terms in the spherical harmonic expansion n is equal to 0. There is only 1 m is also equal to 0 that represents a source in which there is a net energy flow out from the from this the origin, from the source. If you have a combination of a source and a sink there is no net generation of energy. That means, that the temperature has to decay faster than 1 over r. If you have a combination of a source and a sink then the temperature decays is 1 over r square and the dipole moment has dimensions of energy per unit time times length because that dipole moment by 4 by k times p 1 0 of cos theta by r square is the temperature field due this dipole .

So, this was for n is equal to 0 and m is equal to i am sorry for n is equal to 1 and m is equal to 0. As you know, there are three solutions at n is equal to 0 for m is equal to minus 1 0 and plus 1 for m is equal to 0.

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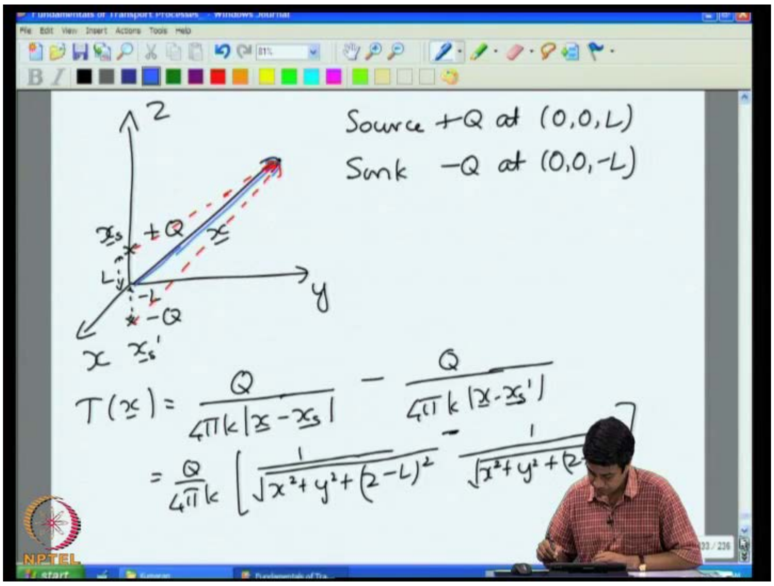

We had a dipole that was aligned along the z axis. That is this the separation between the source and the sink moves along the z axis for m is equal to 0.

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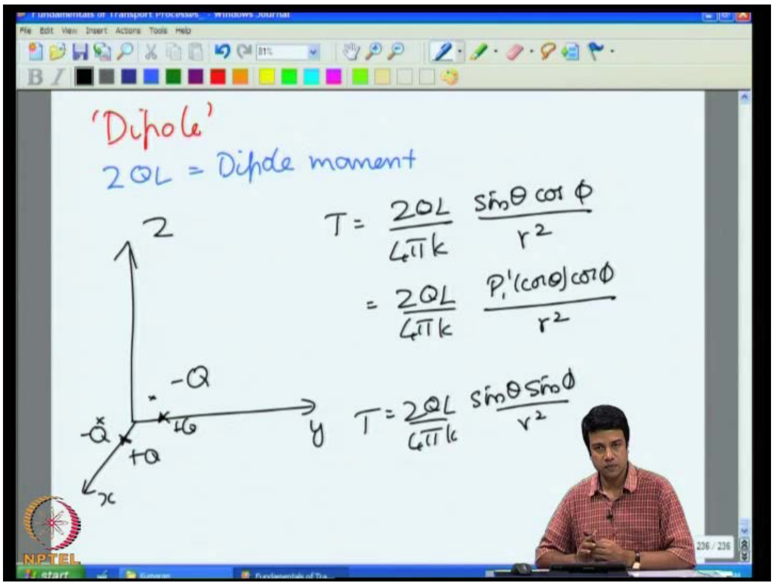

One could solve equations for n is n is equal to 1 and m is equal to plus or minus 1 for n is equal to plus 1 1 will get the solution as t is equal to 2 q l by 4 pi k into sin theta cos phi by r square is equal to 2 q l by 4 pi k into p 1 1 of cos theta cos phi by r square.

So, for n is equal to 1 and m is equal to 1; the solution is a source and a sink separated along the x axis for n is equal to 1 and m is equal to 0. It is along the z axis and the final 1 for m n is equal to 1 and m is equal to minus 1. The temperature field in that case is equal to 2 q l by 4 pi k sin theta sin phi by r square. So, this gives you the physical interpretation of the terms for n is equal to 1 in the spherical harmonic expansion. n is equal to 1 and m is equal to 0 a dipole separated along the z axis minus 1 plus 1 are dipole separated along the x and the y axis. And there are only three ways to separate them along the x y or the z axis. Therefore, there are three terms, three dipole terms. These are vectors. These are what are called dipole moments. We have a direction associated with them. That direction is the direction of separation. So, plus 1 and minus 1 are basically the dipole moments where they are separated along the x and the y axis.

So, that is the interpretation of the dipole terms for m is equal to plus and minus 1.

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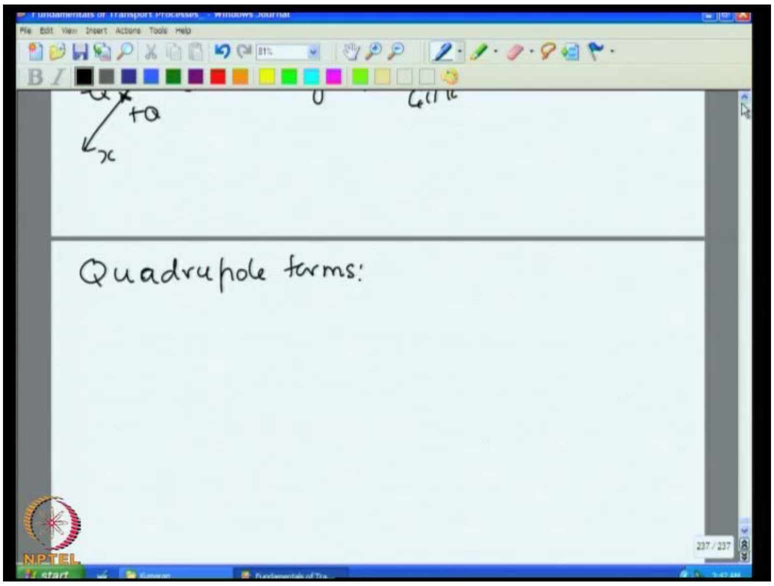

The next higher terms are what are called quadrupole terms. If you recall the source, there was a net source of energy coming out from the origin. The dipole terms were a combination of a source and a sink in such a way that the total amount of energy coming out is identically equal to 0. However, there was a net dipole moment for the, that is the source and sink were separated by a small distance in the limit as in the limit as l goes to 0 ok.

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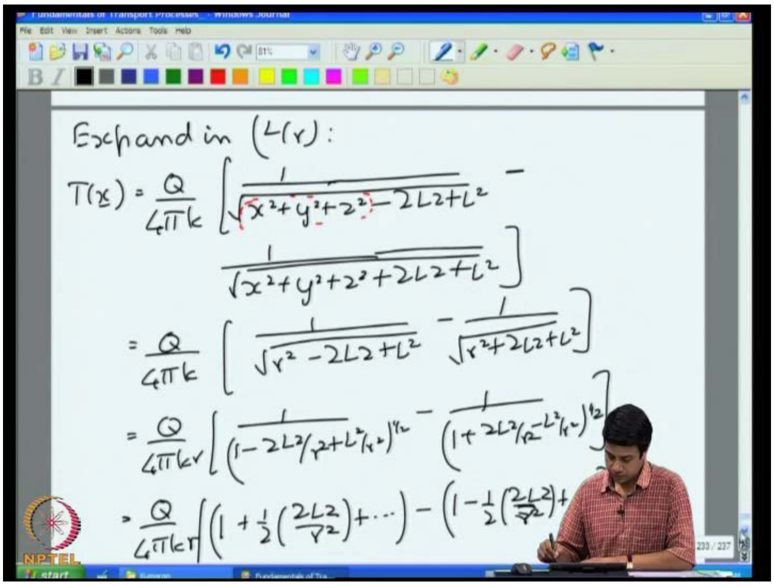

It recall that when we calculated the temperature field due to the dipole; we took an expansion in l by r. That means that the separation of the source and sink was going to 0 compared to the distance from the origin. That is the separation between the source and sink was small compared to the separation between the observation point and the dipole.

So, in the limit as l over r goes to 0; the source and sink are coming closer and closer to each other. Even though the source and sink are coming closer and closer to each other. l is going to 0.

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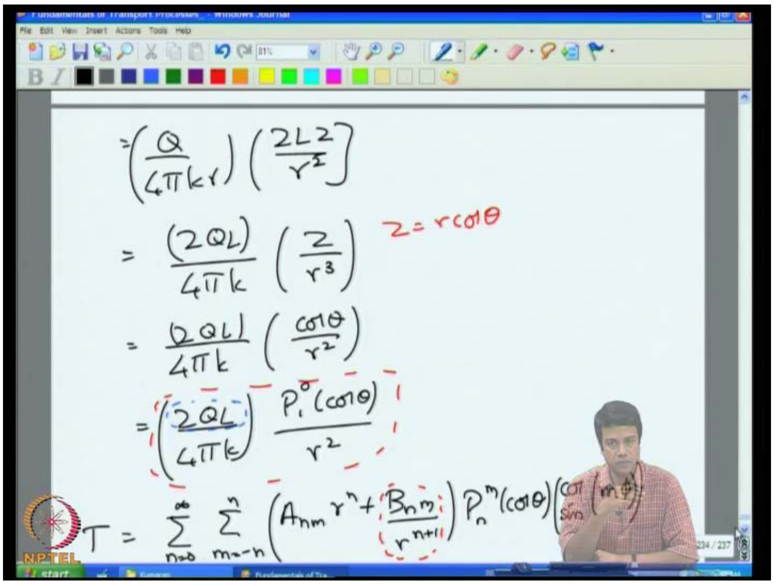

However, this dipole moment 2 Q l has to be a constant. So, that means, that q has to increase l has to decrease in such a way that the product Q and l remains finite even in the limit as l goes to 0. So, you get a dipole when you have a source and sink. The separation between those two goes to 0. The amount of heat generated or absorbed goes to infinity. The product of the two remains finite in that limit. You get a dipole. The temperature field for that dipole decreases as 1 over r square. When you go far away and you can have three different alignments; 1 is along the x, the other is along the y and the third is along the z axis and that corresponds to the three solutions for n is equal to 1 m is equal to minus 1 0 and plus 1. So, there are three solutions for the dipole moment. The next higher term is what is called a quadrupole.

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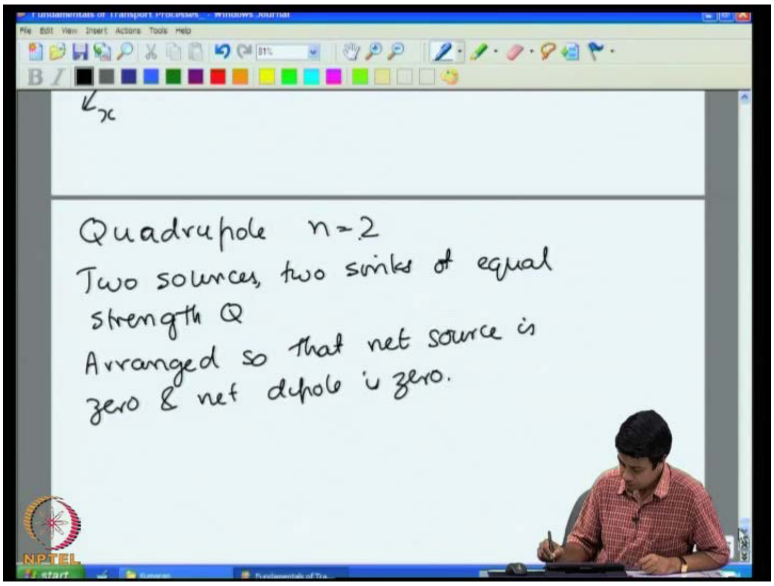

This is the spherical harmonic expansion for n is equal to 2. We have two sources and two sinks of equal strength Q arranged. So, that net source is 0 and net dipole is 0 .

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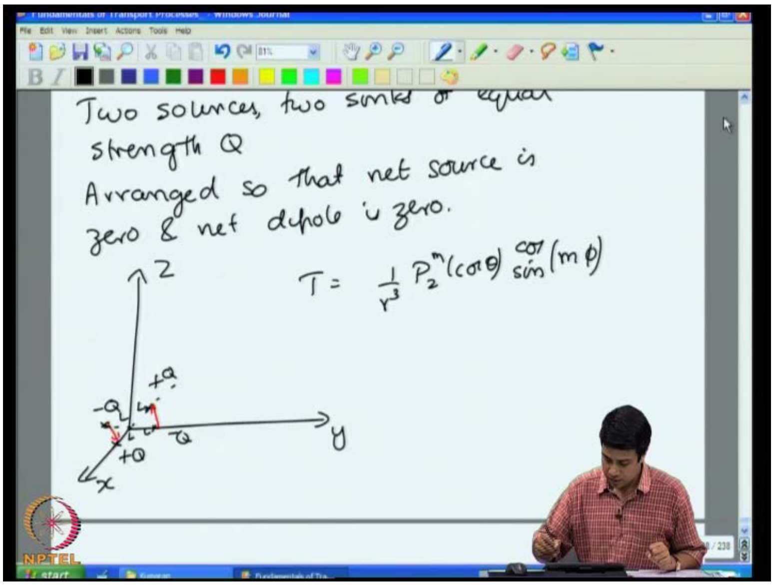

So, for example, in this case for n is equal to 2 m can vary between minus 2 to plus 2. So, there are five solutions for n minus 2 minus 1 0 plus 1 and plus 2. So, these five solutions have the following forms; the first one is when you have sources and sinks along the x and the y direction. So, you can have plus Q minus Q plus Q and minus Q you have two sources of plus Q. That means and two sinks of minus Q. That means that the net source is identically equal to 0. However, in addition you have two dipoles here which are actually in opposite directions. So, this dipole is in this direction, this one is in this direction. That means that if you go sufficiently far away the net dipole moment is identically equal to 0. Therefore, you the temperature field cannot decay as 1 over r square the temperature field has to decay faster than that because if you had a net dipole the temperature would have decreased as 1 over r square when you went far away. In this case, the temperature that there is no net dipole. So, the temperature field has to decay faster than 1 over r square. In this case, the solution of the temperature goes as 1 over r cubed right p 2 and of cos theta times cos and sin of m phi. Therefore, at n is equal to 2; the solutions are in the form of quadrupoles. Each of these are separated by equal length from the origin. The length of each of these from the origin is is is the same.

In the case of the dipoles we saw that we can have three different arrangements; one is the source and sink along the z x and y axis. In this case one can have five different arrangements. Corresponding to n is equal to 2 and m is equal to minus 1 minus 2 0 1 and 2.

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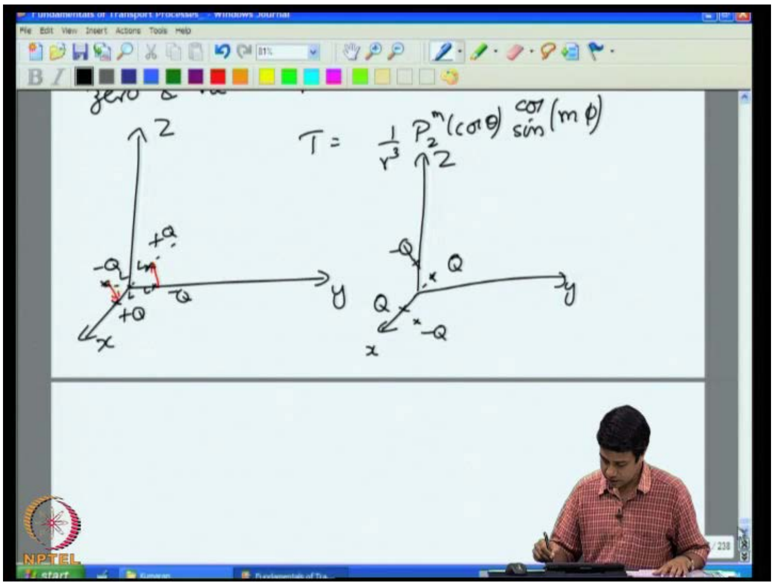

So, three of these correspond to the sources and sinks that are placed along the coordinate axis along the x y plane is 1. The other one is along the x z plane. So, this is Q minus Q sorry this is Q the other one is along the x z plane.

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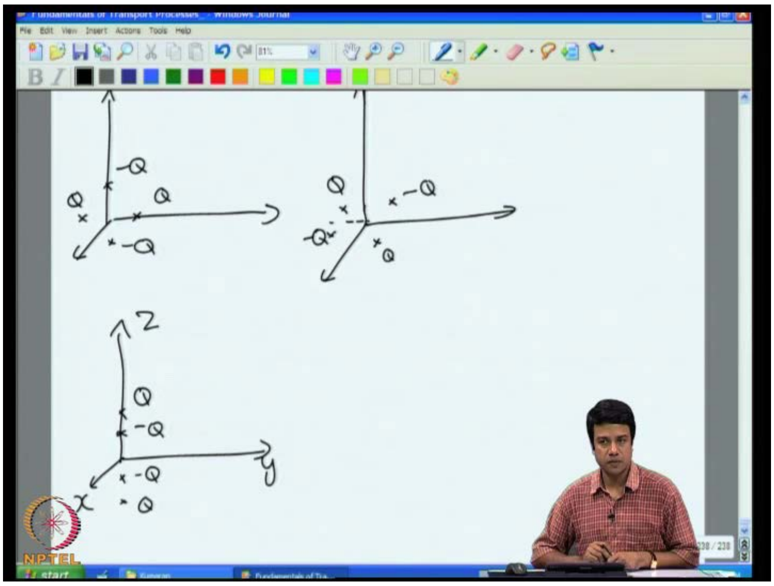

There is a third one along the y z plane. that is. So, these are three along the three different planes. If you arrange two sources and two sinks equidistant from the origin along two of the axis in such a way that the net source is 0 and the net dipole moment is also 0, the other one's that is 1 along the x y plane, but not along the axis. So, this is Q minus Q Q minus Q. Once I can equidistant, but not along any of the axis and the final one has sources and sinks are exclusively along the z axis.

So, these are the spherical harmonic solutions for n is equal to 2 and m going from minus 2 minus 1 0 plus 1 and plus 2. You will note that these are actually reminiscent of the symmetries that I had written for you a little earlier. These are actually identical to these solutions. So, y 0 0 corresponds to a source.

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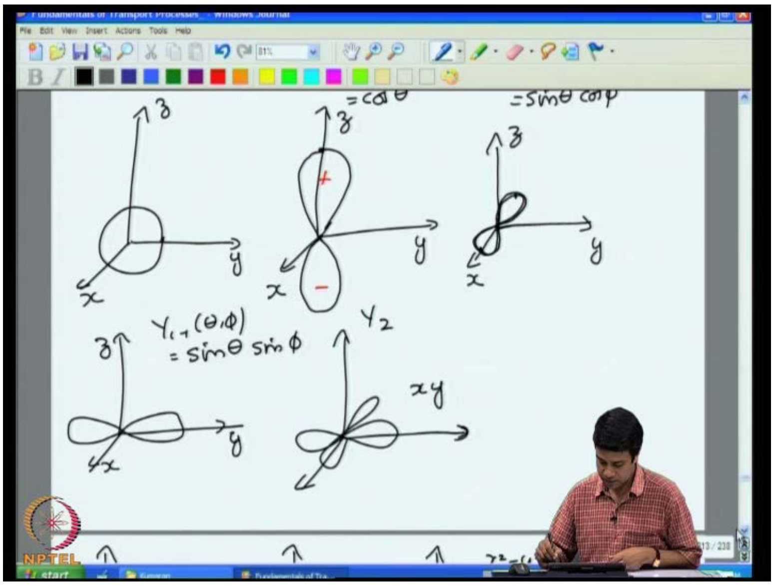

Y 1 0 Y 1 1 and Y 1 minus 1 corresponds to dipoles combination of a source and a sink in the limit as the distance between the 2 goes to 0.

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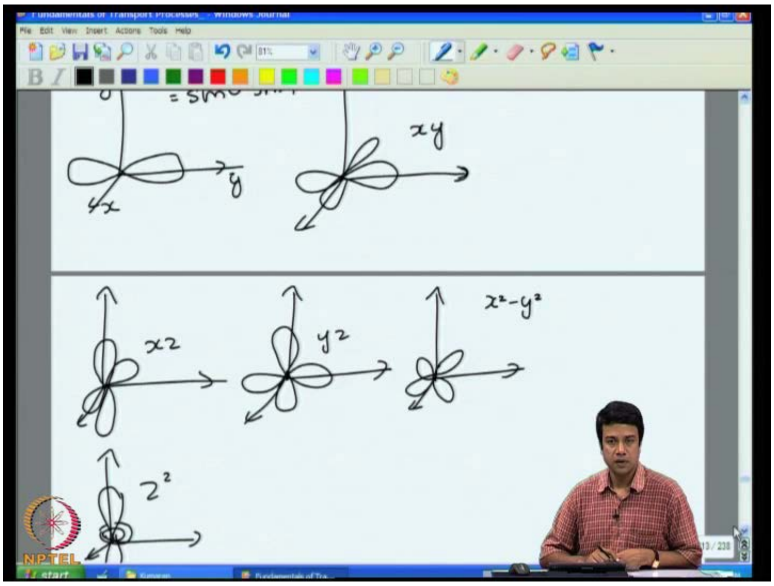

Similarly, the orbital’s that is x y y z z x x square minus y square and z square correspond to quadrupoles, quadrupoles the combination of two sources and two sinks in such a way that there is no net source, no net dipole. The temperature field due to these decreases as 1 over r cubed.

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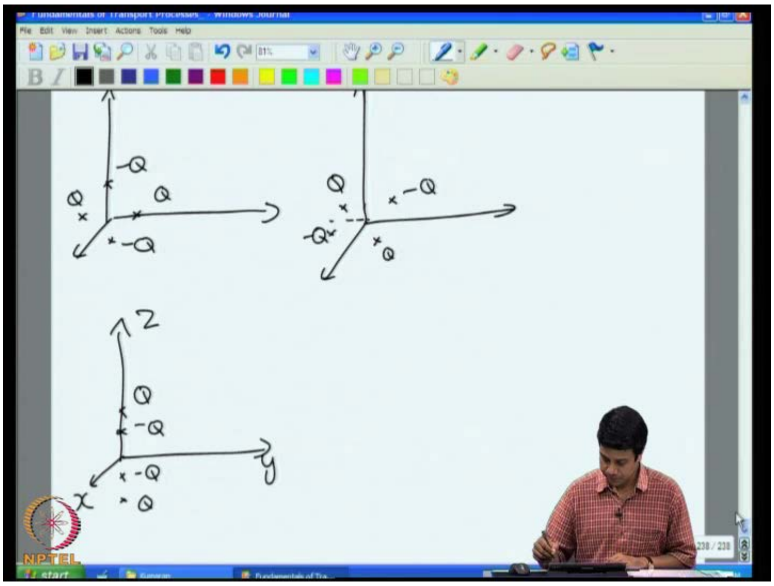

The next higher terms in the series of for n is equal to three and in that case there are seven solutions for m is equal to minus three minus 2 minus 1 0 plus 1 plus 2 and plus 3. Seven solutions called octopoles. They are generated by super posing four sources and four sinks in such a way that there is no net source, no net dipole moment and no net quadrupole moment. Either the temperature field due to those decreases is 1 over r 2. The 4th power that is for n is equal to three the temperature field decays is 1 over r to the 4th power.

So, the terms in the spherical harmonic expansion that we had solved by separation of variables are identical to these terms. That you would get by having a single point source for n is equal 1 a dipole for n is equal to 2 quadrupole for n is equal to three and so on. And if you recall when we did the problem of heat conduction around a particle for constant temperature on the surface there was a net source. On the other hand we had solved the problem of the conduction around a spherical inclusion. In that case, there was a linear temperature gradient far away. There was no net source on the particle ok.

So, the solution that we got finally, for the temperature we said that since far away the temperature goes as t prime r times cos theta the temperature everywhere has to go as t prime r times cos theta and from that we got solutions of this kind a p n into r power n plus b p n by r power n plus 1 p n of cos theta this a p n for n is equal to 1. It was equal to t prime to a 0 for all other values.

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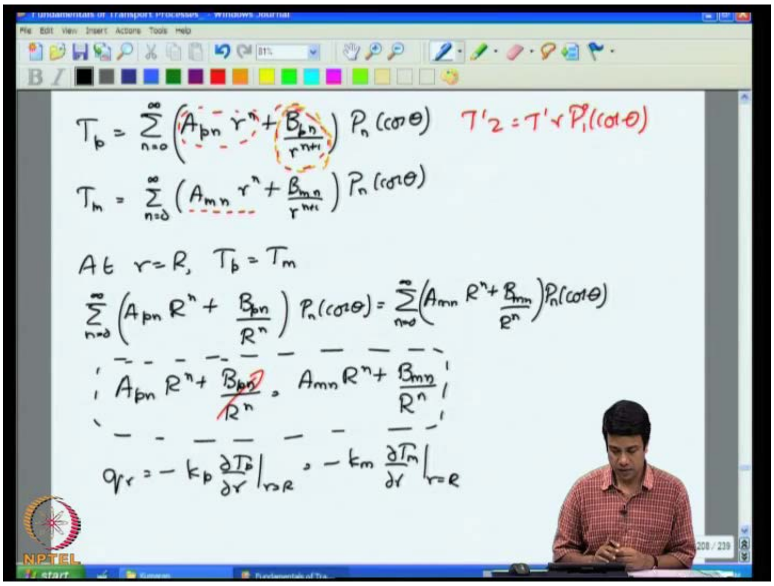

And. So, the final solution of the temperature field that we got was of this kind kind.

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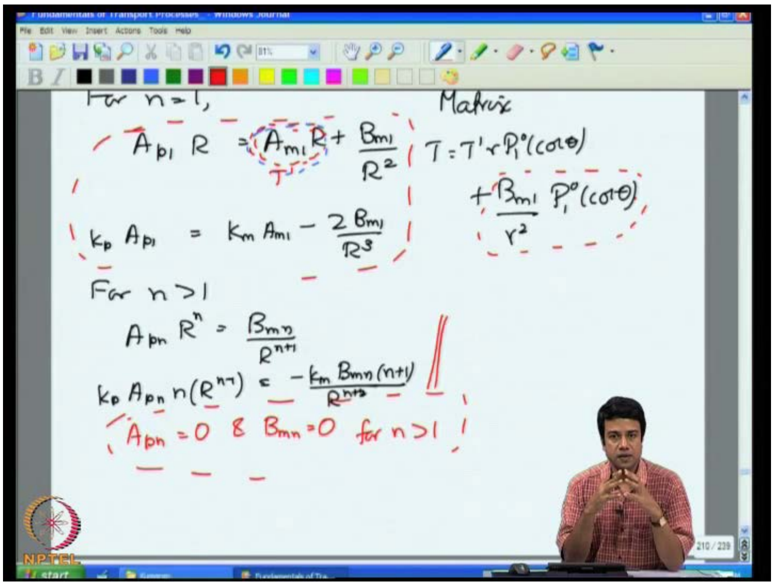

In the matrix t is equal t prime r p 1 0 of cos theta plus b m 1 by r square p 1 0 of cos theta.

So, in this particular example, the temperature field around the particle in the matrix was the combination of the linear temperature gradient far away as well this term here which is basically a dipole term which is basically a dipole term the particle itself did not have any net heat coming out. So, the source of energy within the particle is 0. The next higher term that can be non zero is the dipole term.

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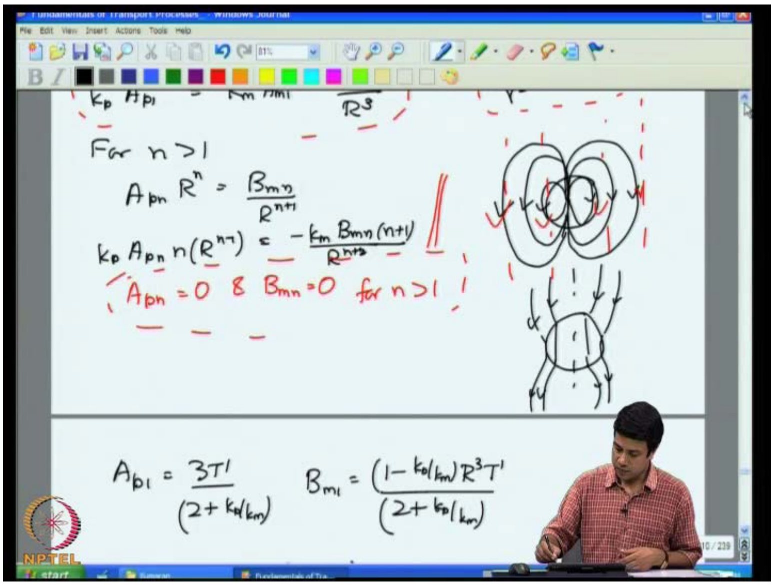

And if I plot the flux lines due to this dipole term. So, there is a source and a sink which are separated by a very small distance energy coming out of the source and going back into the sink. So, that flux lines that I will get due to this dipole look something like this. So, these are the flux lines due to the dipole which is aligned along the z axis. In the case it has to be aligned along the z axis because that is the inhomogeneous direction. That is the direction along which there is a temperature variation.

So, what I have basically done here in constructing this solution for the flow for the the the temperature field. Around this sphere, I know that the solution far away has 1 0 symmetry and because of that solution far away with 1 0 symmetry I have flux lines which are a constant. If I do not have the particle, I would have just constant flux lines, a linear temperature profile. However, I have a particle here as well and I need to satisfy the temperature and the flux boundary conditions on the surface of the particle.

So, I need to add a disturbance to the temperature field to due to the presence of this particle from symmetry. That disturbance can only be in the form of a dipole and. So, I add a disturbance in the form of a dipole to the particle outside of the particle. Note that this solution is only in the matrix. I had a dipole solution outside of this particle in such a way I choose the constant in such a way that the boundary conditions at the surface of the particle are satisfied. So, that I get the correct temperature field outside of the particle ok.

So, the combination of this constant temperature gradient and straight lined flux and this dipole will give me the necessary solution which satisfies all boundary conditions. Of course, this dipole moment decreases as 1 over r square. So, I cannot use this within the particle itself because at the origin itself r goes to 0 and therefore, this goes to infinity. So, this can be used only outside of an object. This is a decaying harmonic. It can be used outside of an object and I add the constant solution and the decaying harmonic in such a way that I get the correct boundary conditions on the surface.

If on the other hand I had a heat flux in the x direction or in the y direction I would have chosen the appropriate spherical harmonic term that would have corresponded to a dipole along the x axis or along the y axis. One of these two dipoles and similarly 1 can in in problems which have the symmetry of quadrupoles. One can add the quadrupole term in such a way that 1 gets a correct symmetry.

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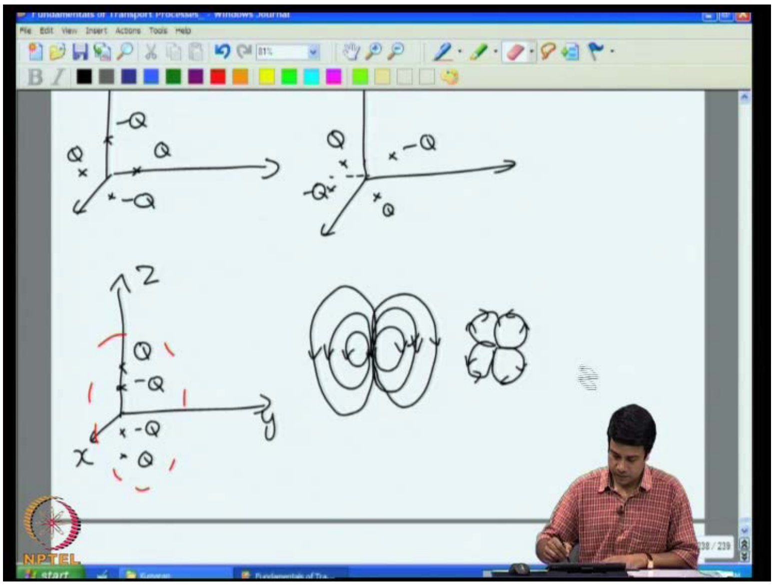

So, the flux lines due to a dipole will look something like this, the quadrupole since it has two sources and two sinks. It looks something like this this is for two sources and two sinks and this of course, these two are separate. Rather along the x y x square minus y square as well the z square. So, in this case alone you will get a linear quadrupole. In this case alone, you will get linear quadrupole. You get something that goes like this. That is in all other cases there will be plane or quadrupole moments and of course, in the case of octopoles you will have two sources and two sinks and so on.

So, this is the relation between point sources and the spherical harmonic expansion by suitable combination of point sources. I can generate the entire all the terms in the spherical harmonic expansion. So, it is another interpretation of the solution of the Laplace equation. We got this the the spherical harmonic expansion by doing separation of variables in a spherical coordinate system. I showed you that you get the exact same relation. If you just super pose point sources source and a sink such a way that there is no net source gives you n is equal to 0 i i am sorry the source and sink in such a way that there is no net source gives you n is equal to 1. Three solutions dipoles aligned along the x axis, y axis and z axis. N is equal to two is quadrupoles. There are five solutions and the source temperature decays as 1 over distance for a dipole. It goes as 1 over r square for a quadrupole, it goes as 1 over r cubed and in that way we get all of the terms in the spherical harmonic expansion.

So, to briefly summarize this section of of solutions in the Laplace equation; you are basically trying to solve del square t is equal to 0. In some domain one method that we use to a separation of variables in a spherical coordinate system, we separated out the variables into the r theta and phi directions and I showed you that just from symmetries you get discrete Eigen values in both the theta and phi direction. In the phi direction the solutions are in the form sine and cosine functions, in the theta direction they are in the form of polynomials. Then I defined the point source which is non zero only at the origin and a zero everywhere else and with a specified heat coming out per unit time, the solutions of that goes as 1 over r and that solution exactly corresponds to the solution obtained by separation of variables for n is equal 0. Then we looked at n is equal to 1 and I showed you that the solution due to a dipole exactly corresponds to the solution for n is equal to 1.

There are three solutions; dipoles along the x y and z coordinates, n is equal to 2 corresponds to quadrupole, two sources, two sinks in such a way that the net source is equal to 0. The net dipole moment is equal to 0. It goes over r cubed and so on and higher and higher terms. So, this basically covers the relationship between the two methods of solution that we looked at so far, separation of variables and the the delta function solutions.

So, this is all in the limit of very low peclet number when we can neglect the convective terms in the equation, when you got two very high peclet numbers. When you would expect convection to dominate? We will use other methods of solution where diffusion is restricted to small regions near the boundaries. So, we will continue this and then go on to diffusion convection dominated flows a little later. So, with this I complete my discussion of sources and sinks in three dimensions and I will continue a little bit on on green's functions methods and then we will go on to discussing high peclet number transport. So, will see you in the next lecture.