Fundamentals of Transport Process
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Module No. # 06
Lecture No. # 33
Diffusion Equation Delta Functions

So, this is lecture number 33 of the course on fundamentals of transport processes. Welcome, we were discussing solutions of the diffusion equation in the last class. So, just to recap where we were. We had initially derived conservation equations for mass and energy conservation. In different coordinate systems: in a spherical, cylindrical as well as Cartesian coordinate system.

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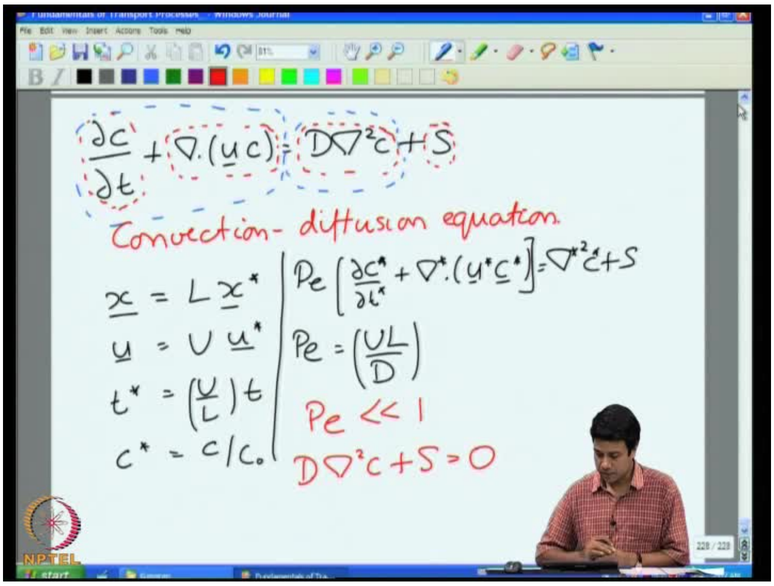

And all of these had a similar form. They all had the form dc by dt plus the divergence of u c is equal to D del square c plus the source or sink of energy.

So, this was the form of the conservation equations both for mass and energy only in the case of energy conservation. You substitute the temperature instead of the concentration and the thermal diffusion coefficient instead of the mass diffusion coefficient. So, this is the common form of the diffusion equation. This is commonly called as the convection-diffusion equation.

The left side of the equation contains first, a time derivative the rate of change of mass within a differential volume respect to time divided by the volume itself. So, c is the concentration. And then there is the convective term the rate of transport of mass due to the mean fluid flow. This takes place because the mean fluid velocity u which has different components and different coordinate systems.

So, this is transport along the direction of the main flow. And then there is this diffusion term, which is the transport of mass due to the fluctuating velocity of the molecules. Recall that when we derived this term, we had assumed that the diffusion coefficient is independent of position. So, that is an assumption that goes into all of these equations. And then there could be a source or sink of energy depending upon whether mass is produced or consumed in a reaction due to reactions. Similarly, energy could be produced or consumed either due to exothermic or endothermic reactions or due to phase change of various other reasons.

So, this is the general form of the convection diffusion equation. The left hand side contains the rate of change and the convective term. Well, there is a diffusion term on the right hand side. and if I scale my coordinates by a characteristic length and a characteristic velocity. So, if I define the length x vector is equal to the characteristic length times x star. where x star is a dimensionless coordinate u is equal to capital U times dimensionless velocity. In the case of flow around a particle the length would be the particle diameter and u would be the free stream velocity. In the case of flow path surfaces or through channels and tubes L would be the radius of the tube or the the width of the channel u would be either the maximum or the average velocity.

And 1 can define the dimensionless time quite easily, this has to be U by L times t. If I put all this in to the conservation equation along with a dimensionless concentration, where C naught is some characteristic concentration then, my equation becomes P e times partial c by partial t plus del dot u c is equal to del square c plus the source. Where the Peclet number is equal to UL by D gives you the ratio of convection and diffusion. Dimensionally the diffusion coefficient has dimensions of length square by unit time. And therefore, this is a dimensionless number the velocity times the length divided by the diffusion coefficient in the limit, Where the Peclet number is small compared to 1. The equation just becomes D del square c plus S is equal to 0. So, in the limit where the Peclet number is small compared to 1. We have the diffusion equation as d times the

Laplacean of the concentration field plus the source or sink is equal to 0. And in the limit of low Peclet number diffusion dominated transport, we were examining strategies to solve the equation of the form the Laplacean of the concentration is equal to 0.

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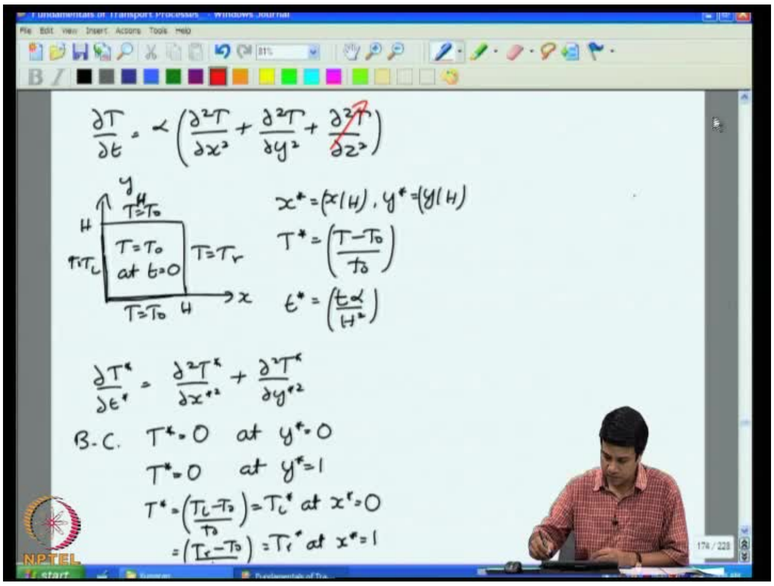

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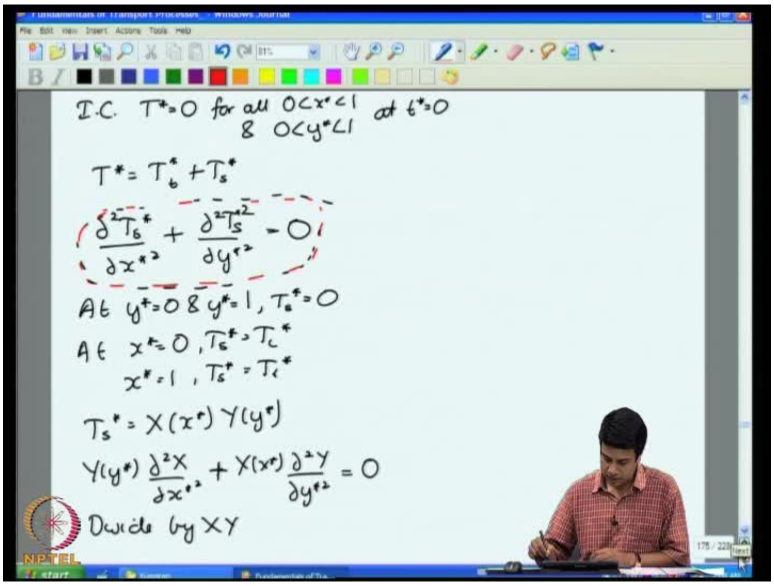

We already saw, one way to do this. And that is by separation of variables. if you recall; when we solve the problem of flow, In a cubic In a cartesian coordinate system. For the flow in a cartesian coordinate system, we derived the equations for euro grad is equal to D del square c. And then we had solved the problem of flow of heat transfer in a cube.

This cube had different temperatures on the different surfaces and, we use the method of separation of variables. To separate the temperature dependency into 2 components one function only of X the other only function of Y.

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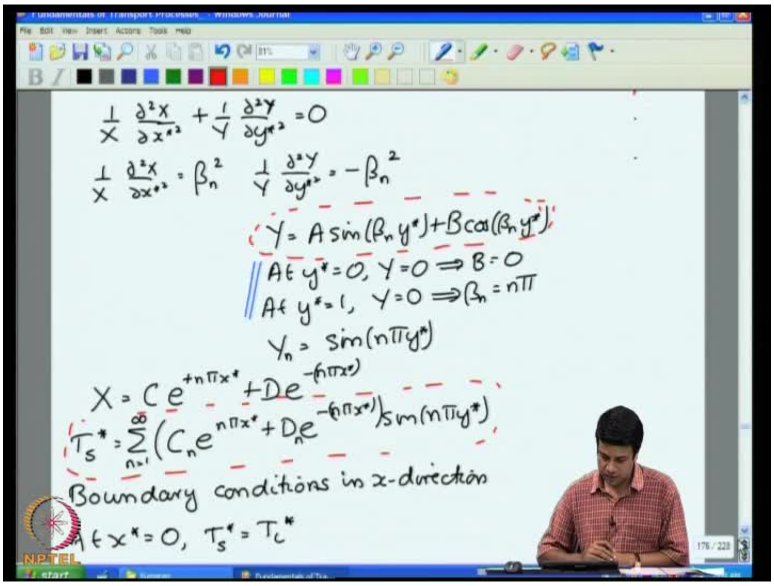

And we got eigen basis functions and eigen values in the form of sine and cosine functions in the coordinate, where there were homogenous boundary conditions. So, in this particular case: in the Y coordinate, there were homogenous boundary conditions and we got basis functions in the form of sin functions in that direction. Exponentials in the X direction where, they were inhomogenous boundary conditions and we saw how to construct a solution as a sum of these basis functions which are all orthogonal to each other.

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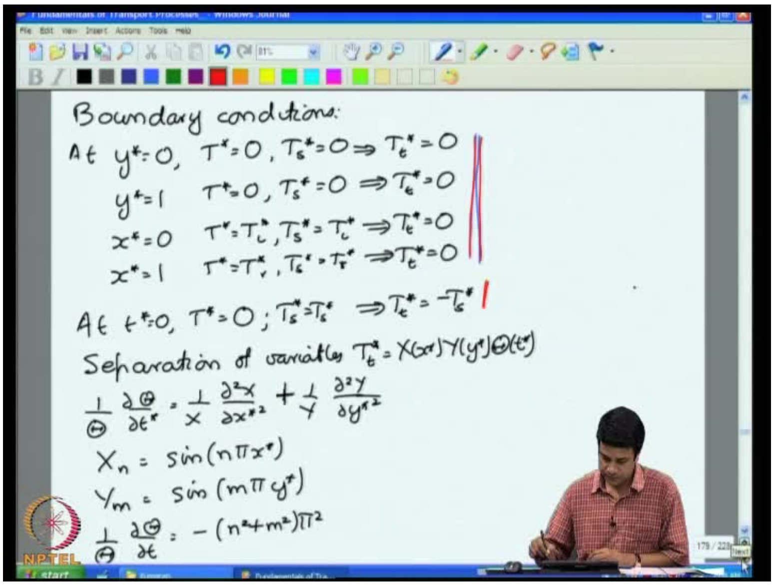

And then, we did the same thing further time dependent problem. In this case: one has to be careful because, when one solves this in 3 independent coordinates, one has to ensure that there are homogenous boundary conditions in 2 out of the 3, And there is inhomogeneous boundary conditions only in 1. In this particular case: we are defined the transient temperature as, the difference between the actual temperature and the steady state temperature. Once we did that we got homogenous spatial boundary conditions for the transient temperature field. There was an inhomogeneity only in time and due to that reason we got exponential dependness in time and sine and cosine functions in both the spatial coordinates.

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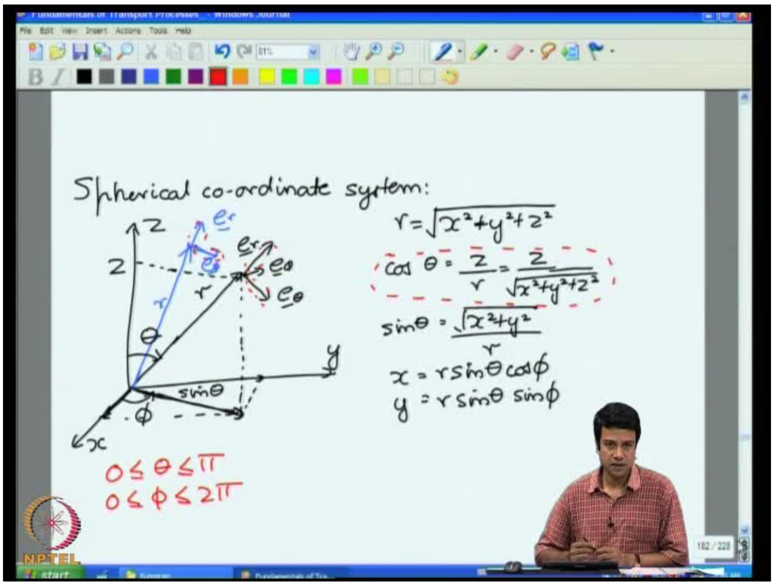

I showed you, how to do that for a spherical coordinate system in this particular case. We have r as the distance from the origin theta is the angle made by the radius vector with respect to the z axis and phi is the the angle made by the projection with respect to the x axis. So, we have x is equal to r sin theta cos phi y is equal to r sin theta sin phi and z is equal to r cos theta and by defining the surfaces of a differential volume. We have to choose the surfaces to be of constant coordinate. 2 surfaces in the r direction 2 in the theta direction and 2 in the phi direction and by defining the surfaces of constant coordinate. We derived the differential equation for the concentration field.

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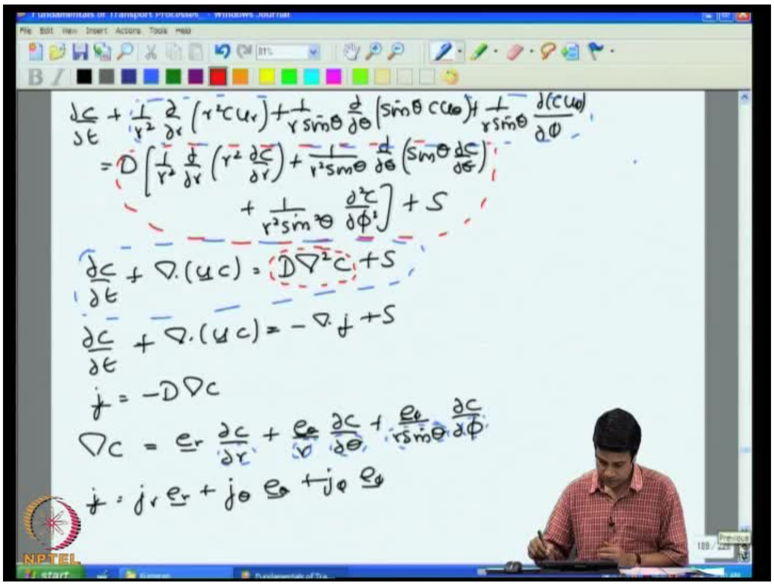

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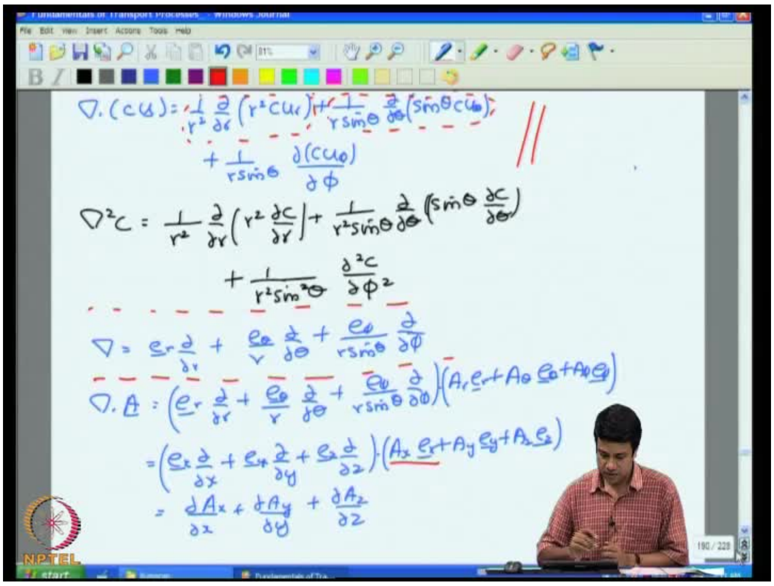

Once again, it had exactly the same form as that for the Cartesian coordinate system. Except that these operators the divergence on the left the Laplacean on the right are different in a spherical coordinate system. The reason is because the surfaces of constant coordinate are curved surfaces and the unit vectors perpendicular to these change with position. So, we have defined, the divergence in the Laplacean coordinates in this coordinate system as well.

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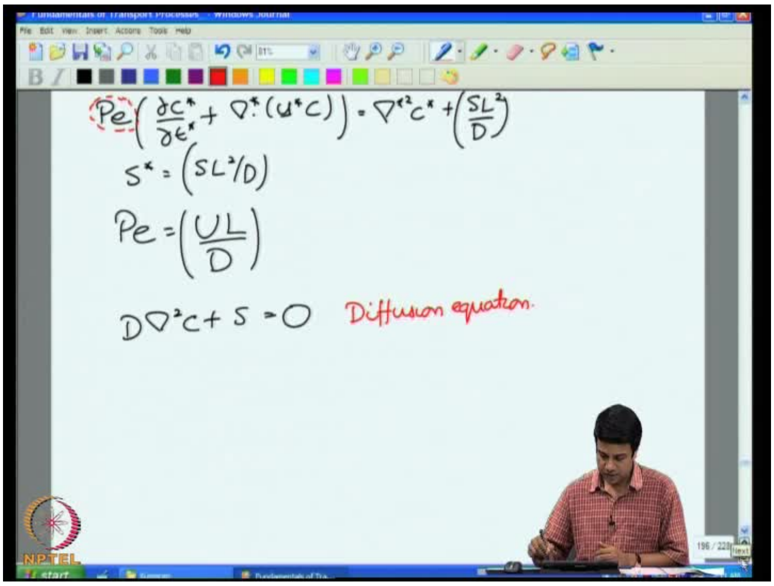

And I briefly showed you the equations in the cylindrical coordinate system. They have a similar form once again. And after that we had come down to this convection diffusion equation and taking the limit of peclat number becoming small to get to the diffusion equation.

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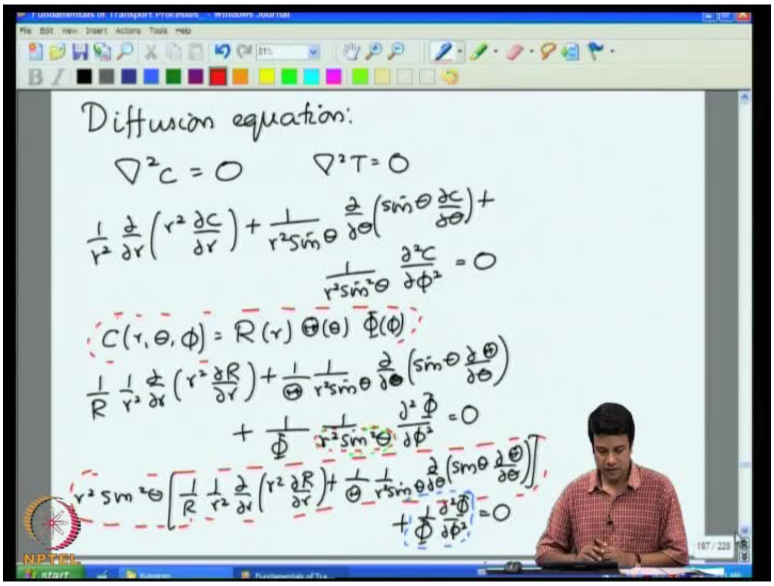

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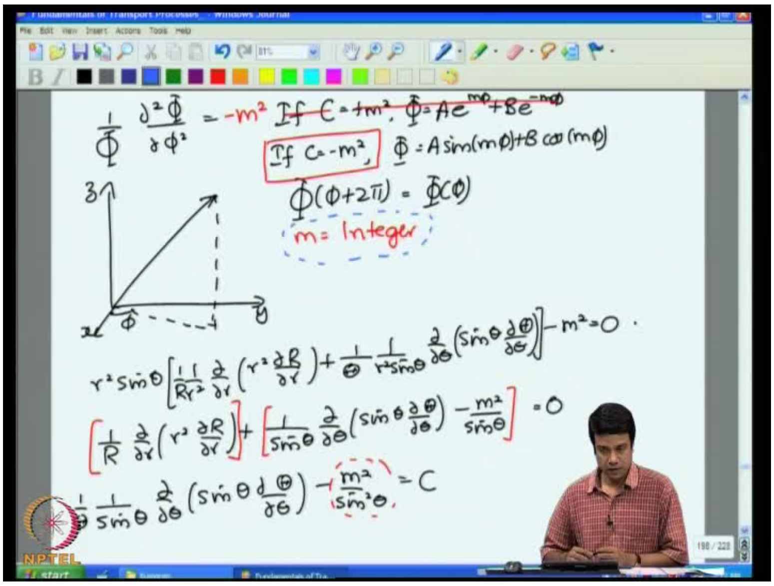

Now, we solved the diffusion equation in a spherical coordinate system by separation of variables. In the phi direction you got Eigen values integer. Eigen values in the phi direction from the requirement that, when you go around by an angle of 2 pi you come back to the same physical location in space. Therefore, in the sine and cosine solutions along the phi coordinate the coefficient has to be an integer. Because without that you would not get back the same value, when you go around by an angle 2 pie.

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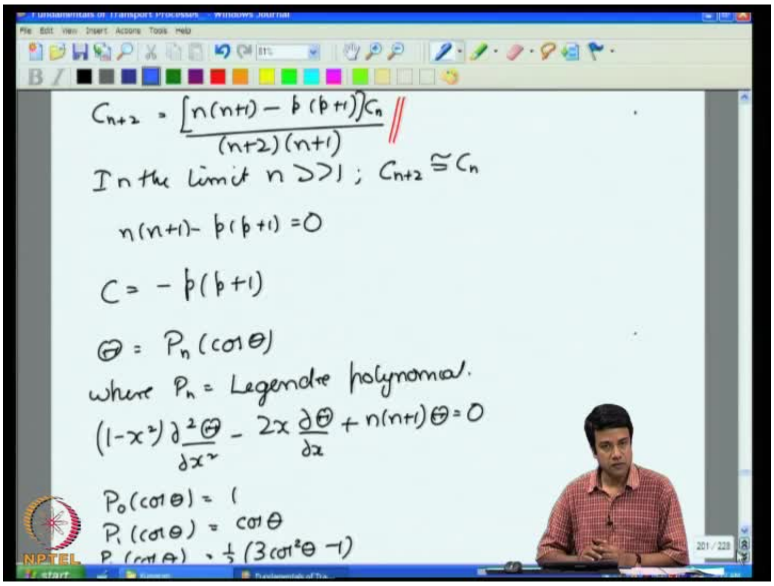

Then we looked at the theta equation and I showed you that the solutions are the the equation that results for the theta coordinate is in the form of what is called as legendre equation? And, the solutions for these are what are call legendre polynomials these solutions turn out to be finite, only if the constant in the theta equation is of the form n into n plus 1. Otherwise the solutions become infinite at theta is equal to 0 or at theta is equal to pi.

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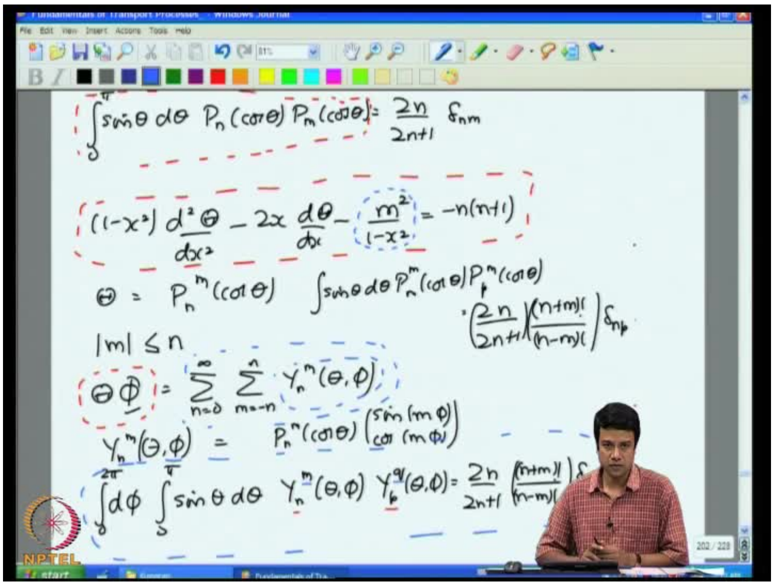

So, you get discrete eigen values in the theta direction as well. From the requirement that the solution has to be finite both at theta is equal to 0 and theta is equal to pi. So, combining the r and theta directions we get solutions in the form of what are called spherical harmonics, these are products of P and m of cos theta legendre polynomials and sine or cosine of phi. These have their own orthogonality relations, which can be used in order for determining coefficients in the differential equation.

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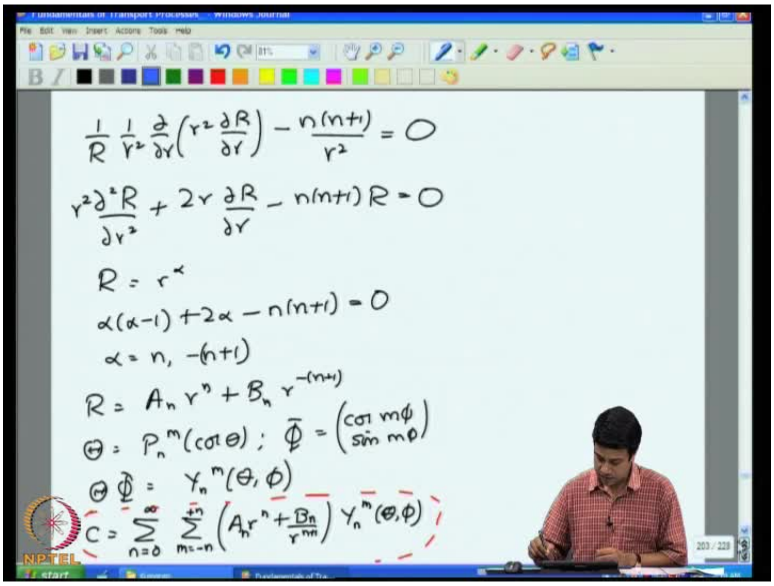

And then finally, we looked at the radial part of the equation and this just has 2 sets of harmonics. A growing harmonic proportional to r power n and the limit as r, becomes large and a decaying component which decreases r power minus of n plus 1. So, it goes 1 over r power n plus 1 and combining these as you can see, in the red at the bottom, we have the final solution in terms of spherical harmonics and the r dependence is in the terms for r power n and 1 over r power n plus 1.

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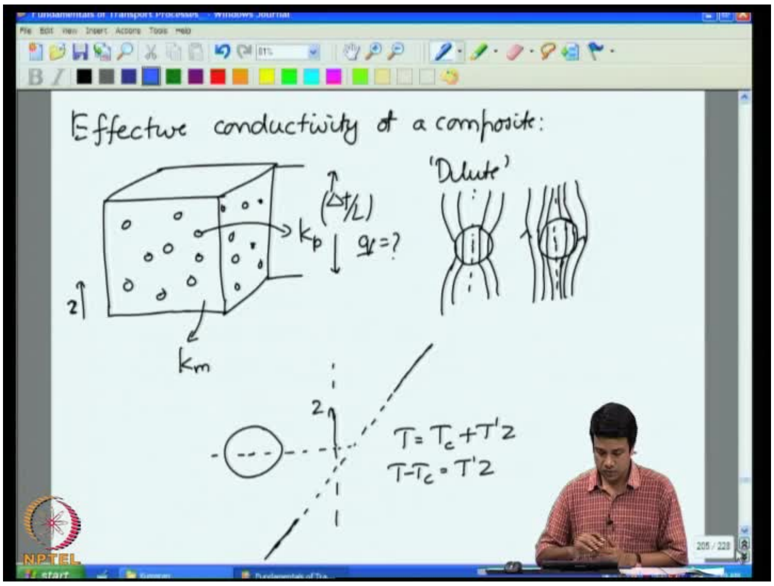

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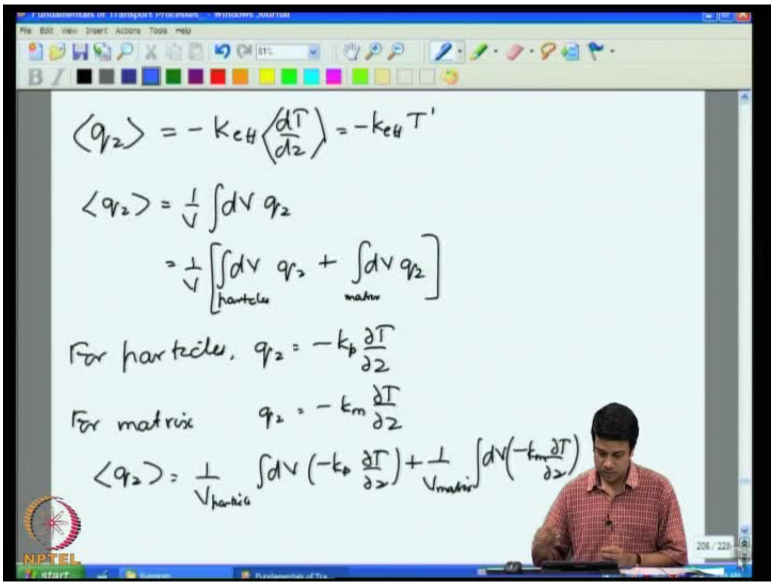

And I showed you how these orthogonality relations can be used for this specific case of the effective conductivity of composite. We looked at this only in the dilute limit, Where the number of particles is small such that the temperature field around 1 particle does not affect the temperature around another particle.

So, this effectively reduce the problem to finding out the flux within a particle, which is placed in a temperature field, which has a linear temperature gradient far from the particle.

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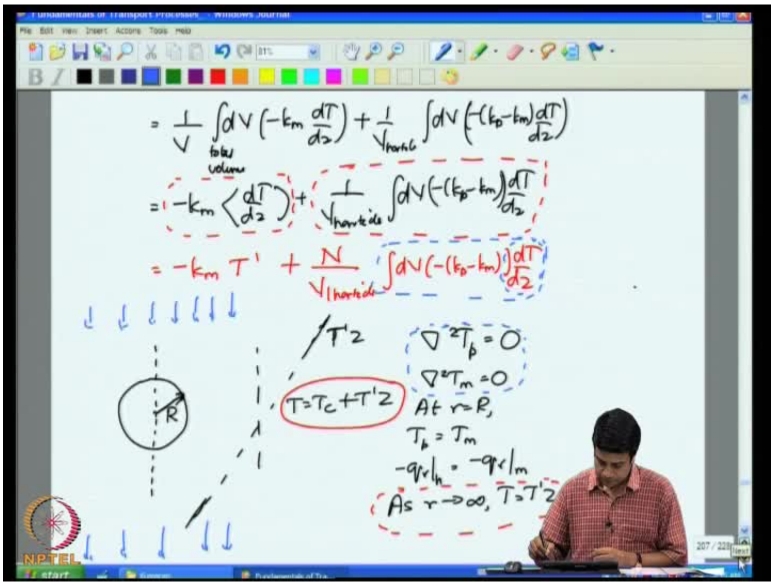

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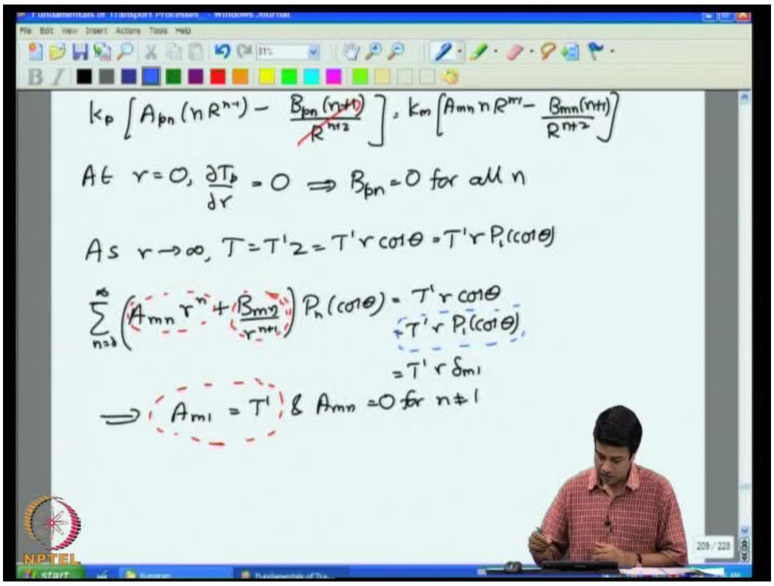

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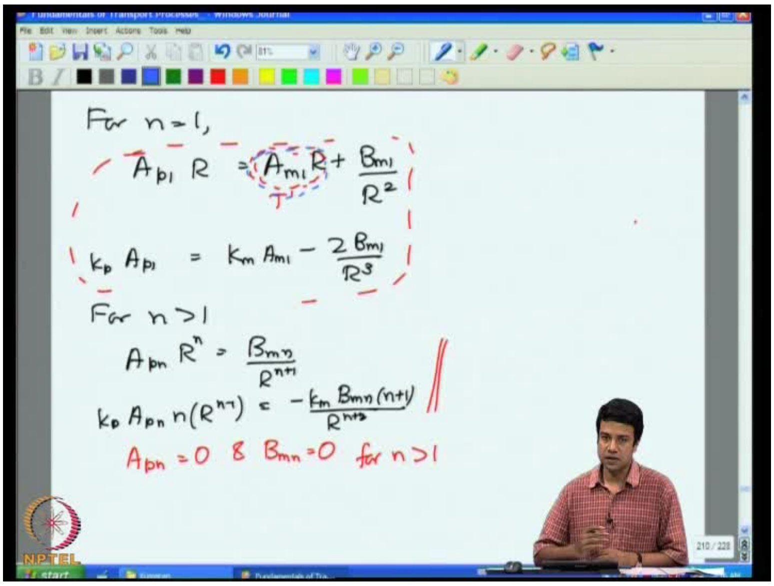

And, we solved this problem using the spherical harmonic expansion and I showed you that since, that temperature far away from the particle is proportional to T prime times P 1 of cos theta. The temperature far away is equal to T prime times P 1 of cos theta. That means that the only solutions that will, which are non zero are those proportional to P 1 of cos theta itself. Because all of the spherical harmonic functions are, all are orthogonal to each other. Therefore, if the forcing is of the form P 1 of cos theta the solution, will also have that exact same symmetry and using that condition quite simply, we manage to get the coefficients in the equation and from that find out the effective conductivity of the composite.

Only for n is equal to 1 will you get nonzero solutions. Because there is this inhomogenous term in the equation for all other coefficients there is no inhomogeneous term. 2 linear equations 2 unknowns both of them homogenous the only solution is for both of these coefficients to be 0.

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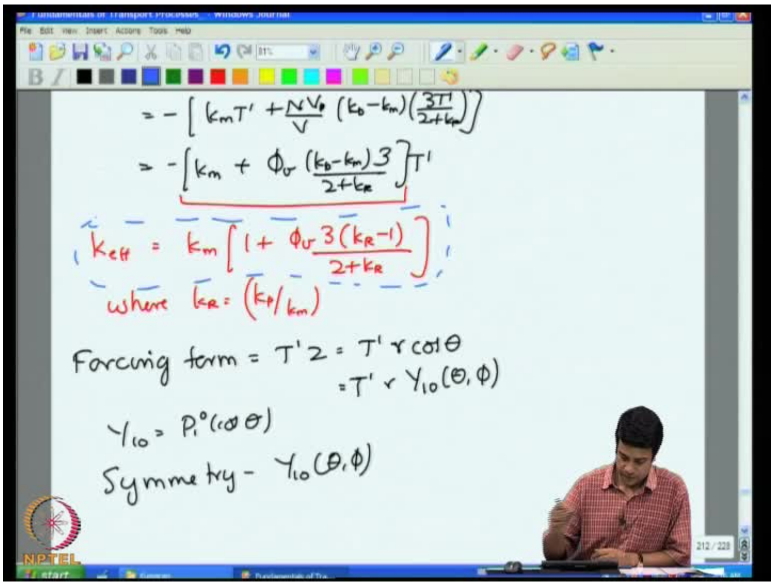

A result of the orthogonality condition, that all the spherical harmonics are all orthogonal to each other. Therefore, the symmetry of the solution is the same as the symmetry of the force in that is applied. In this case proportional to P 1 of cos theta far from the sphere and from that we manage to get the effective conductivity of the composite in the dilute limit.

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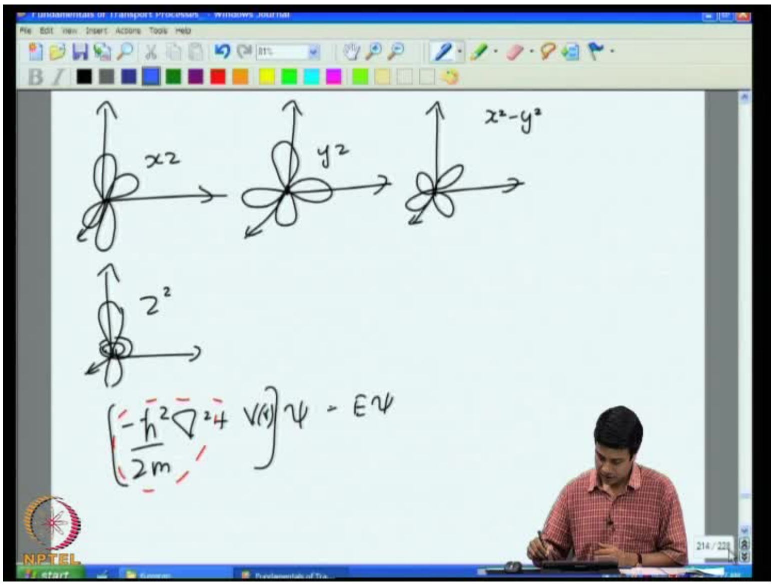

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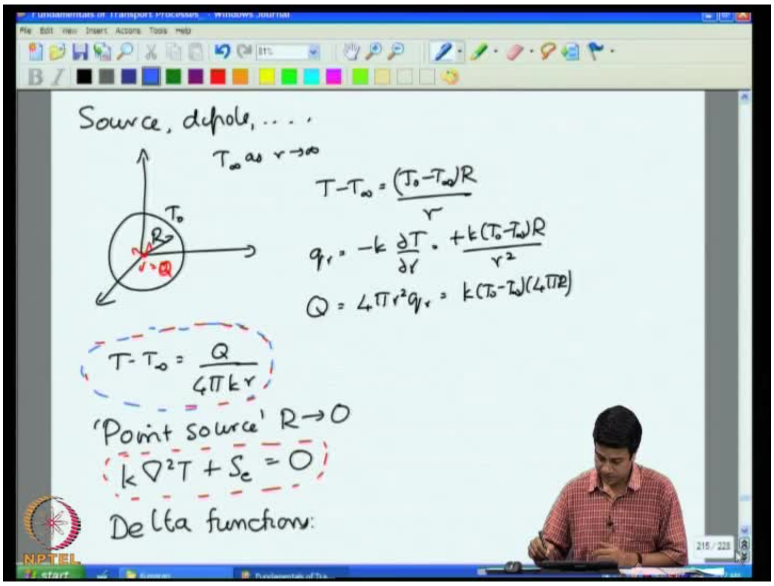

I had explain, to you the symmetries that arise from the spherical harmonic expansions briefly. And then we went onto looking at point source, a delta function source. I defined for you the point source in the previous lecture.

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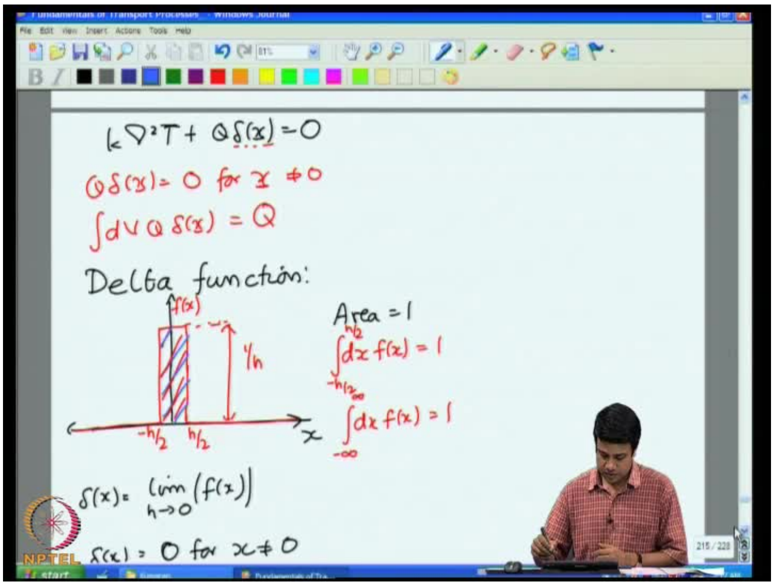

A delta function is defined as a function which is 0 delta of x is defined as to be 0 for x not equal to 0. The area under the curve of the delta function is equal to 1. And if you multiply the delta function by any function g and integrate over all, over the x axis from minus infinity to infinity you will get g of 0.

This delta function is the limiting case of a function. Such as this it has a value 1 over h when x is between minus h by 2 and plus h by 2 0 otherwise. So, the in the limit as h goes to 0 the height increases the width decreases to 0 and you get a delta function, which is 0 for all x naught equal to 0. And it is non zero only at x is equal to 0 in such a way that the area under the curve has to be equal to 1.

This is not a unique choice, there are other functions as well. Which can be reduced to delta functions in the limit as h goes to 0, but further moment we will restrict ourselves to just this form of the delta function.

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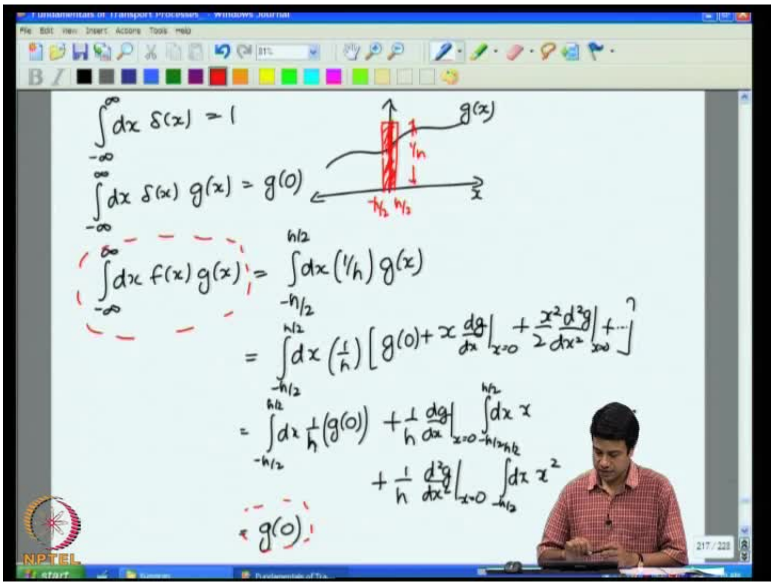

With this form of the delta function, I showed you that the integral of delta of x times g of x from minus infinity to infinity is equal to the value of the function at 0 itself. Similarly, delta of x minus x naught times t of x has to be delta at the location x naught. You just shifting the origin to x naught in that case delta function is located at the location x naught, as defined here integral minus infinity to infinity of dx times delta of x has to be equal to 1. That means that delta function has dimensions of 1 over length in this case.

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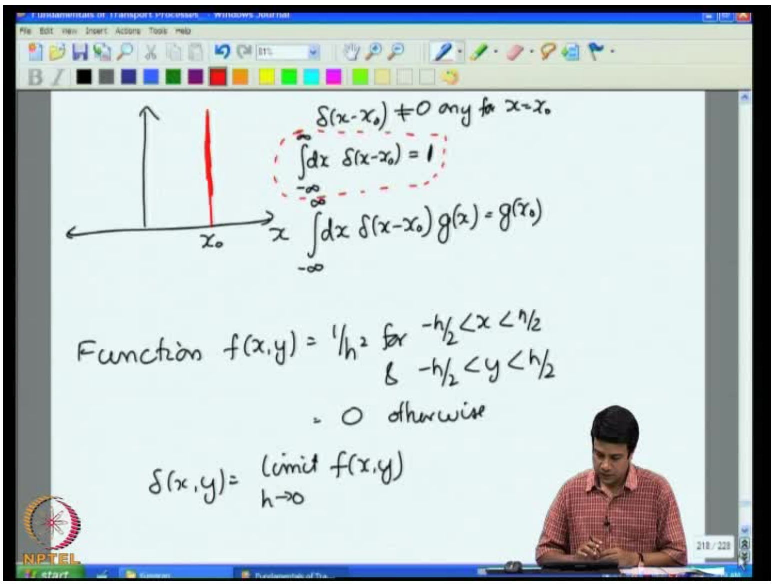

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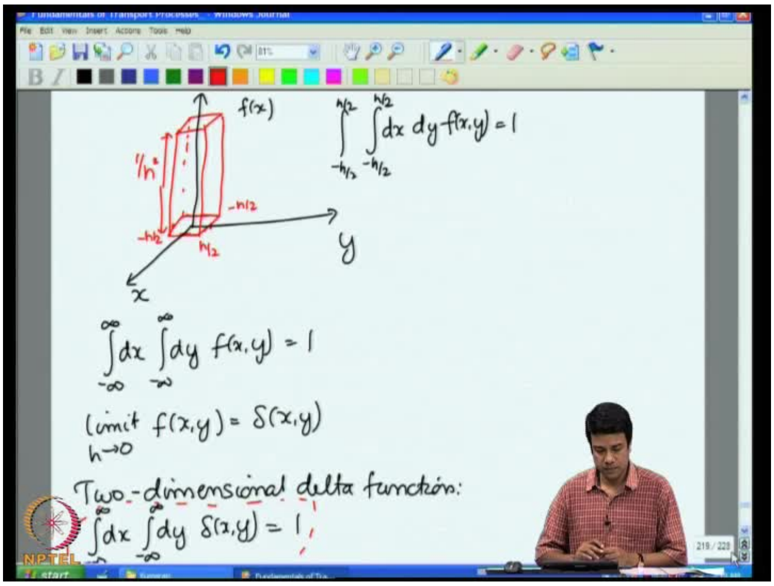

In a similar manner, one can define a 2 dimensional delta function. I had in fact drawn it for you in the previous class, x y axis on the plane f of x is in the perpendicular to the x and y directions. And this delta function is defined, such that f of x y is equal to 1 over h square for minus h by 2 less than x less than h by 2 and minus h by 2 less than y less than h by 2 and its equal to 0 otherwise.

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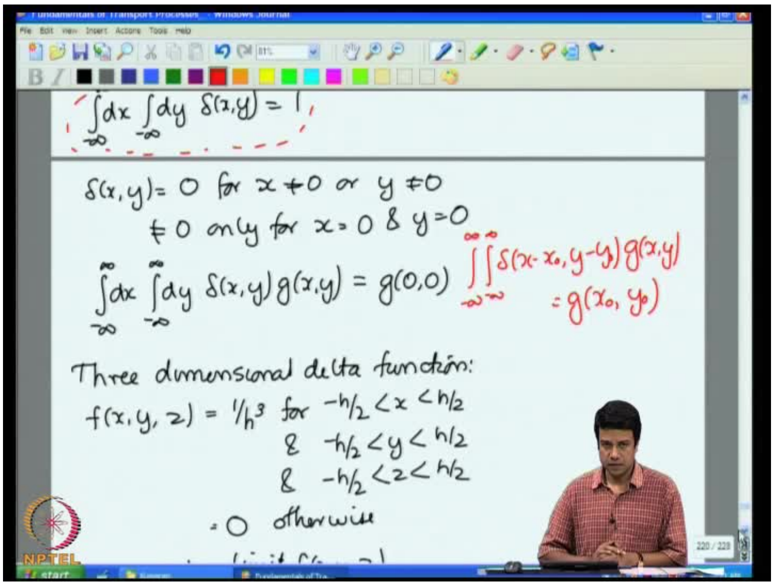

And the integral of the delta function over x and y integral dx dy f of x y has to be equal to 1. In the limit as h goes to 0 one gets the delta function f of x y is equal to delta of x y. So, this 2 dimensional delta function has the property that the integral of delta of x y over x and y from minus infinity to plus infinity is equal to 1 delta is equal to 0 for when x is not equal to 0 or y is not equal to 0. It is non zero only when x is equal to 0 and y is equal to 0 and the integral minus infinity to infinity of dx dy delta of x y g of x y is equal to g of 0 0.

So, basically this delta function picks out the value of the function g exactly at 0 0 at the location 0 0 so delta of x minus x naught y minus y naught. So, this the state extension of this is minus infinity to infinity. This will be equal to g of x naught y naught just shifting the origin to the location x naught and y naught. In a similar manner the 3 dimension delta function is defined as 1 over h cubed for x between minus h by 2 and h by 2 and y between minus h by 2 and h by 2 and z between minus h by 2 and h by 2 equal to 0 otherwise. Limit as h goes to 0 you get the three dimensional delta function.

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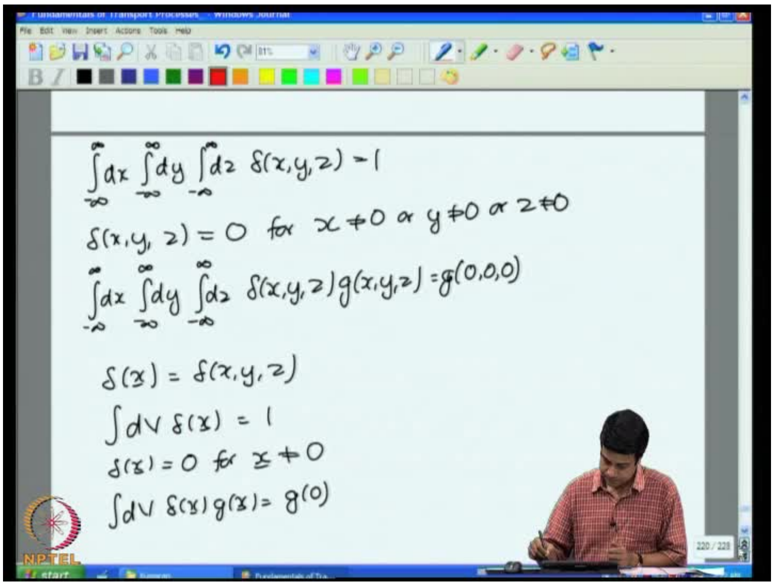

And this has the properties integral over the entire volume of delta of x y z is equal to 1 and its equal to 0 for x naught equal to 0 or y naught equal to 0 or z naught equal to 0. And if you take the delta function multiplied by any other function and integrate over the entire volume, it will pick out the value of that function at the origin.In the remainder of these lectures, I will use the short hand notation x vector for the three spatial coordinates x y and z and dv for dx d y dz and, we will integrated over the entire volume.

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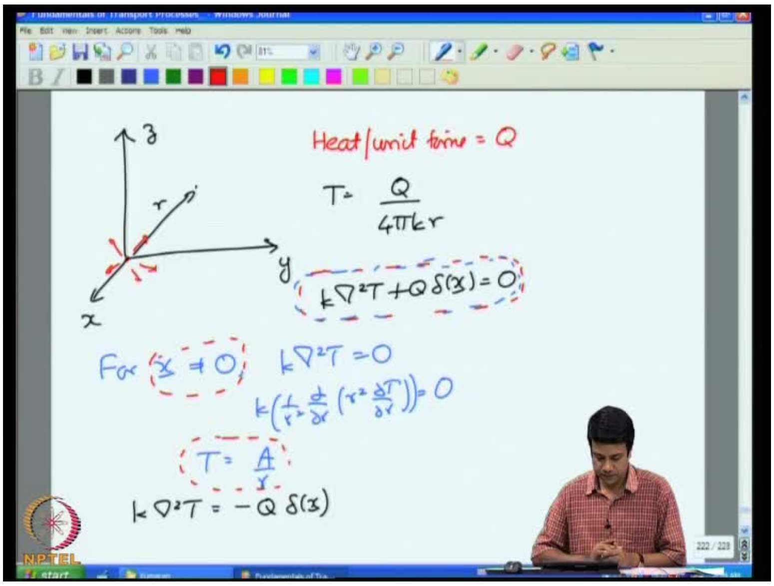

Note that has defined here delta function in 3 dimensions has dimensions of 1 over volume because when I take the delta function multiplied by the volume and integrate I get a result that is dimensionless therefore, the delta function has dimensions of 1 over volume. And we had looked at how to use the delta function to get solutions of the diffusion equation for a point source. which is emitting heat by unit time equal to Q the solution for the temperature is equal to Q by 4 pi kr.

So, that is the solution for the temperature field due to this point source located at the origin. I showed you that this temperature field is also obtained using a diffusion equation of this form k del square t plus q delta of x is equal 0. Since, delta is non zero only at x is equal to 0 for everywhere x is not equal to 0. I can solve the Laplace equation k del square T is equal to 0, and I will get the temperature as some constant divided by r. This constant is determined from the condition that the total heat coming out of the point source has to be Q. So, if I take an equation of the type k del square T is equal to minus Q delta of x and then integrated over a small volume around this point source.

The integral over the volume of integral of k del square T over the volume is equal to minus Q, because integral of the delta function just gives me 1. Where Q has defined is the heat generated per unit time from the source.

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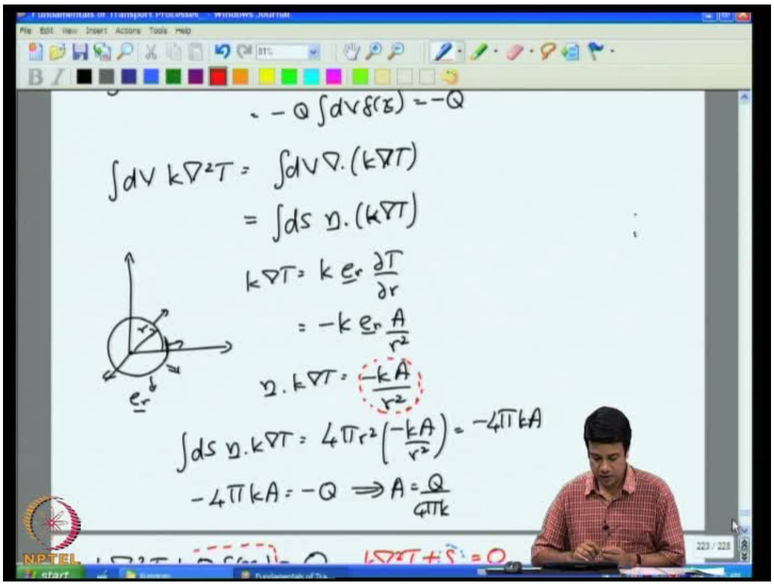

The first term can be simplified a little integral over the volume of k del square T is equal to integral over the surface surrounding, this volume of the unit normal dotted with k grad T where grad int of T the k is in the radial direction because T is only a function of radius. So, if I take k grad T and multiplied by the unit normal an integrated over any surface. The gradient of the temperature goes as 1 over r square, it decreases 1 over r square the surface area increases proportional to r square. So, if I take k grad T an integrated over the surface. I will just get something that is independent of radius turns out to be equal to minus 4 pi kA and from that I get the constant k is equal to Q by 4 pi k.

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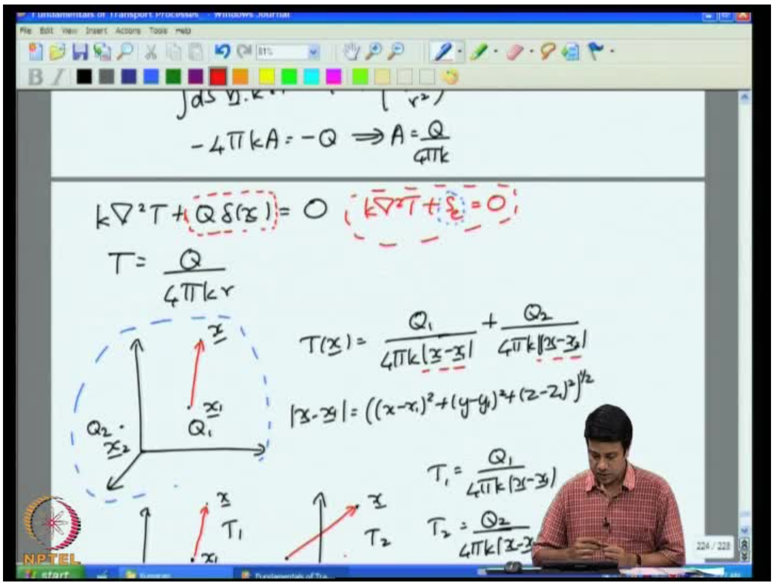

So, therefore this is the solution for the point source. The solution for the equation of this kind minus k del square T k del square T plus the source of energy is equal to 0, the solution for that is T is equal to Q by 4 pi k r. At the end of the last lecture we would discussing, that if you have one point source. The temperature is Q 1 by 4 pi k times the distance between the source and the observation point x 1. In this case is the point source at, which the source is located x is the point at which, you are taking the temperature. So, that is the observation point.

So, the temperature field due to a source is equal to the the energy generated per unit time divided by 4 pi k times. The distance between the source point and the observation point If, I had2 sources I can write down the temperature fields individually due to each of the sources and add them up to get the temperature field due to the 2 sources.

So, this is the temperature field due to 2 sources together where I just take the temperature end of the individual sources and add them. So, this is the procedure known as linear super proposition. The equation for the temperature field is linear. Del square T is equal to source or sink is linear in the temperature.

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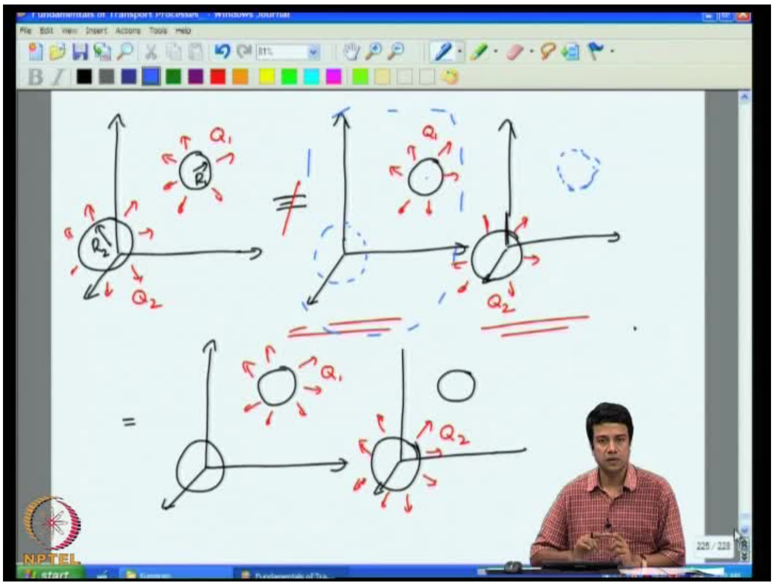

And therefore, if I have 2 sources I can define the temperature fields due to the each sources individually and add them up. One has to be careful while doing linear superposition in that when one adds up the temperatures the boundary conditions. The configuration has to be kept a constant. I explain this in the last lecture, but its worth explaining again. If for example, I have solving the temperature due to 2 spherical particles. One of them was generating heat Q 1. The other was generating heat Q 2. I cannot write it as the sum of 2 problems. Where in the first problem I have only 1 particle present and the second problem I have only the second particle present.

This is not correct because the boundaries between the original problem and the 2 sub problems are different. In the original problem, there are boundaries on both the spherical particles in the 2 sub problems. There is only 1 spherical particle each. So, this is not the correct way to do the linear superposition.The correct way is to divided into 2 sub problems each of which has both the spheres present in 1 problem, I can have only 1 of the spheres emitting energy in the second problem can have only the second sphere emitting energy. Therefore, in both of these problems boundaries are exactly the same as in the original problem. The source or sink of energy is different, but those can be added up to give the original problem.

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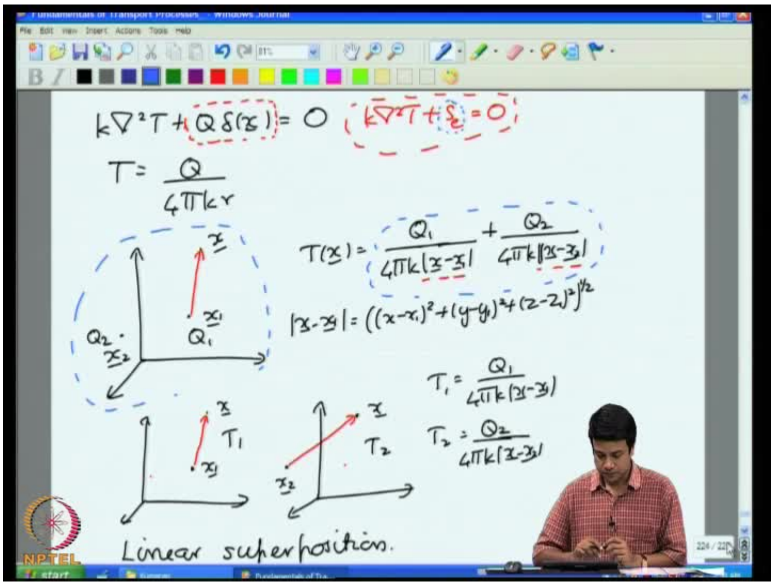

So, this kind of linear superposition is possible. So, long as you do not change the boundaries in the sub problem. And that is why the concept of a source a point source is such a valuable one. Because the point source has no dimension, it has no size. Therefore, whether I have a point in the field or is not really important. What comes out of that point is important. So, in this case I am able to superpose 2 problems in which there is one point source in one case the other one in the other case. Since, the point has 0 volume anyway it does not matter, if a second point is present. So, longer if it not emitting energy it does not enter into the problem and that is why it is form or usefull to use point sources in order to represent the continuous fields.

So, that is the big advantage of point sources. I can remove and add sources and still, we able to use linear superposition. Whereas the finit says particles it is not possible to do that.

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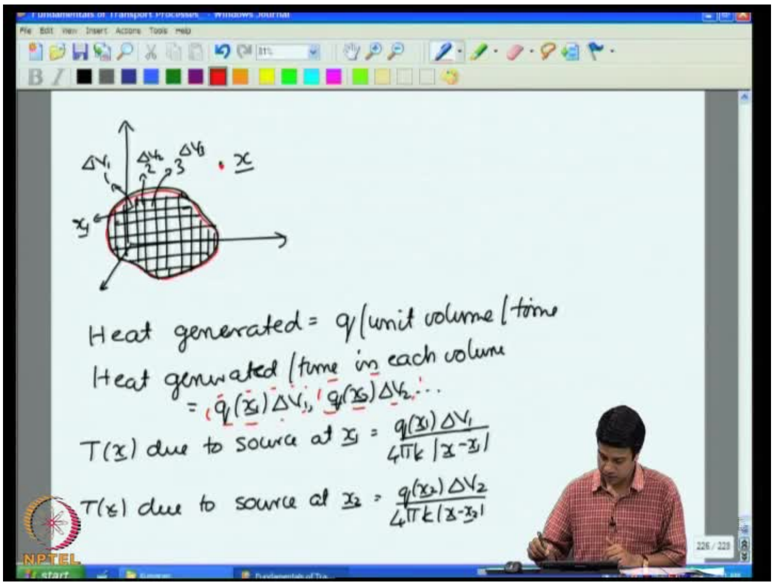

And briefly at the end of the previous lecture, I had said that if you had a distributed source. I can still divided into a large number of point sources. Let us say, I have a distributed source in which the heat generated is q for unit volume for unit time. This distributed source can be divided into large a number of smaller volumes in a space filling manner. In such a way that each volume has its source at the centre of that volume in the limit as the volume goes to 0. This will reduce to the distributed source. So, I have heat generated per unit time in each volume as the heat as the q, which is the heat per unit volume per unit time the volume itself for each of these sub volumes.

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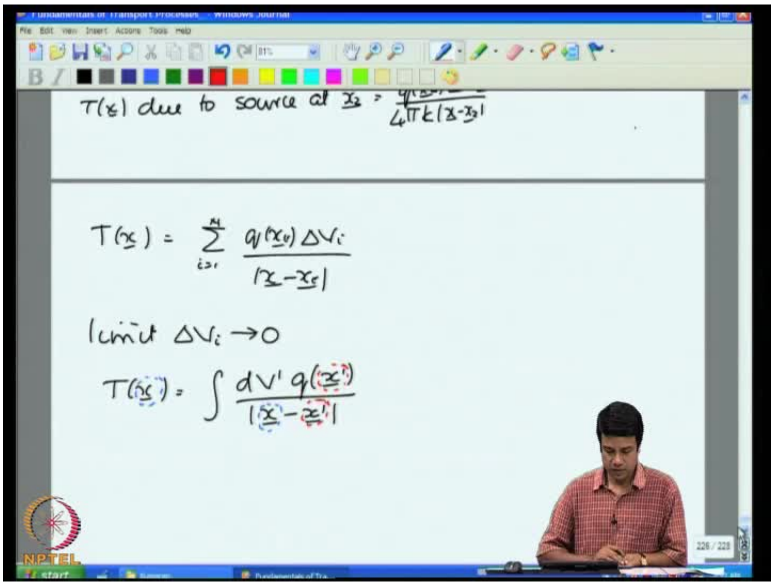

So the temperature at the observation point due to each of this sub volumes is q times delta V divided by 4 pi k into x minus the source point the distance between the source and the observation point. So, to get the total temperature you just add up all of these individual source points to get the total temperature in the limit as delta V goes to 0. You get, you can convert this from summation to an integral over the entire volume.

So, temperature at the location: x is equal to integral dV prime q of x prime divided by x minus x prime the modulus that is the distance between the source and observation point. Note that x prime is the source point is the location at which the source is and x is the observation point. So, I am calculating the temperature t at the location x and the function of, where the source is located. And how much heat is coming out from the source and the distance between the source and the observation point.

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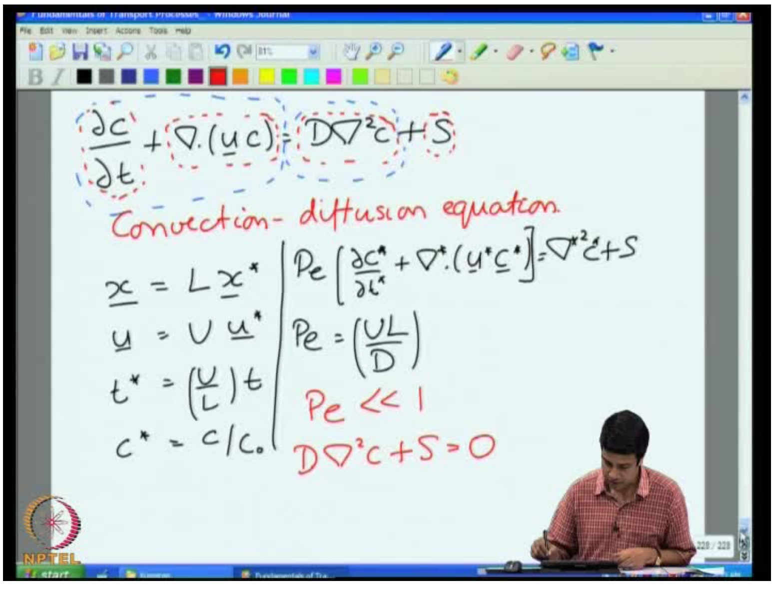

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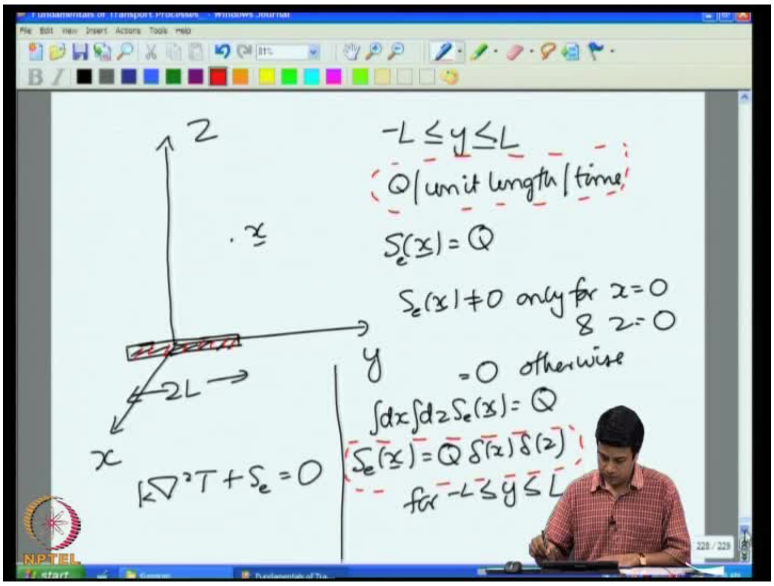

So, how do we use this for solving actual problems. So, let us just solve an example, let say that I have a heated wire and it has a distance 2L in the length of the wire is 2L and it goes from minus L less than or equal to y less than or equal to L. So, this is the wire that is emitting heat which is Q per unit length per unit time. So, this is the heat that is being emitted by this wire and at any location x. One would like to find out: what is the temperature due to the heat emitted by this wire.

So, one would like to find out what is the temperature field due to this wire. So, this is the source, the source of heat is this wire it is a thin wire along the y axis and its emitting heat Q per unit length per unit time. That means the source it is non zero only, when x is equal to 0 and z is equal to 0 and y is between minus L by 2 and plus L by 2. So, this is equal to Q, note that in my differential equation k del square T plus S e is equal to 0. This source has dimensions of heat generated per unit volume per unit time. This wire is emitting heat Q per unit length.

So, that heat emitted per unit volume per unit time. Note it is non zero only when x is equal to 0 and z is equal to 0 it is 0 otherwise. So, this this Q is not equal to 0 only for x is equal to 0 and z is equal to 0. Because, it is a wire that is along the y axis is equal to 0 otherwise. And I also require that integral dx integral dz S e of x this integral integral dx integral dz times S e of x should give me the total heat coming out per unit length per unit time. So, this has got to be equal to Q.

Now, what is the function that satisfies all of these properties. A function is 0 if x is not equal to 0 or z is not equal to 0. It is non zero only, when x is equal to 0 and z is equal to 0. Total integral of that over x and z has to be equal to Q, which means that S e of x is equal to Q times delta of S x delta of S z. So, this gives me the expression for the source the equation S e of x is equal to Q times delta of x times delta of z. Because, this function is non zero only when x is equal to 0 and z is equal to 0 integral of this function. Over both x and z is going to give me Q for minus L less than or equal to y less than equal to L. So, in over the length of the wire this source is non zero otherwise it is 0.

So, I have framed the equation for the heat conduction as well as the source s in terms of the heat generated per unit length per unit time. I recall, that S e is in is has dimensions of energy per unit volume per unit time. Q has defined here has dimensions of energy per unit length per unit time. Q as defined here has dimension of energy per unit length per unit time.

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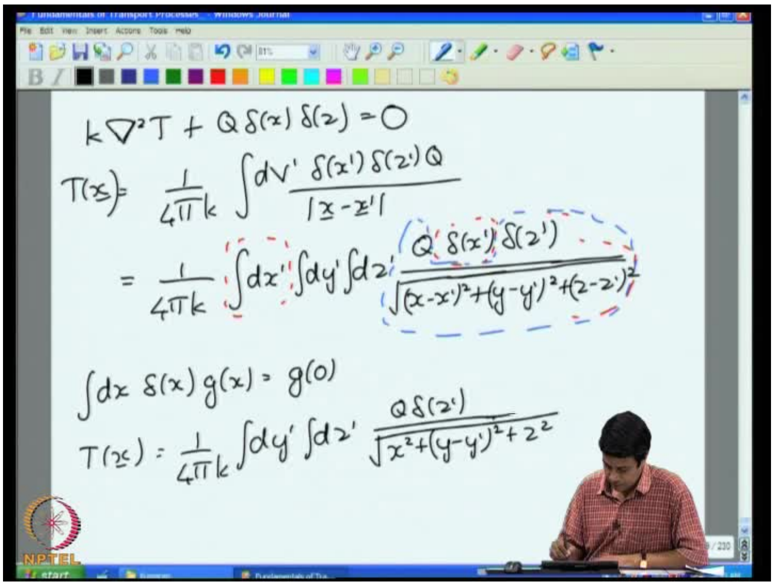

Therefore, if I take Q and multiply it by delta of x and delta of z. I get something dimensions of energy per unit volume per unit time. Therefore, I have to solve now k del square T plus Q delta of x delta of z is equal to 0. We already know what the solution to this equation. The solution T is equal to 1 by 4 pi k integral dV prime delta of x prime, delta of z prime Q by the distance. Let me just write this for completeness as T of x. So, x here is the observation point, x prime is the source point. Vector x prime is the source point vector x is the observation point. The source is Q times delta of x times delta of z. That means at the temperature is one over 4 pi k integral of the volume dV prime delta of x prime times delta of z prime times Q that is the source x prime by x minus x prime.At a given location in order to evaluate this I need to actually carry out the integral 1 over 4 pi k integral dx prime dy prime dz prime Q times delta of x prime delta of z prime divided by the distance. The distance is square root of x minus x prime the whole square plus y minus y prime.

So, that is the distance the modulus of x minus x prime that is the distance, 1 over x minus x prime which is square root of 1 over x minus x prime square, plus y minus y prime square, plus z minus z prime square. So, now how do we evaluate this function. We already know what the properties of the delta function are integral dx delta of x minus x prime, delta of x, g of x is equal to g of 0. The value of the function at x is equal to 0. Here, I have 2 integral with respect to the delta function. One is integral x prime of delta of x prime time some function. The second function is some function of x prime.

So, the integral is just going to be equal to the value of that function at x prime is equal to 0. So, using this property I can rewrite the temperature x integral dx prime times delta of x prime divided by some function of x prime. This is just going to be equal to integral the value of this function at x prime is equal to 0. So, this square root of x square plus…(Refer Slide Time:37:22).

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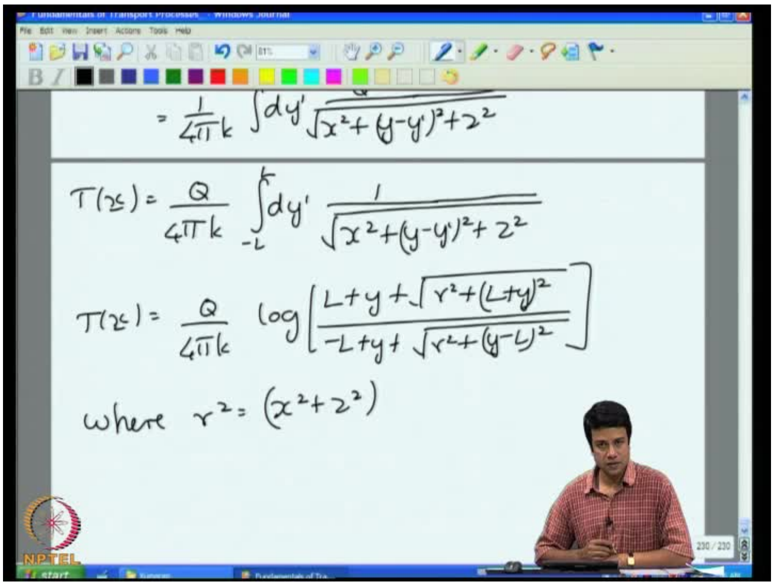

Now once again, I have an integral over z of delta function times this other thing. The delta function times the other whole thing. So, this once again I can write this as 1 over 4 pi k integral dy prime Q by square root of x square plus y minus y prime. So, because of these 2 delta functions the final equation for the temperature has been reduced to just an integral over the y coordinate alone.This can further be simplified note that there is a source of energy only, when y is between minus L and plus L. Therefore, I can write this as Q by 4 pi k integral between minus L and L dy prime 1 over root of x square plus y minus y prime plus z square.

So, this integral can actually be evaluated analytically the final result that we you get we evaluate this integral an analytically is that T of x is equal to q by 4 pi k log of L plus y plus square root of r square plus L plus y the whole square by….(Refer Slide Time: 41:40)

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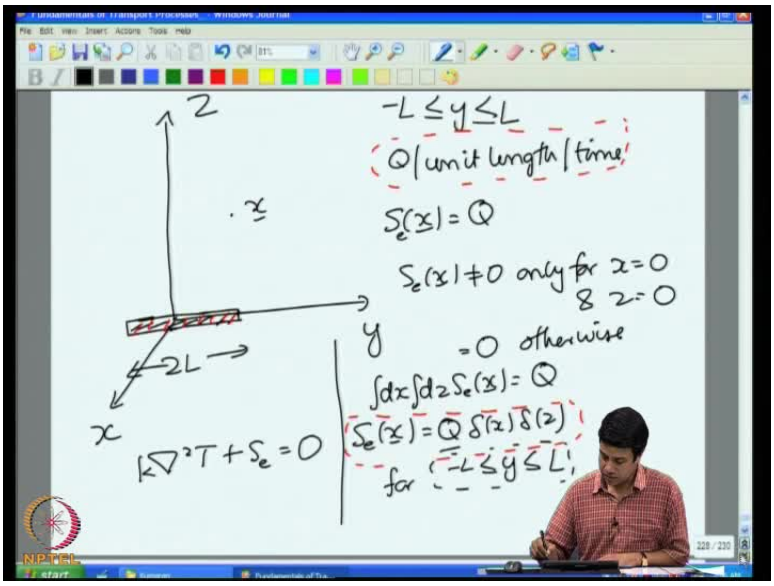

The r in this solution is the radial coordinate in the cylindrical coordinate system. In which the axis is the y axis. It comes out naturally in this form. The reason is because if I go back to the original problem. This problem: there is a wire in the y direction along the y axis. Therefore, as you go around the y axis in the x z plane. There is no change. Therefore, it will be more convenient to analyze this in a cylindrical coordinate system.

In, which y is the axial direction and r is the distance from this axis and theta is the angle around. Because, there is no variation in the theta direction as you go around this.

Therefore, you get the solution to be symmetry with respect to x and z, the final solution form square root of x square plus z square.

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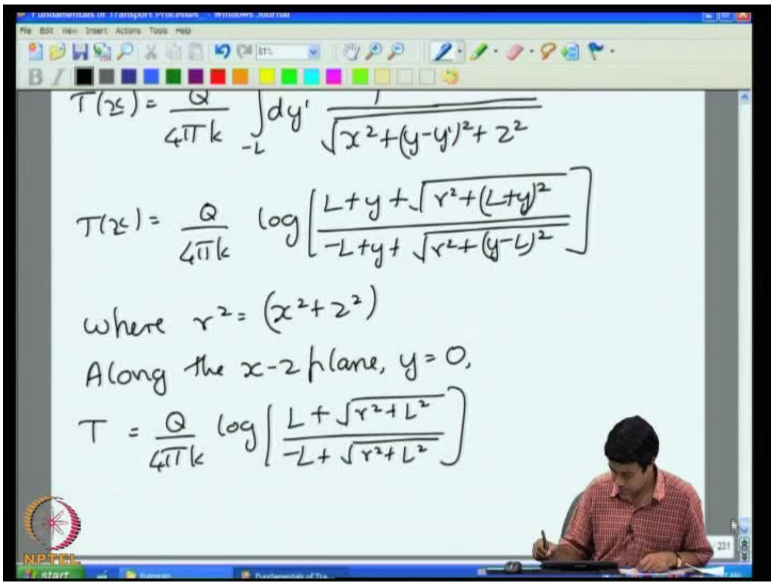

So, this of course gives us the analytical solution the question is what physically does this mean. So, as usual let us take 2 limiting cases. To simplify the problem, I will assume for the moment that we are along the x z plane y is equal to 0. Therefore, T is equal to Q by 4 pi k, log of L plus root of r square plus L square…(Refer Slide time: 45:09)

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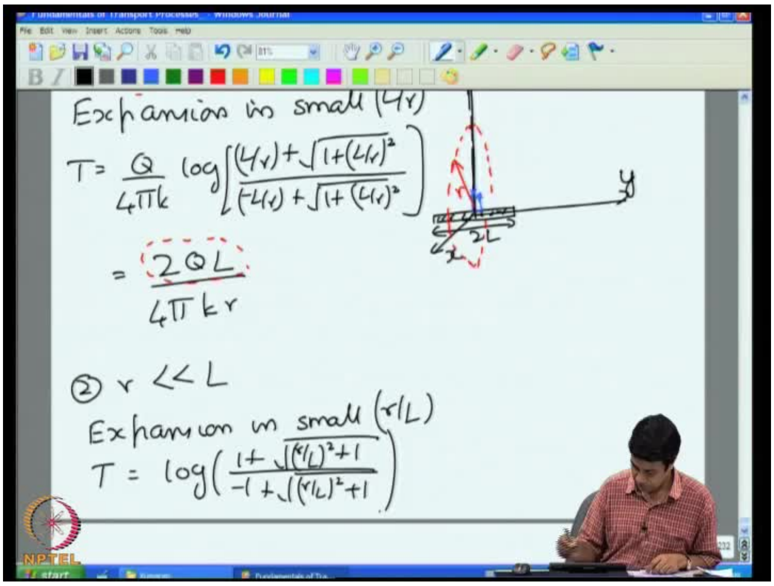

So, I have my wire along the y coordinate along the x z plane. So, I am taking the distance an observation point. That is somewhere, along this x z plane and for this observation point. The distance from the origin is the radius r and of course the solution is symmetry with respect to r. It depends upon only the distance from the axis. So, this is the solution along the x z plane.

One can consider 2 limiting cases. Here the first case is, when r is large compared to L this wire has thickness 2L. So, when r is large compared to L that means that the distance from the origin is large compared to length of the wire itself. In that case I can do an expansion in small L by r. I can rewrite this equation as T is equal to Q by 4 pi k log of L by r plus square root of 1 plus L by r the whole square by minus L by r plus square root of 1 plus L by r the whole square. And then expand the log function the small parameter L by r. if we do that expansion the final result turns out to be equal to 2 Q L by 4 pi k r, where r is the distance from the origin.

Now this looks exactly like the solution that we got for a point source. Because this 2 Q L you recall for a point source that is generating Q amount of heat per unit time. We got this solution as Q by 4 pi k r. In this case Q is the amount of heat generated per unit length per unit time. And the temperature we are getting when the distances far from, when the distance from the source is large compared to the length of the source is 2 Q L by 4 pi k r. The length was 2L. The amount of heat generated per unit length was Q, total amount of heat generated is 2 Q times L.

So, we getting the exactly the same solution that we would get for a point source except that Q In this case, is the total amount of heat generated per unit time and this is physically the reason is because, where we go sufficiently far from the source. The distance is large compared to the length of the source. If you go sufficiently far the source will always looks like a point source. If you are sufficiently far away we will not see the details geometry of the source. It will appear to us to be just a point and the solution that we get will just be equal to the solution due to the point source.

So, this is just the leading order term in the expansion. That are higher terms of course we will see that in the next lecture: the dipole term, the quarter pole term and so on. But, this is just the leading term in the expansion and physically. This is because, if the distance from the source is large compared to the characteristic dimension of the source.

Then, it looks just like a point. And the solution that I get for the temperature field is exactly the same solution that I would have got for a point.

So, physically that is what happens when the distance from the source is large compared to the characteristic dimension of the source. We can consider the opposite limit r is small compared to L. So, in that case the actual distance from the source is small compared to the overall length of the source. if I take the distance from the source being small compared to the overall length of the source. In that case I can use an expansion in small r by L the other way. And if I take this equation I can do the expansion in small r by L. I get T is equal to log of 1 plus square root of r by L square plus 1 by minus 1 plus square root of r by L whole square plus 1 and 1 over 4…(Refer Slide Time: 46:15) .

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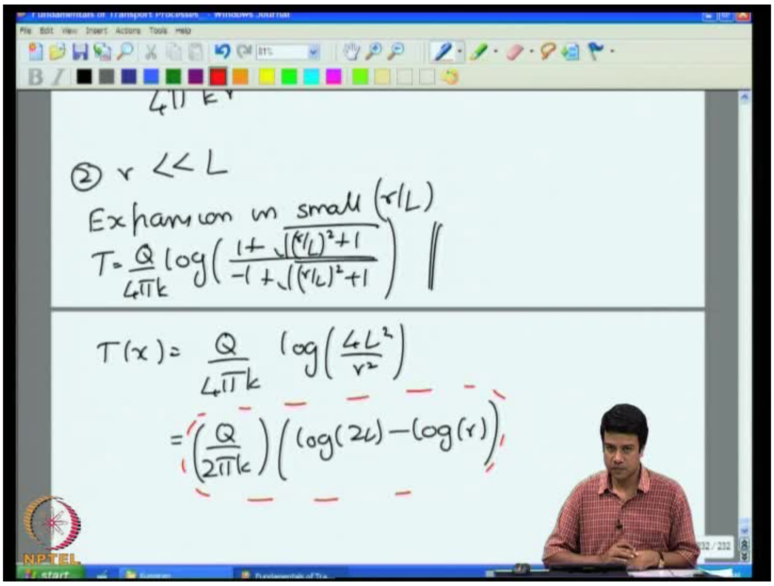

And if I expand out this log function in small r by L. Then, I will get the temperature Q by 4 pi k log of 4L square by r square. Just expanding out this function in small r by L and I can further write this as Q by 2 pi k into log of 2L minus log of r.

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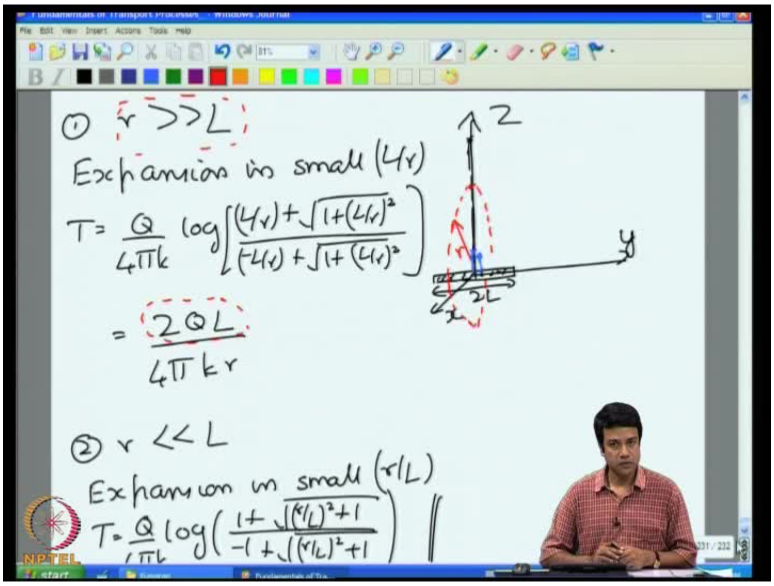

So, when the distance from the wire is small compared to the length of the wire. I get a log function as the solution of the differential equation. The temperature decays logarithmically with distance. This as we will see a little later is also the solution for a point source in 2 dimensions. If I am sitting very close to the wire the distance from the wire is small compared to the length. It looks like the conduction from a wire of infinite length close to some long linear object. The distance of that object is large compared to my distance from that object. The length of that object is large compared to my distance from the object. It looks like an infinite object infinite line.

So, this looks like an infinite line source in 3 dimensions or a point source in 2 dimensions in the x z plane, and in 2 dimension the solution for the point source is a logarithmic function.

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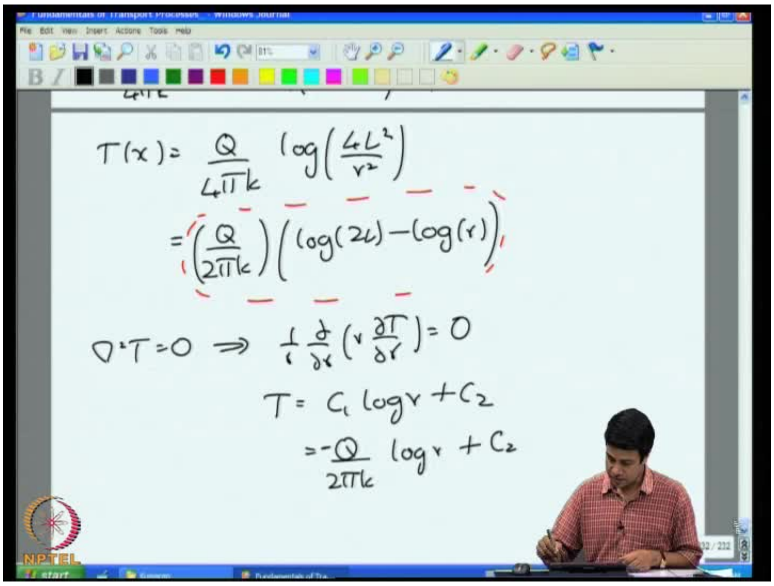

If you recall we had solved the problem del square T is equal to 0 in 2 dimensions. Which in 2 dimensions is 1 over r d by dr of r dT by dr is equal to 0. The temperature t end out to be C 1 log r plus C 2 and if, you write this in terms of the amount of heat coming out per unit time per unit length along the line. This becomes equal to Q by 2 pi k log r plus some constant C 2.

So, this is the amount of heat generated due to a 2 dimensional source. Logarithmic function does not really indicate some problem with the formulation. In 2 dimensions the solution for the Laplace equation is the fundamental solution is the log function itself. And the of course with the log function one cannot really satisfy the boundary conditions at infinity. At some point one has to recognize that the object is actually a finite length. In order to find out, what is the order to match the temperature fields in any case? So, this log function is a solution of the equation and if we are sufficiently close to the wire. If the distance from the wire is small compared to the extent of the wire itself, this appears to me to be conduction from a wire of infinite length in 3 dimensions or a point source of energy in 2 dimensions. The solution for that is a logarithmic function.

So, therefore, this example illustrates how one can use the method of delta functions, in order to solve problems for objects of finite dimension. So, with this is an illustrative example, of the use delta functions. It also turns out that the delta functions solutions can be related to the spherical harmonic expansion that we studied, when we did separation of variables in spherical coordinates.

So, in the next lecture, I will start with that how does one relate these delta function solutions to the spherical harmonic expansions that we did in. When we look at spherical coordinates and then we will look at general methods of using the delta functions to actually solve problems in more complicated geometries. So, the delta function have a physical meaning which is more than just an idealization for spherical systems. That is the reason why spherical system coordinates are so important. We spent a lot of time on this on both the delta function solutions as well as the solutions in terms of spherical harmonic expansions.

And the reason is because these point sources can be used in various contacts not specifically for a given problem, but in any problem that you have. If you can write down the source or sinks in some of the delta functions. That is an easy way to formulate the solution. So, next lecture, we will start on the relation between delta functions and the spherical harmonic expansions wil see you then.