Fundamentals of Transport Processes
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Lecture No. # 22
Unidirectional Transport Cylindrical Co-ordinates - VII (Oscillatory Flow in a Pipe Singular Perturbation Expansion)

So, welcome to this lecture 22, Fundamentals of Transport Processes, where we were looking at ways of getting approximate solutions for the oscillatory flow in a pipe. These solution procedures will be used commonly afterwards for various kinds of other problems. I am using this to basically illustrate how it is done so that later on, we can use similar procedures for solving problems in an approximate manner.

One of the objectives of this is to show how one can obtain approximate solutions and more importantly, how one can obtain physical insight while trying to solve a problem in transport phenomena.

Very often, problems are too complex to be solved exactly. In fact, the amount of effort you need for an exact solution may be many times larger, (( )) magnitude larger, then the amount of the effort you require for getting an approximate solution, which is sufficient for practical purposes.

Many of the correlations that we saw in the beginning for the dimensionless fluxes as a function of the Reynolds number, the Schmidt number and the Prandtl number; these are obtained from approximate solutions and some physical insight into the competition between convection and diffusion.

And in this particular problem, I was trying to illustrate to you how approximate solutions can be obtained; it depending upon the relative magnitudes of the inertial and the diffusive terms, inertial and the viscous terms in the momentum balance equation. The problem we were solving was an oscillatory flow in a pipe.

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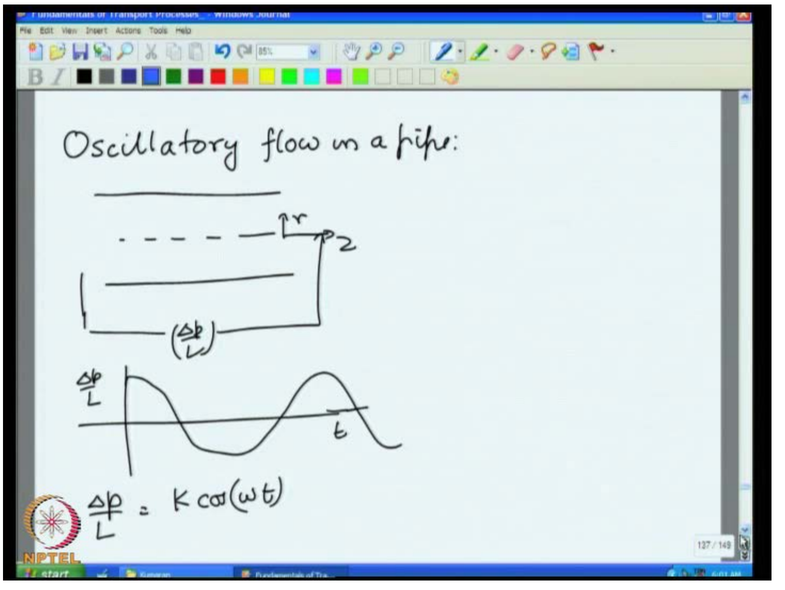

And in this particular problem, the flow is being driven by a pressure gradient which is oscillatory in time.

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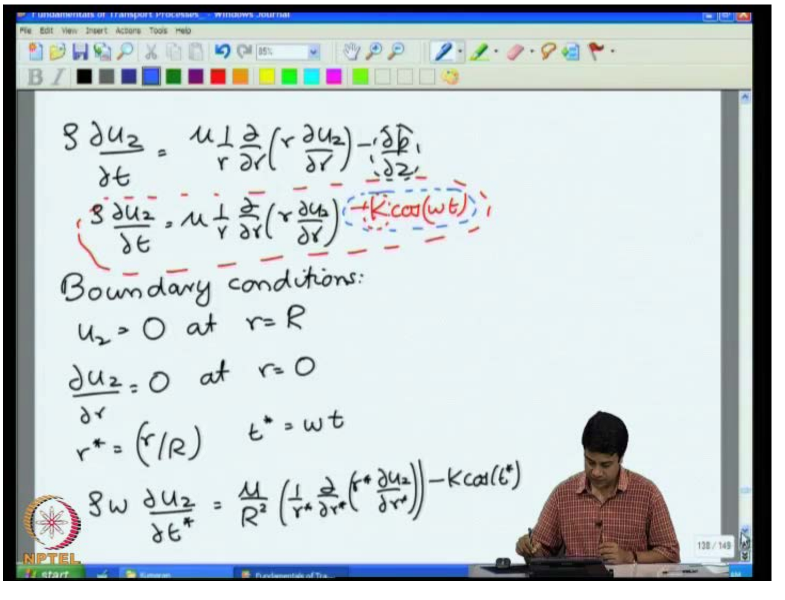

And we derived the momentum balance equation for this using shell balances. And that contains a pressure gradient term, which is versus k times cos of omega t. And we went ahead and tried to solve this using the same procedure we had adopted earlier for the flow pass for the velocity field near an oscillating plate.

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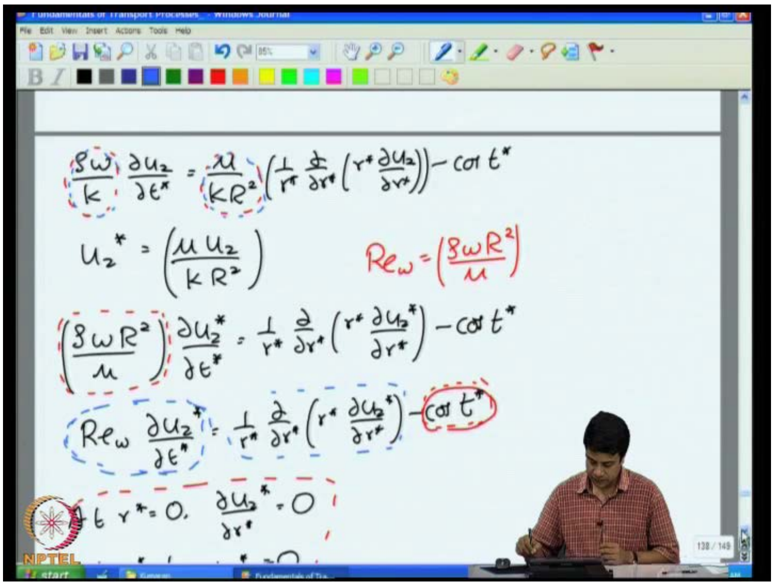

So, in that case, we write down u z for the scaled velocity u z. In this case it was scaled by the viscous scales.

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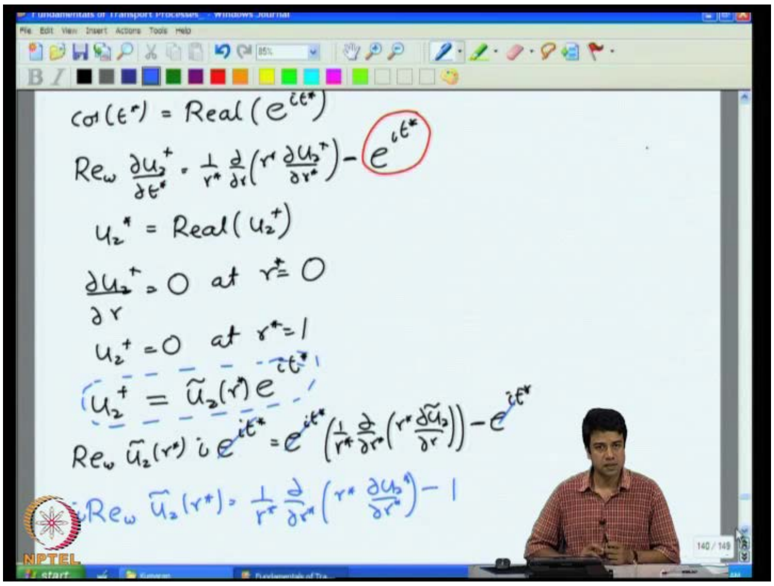

We wrote that cos of t as the real part of e power i t, and we defined the complex velocity u z plus in such a way that u z star is the real part of u z plus. And we wrote an equation for u z plus which contains e power i t as the inhomogeneous, the forcing term.

And of course, we know that, if linear system, in this particular case, the conservation equation is linear in the velocity, if its being forced by a forcing function that is sinusoidal in time with particular frequency, then the response also has the same frequency as the forcing. That was the principle we used to write down u z plus, as u z tilde times e power i t. So, we wrote u z plus as u z tilde times e power i t, and then we tried to obtain a solution for u z plus.

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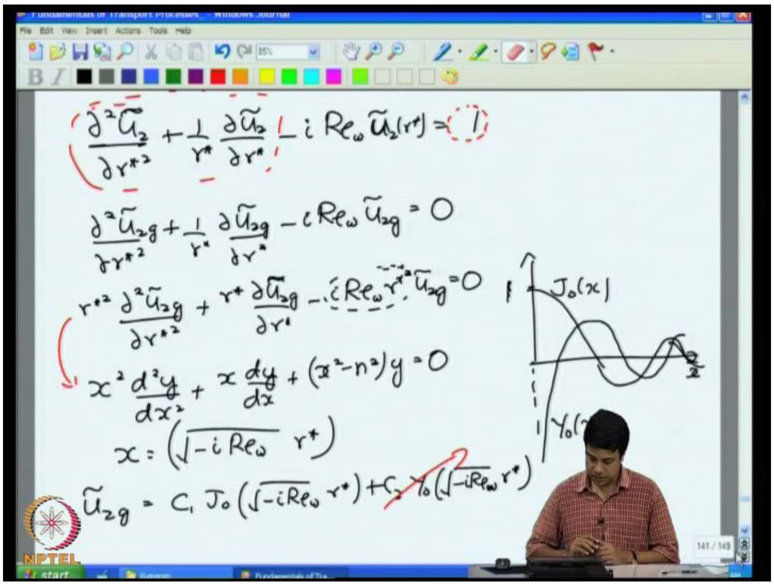

In this case, we were successful. We could write the solution as the sum of a general solution and a particular integral. The general solution had the form of a Bessel function, the particular integral; we could evaluate as just a constant.

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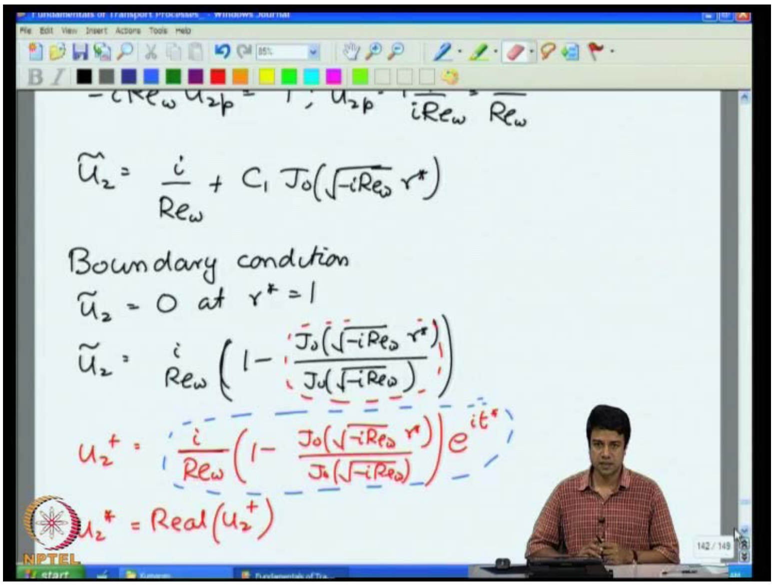

And on this basis, we managed to get a solution, which satisfied the boundary condition that u z is equal to 0 at the wall of the pipe, and the symmetry condition at the center; r is equal to 0, that u z the derivative with respective to z of u z is equal to 0 at r is equal to 0.We managed to satisfy both of these boundary conditions, get a solution in terms of Bessel functions.

However, this is a complicated solution; does not give us much insight into the physical processes within the system. In order to get more physical insight, we decided to look at the limits of low and high Reynolds numbers. Reynolds numbers is the ratio of inertia and viscosity. So, low Reynolds number limit implies that inertial effects are negligible.

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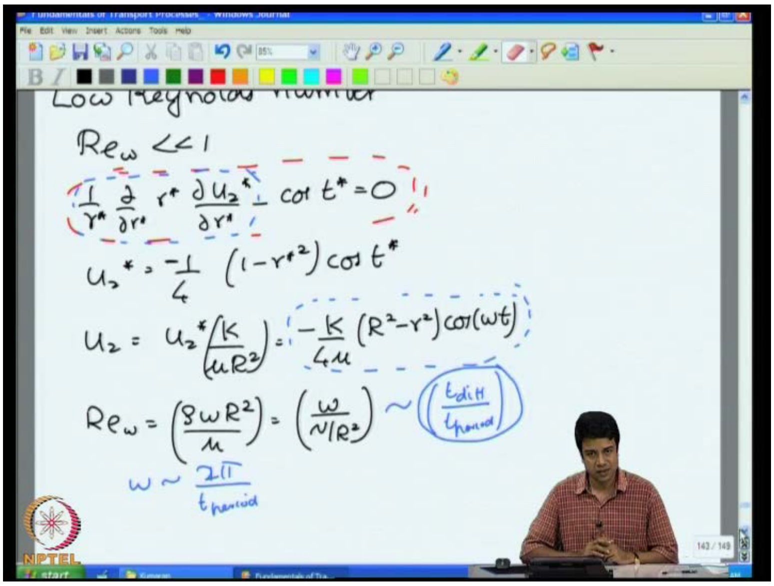

That is, the frequency is small or the time period is large. And in that case, we got a solution, which was identical to the solution for a steady flow, for a steady pressure gradient, except that the pressure gradient in this case is the instantaneous value of the pressure gradient at one particular instant in time. So, if I just take substitute k cos omega t instead of the study pressure gradient d p by d x d z in the Hagen-Poiseuille flow, the parabolic flow for a pipe, I get this particular velocity profile. And I explained to you physical interpretation of this.

Reynolds numbers being small implies that the frequency is small compared to nu by r square. Frequency is 1 over the time period; 1 over the 2 pi divided by the period of oscillation. R square by nu is the time it takes for diffusion over a length comparable to capital R. Nu is the kinematic viscosity or the momentum diffusivity with dimensions of length square per time. Therefore, r square by nu is approximately the time it will take for momentum to diffuse over a length comparable to r.

And therefore, when the Reynolds number is small compared to 1, it implies that the time it takes for momentum to diffuse across the pipe radius is small compared to the time period of oscillation. Therefore, momentum diffusion is instantaneous compared to rate of change of the pressure gradient, and therefore, the velocity field at any instant in time responds as it would to a steady value of the pressure; that steady value is equal to the instantaneous value of the pressure at that particular instant of time.

So, this was when we completely neglected inertial terms. But one can also examine what effect the inertial term has on this solution. That procedure we saw in the last lecture called regular perturbation expansion.

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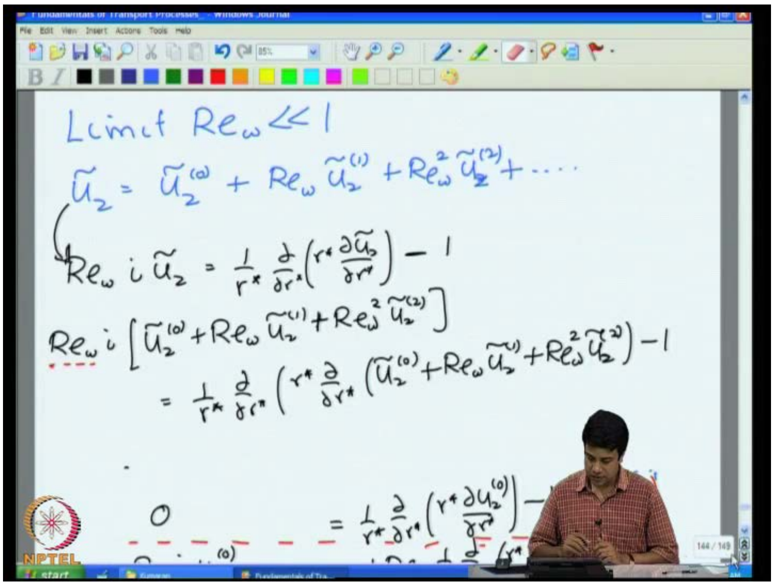

I am considering the limit R e omega small compared to 1; that means that I am considering the limit in the limit R e omega goes to 0. So, as you make R e omega smaller and smaller, I expand the velocity u z in as a series in R e omega; this is the term that you would have if R e omega were identically equal to are 0. All other terms in the series would be identically equal to 0. However, in the limit is R e omega goes to 0; this can be correction to this due to inertial effects. That correction has to be proportional to R e omega itself.

So, there is a correction proportional to R e omega and then there are higher order corrections. So, I can always expand it in this series. I have the advantage that in the limit as R e omega goes to 0; each successive term in this series is small compared to the previous term. So, that is the advantage in the limit of R e omega going to 0. I can of course always do this expansion. There is no problem.

But in the limit of R e omega going to 0, I have the additional advantage that each successive term in this series is small compared to the preceding term. So, I take the series, put into the governing equation and then expand the governing equation in a series in R e omega.

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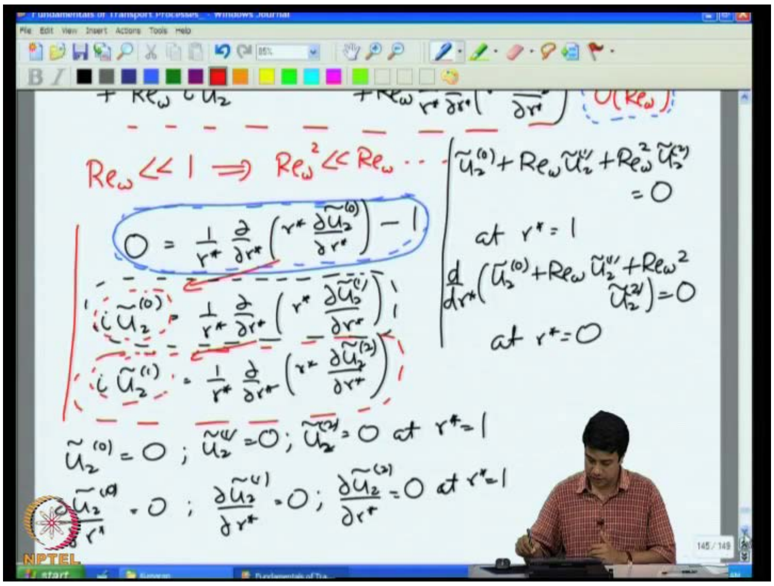

So, I will get an equation of order 1, of order R e omega, of order R e omega, order R e omega square. So, I will get a successive a series of equations. That some of all of those equations has to be satisfied, but in the limit of R e omega going to zero, the additional advantage is that each individual equation in itself has to be satisfied. The order 1 equation has to be satisfied, the order R e equation has to be satisfied, the order R e omega square equation has to be satisfied because if it is not, in the limit is R e omega goes to 0, the order 1 equation is the largest. The order R e omega equation is smaller than that. So, therefore, this leading order equation has to be satisfied.

Once that is satisfied, I can remove this equation, if this identically satisfied. Then I have a series which contains R e omega, R e omega square and so on. That means, the next higher order equation also has to be satisfied, because it is much larger than the succeeding one.

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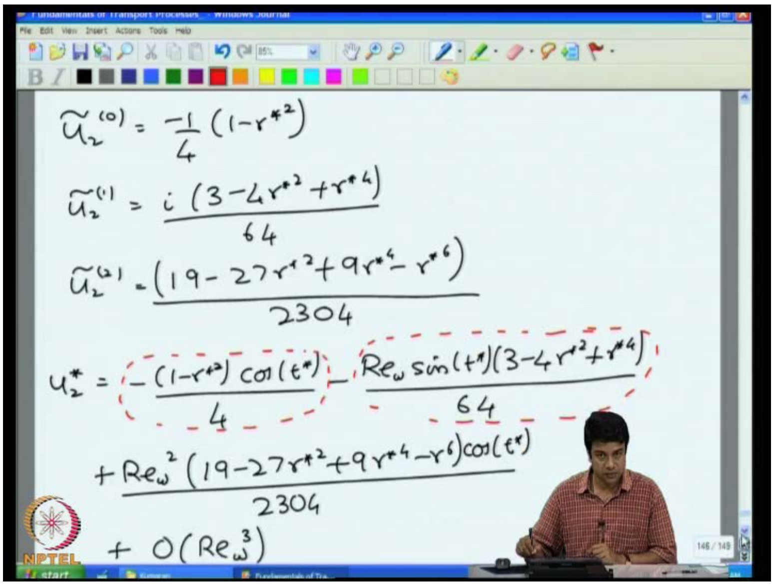

In a similar manner, we expanded the boundary conditions as well. And you end up with the series of equations; the first of which is identical to what I had for R e omega identically equal to 0. With that, I can obtain this leading order solution which we have already obtained before. But that leading order solution appears as inhomogeneous term in the next equation. So, with that I can obtain first correction. The first correction appears an as an inhomogeneous term in the second equation. So, that I can obtain the second correction and I can do that to whatever order that I want. So, that is basically the advantage of doing this regular perturbation expansion.

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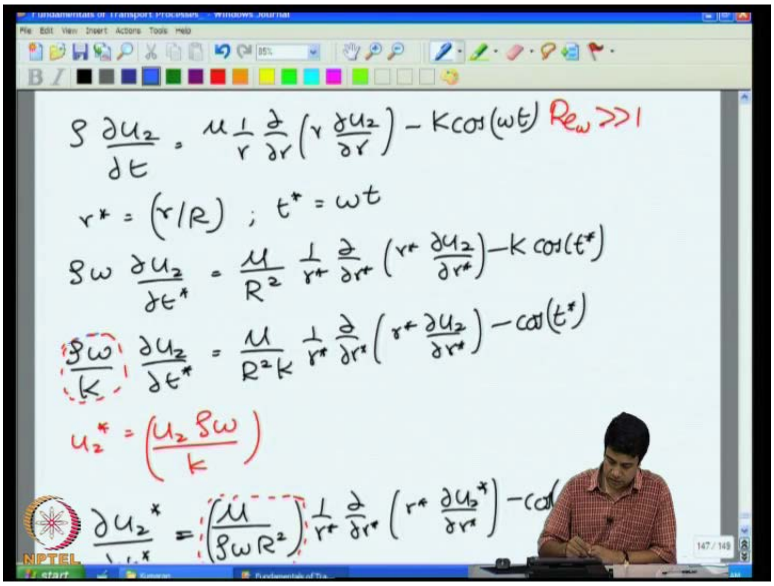

And I wrote down for you the solutions that we got the last time. And the solution that we get for the final velocity field; the first term is just the solution we got at R e omega equal to 0 proportional to cos t, a parabolic velocity profile. The first correction proportional to R e omega was proportional to sin t. So, it was exactly pi by 2 out of phase from the pressure gradient. And then you have a solution for proportional R e omega square and so on. You have higher and higher order terms. So, this was a regular perturbation expansion. (No audio from 11.31 to 11.45)

Then we looked at the opposite limit. We looked at the opposite limit, where R e omega is large compared to1 (( )) the opposite limit where R e omega is large compared to 1. In that case, I have to scale my velocity by the inertial scale.

I have to scale my velocity by the inertial scale. The scaling for the radius and the scaling for time remain the same because r and t are the only length scales in the problem, length and time scales in the problem.

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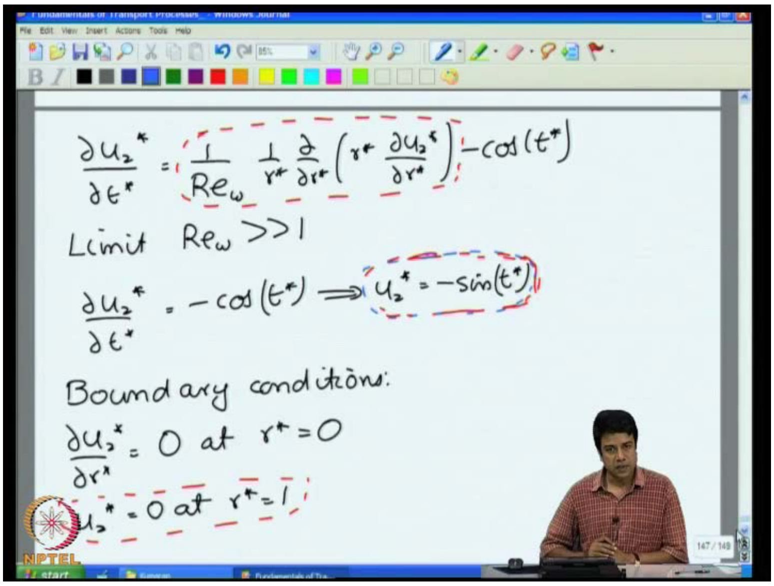

So, once I do that, I get an equation in which there is a viscous term which is proportional to 1 over R e omega, then there is a steady term and pressure and the imposed pressure as well as the inertial term. Since I have scaled velocity by inertial scales, the inertial term is now order 1. Note that R e omega is a large number. Therefore, 1 over R e omega is a small number. The inertial term is order 1. By default, the pressure gradient has to be order 1 because that is what is driving the flow. And the viscous term now is small compared to 1 because it has a factor 1 over R e omega in front of it. So, therefore the viscous term is small compared to 1.

We tried to solve this simplistically by just going ahead and neglecting the small contribution. The small contribution proportional to 1 over R e omega; we just went ahead, neglected it and tried to see if we can get a solution. And we did get a solution. It is quite simple really if the velocity was equal to minus of sin t. However, this solution had to satisfy boundary conditions. The derivative of the velocity is equal to zero at the center, at the access, at r is equal to 0; that is satisfied in this case because the solution is independent of r. So, that is identically satisfied in this case.

The second boundary condition was that u z is equal to 0 at r is equal to 1. u z is equal to 0 at r is equal to 1 at all instants of time because it has to be… the velocity at the wall is 0 at all times. And that we found out; there is no way that this solution can satisfy that boundary condition. So, how does one resolve this paradox? Where is the problem?

Mathematically, it is quite clear with where the problem is. The term proportional to 1 over R e omega actually contains the second derivative of u z with respect to r. Because of that, the original equation versus second order differential equation. Second order differential equation is completely specified if you with two boundary conditions and these were the two boundary conditions.

However, when we simplistically went ahead and neglected the viscous term, I converted this from a second order differential equation in r to an ordinary equation in r. It has no derivative with the respect to r. Because I converted the second order differential equation into just an ordinary equation, I cannot now satisfy boundary conditions in r. Because I have neglected the highest derivative, I am unable to satisfy boundary conditions. So mathematically, that is the problem. I have neglected the highest derivative. Therefore, I am unable to satisfy boundary conditions.

Physically, what is the problem? If you look at this particular system, physically the problem is that I have neglected viscous diffusion at the wall. The boundary condition that I am unable to satisfy is that the velocity goes to 0 at the wall. The velocity can go to 0 at the wall only if the wall is able to exert a shear stress on the fluid to decrease its velocity to 0 at that wall. So, the wall of the pipe has to be able to exert a shear stress on the fluid; which means that it is necessary for the momentum diffusion mechanism to operate very near the wall.

One is only by diffusion of momentum that is stress can be exerted on the fluid in order to reduce the velocity to 0 at the wall. When I made my approximation in the limit of R e large compared to 1, I have neglected momentum diffusion all together. And because of that, physically there is no way to for the velocity it to come back to 0 at the wall. So, because I have neglected momentum diffusion, there can be no shear stress exerted on the fluid and therefore, the velocity cannot come to 0 at the wall. So, physically that is the problem.

Whenever I neglect the diffusion term, convection transports mass momentum energy only along the flow direction. In order to transport mass, energy, momentum to a surface it is necessary to transport it perpendicular to the direction of flow. Transport perpendicular to the direction flow can happen only due to diffusion, and when I neglect it, the viscous term in this equation; I have neglected diffusion. So, there is no way for momentum to be transported perpendicular to the direction of flow and therefore, I cannot satisfy the momentum balance. I cannot satisfy the no slip condition at the surface.

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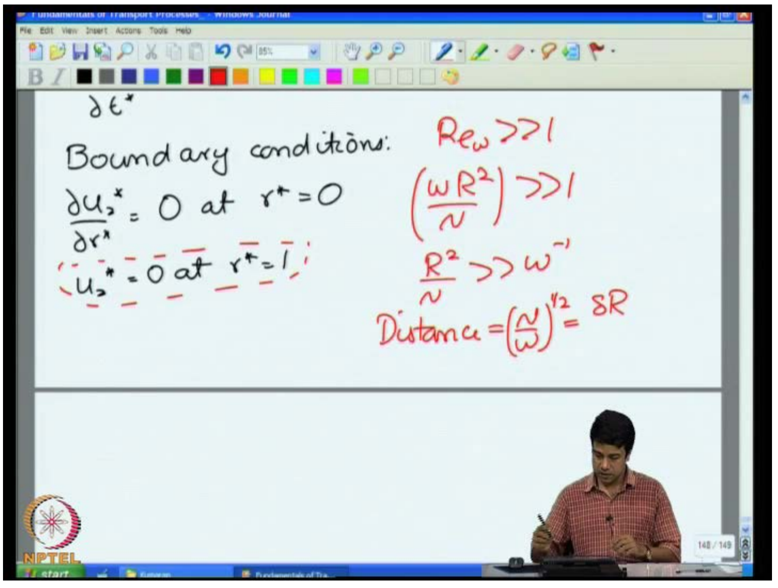

So, let us go and back and look at what is the meaning of the limit R e omega large compared to 1. ((No audio from 18:01 to 18:10)) R e omega large compared to 1; this implies that omega r square by nu is large compared to 1 or r square by nu is large compared to omega inverse. r square by nu is the time it takes for momentum to diffuse over a distance comparable to r. Omega inverse is the period of oscillation.

So, what this is telling you is that the period of oscillation is small compared to the time it takes for the momentum to diffuse a distance comparable to the radius of the pipe; that means, that within one oscillation cycle, there is not sufficient time for momentum to diffuse over a distance comparable to r itself. So, then what is the distance over which momentum would diffuse in this period.

So, this is basically saying that I have scaled my r coordinate with the radius of the pipe. When I scale my r coordinate to the radius of the pipe, I have automatically assumed that the time scale; the fluid time scale in a problem is the time it takes for momentum to diffuse over a distance r. Now the time period of oscillation is small compared to the time it takes for momentum to diffuse over a distance r, but; however, momentum diffusion mechanism is still operator, and the momentum is going to diffuse over a distance which is smaller than r.

And if I know what is that distance over which it is going to diffuse, and if I scale, my distance in the momentum conservation equation by that length over which diffuses, then diffusion and convection will be of the same magnitude. Then my equation will still be a second order differential equation and therefore, I will be able to satisfy boundary conditions.

So, the idea is that the distance that the momentum diffuses is small compared to r. In the limit of high Reynolds number, the flow itself is creating a smaller length scale. It is called an inner scale or a boundary layer thickness over which there is a balance between convection and diffusion. And the thickness of that a distance over which momentum diffusion takes place is determined from the condition that convection and diffusion have to be of same magnitude over this distance.

So, if that distance is small compared to r, then r is no longer a parameter in this problem. And it is quite easy to see what that distance should be; that distance; just from dimensional analysis because r is no longer a parameter in the problem, which distance has to go as nu by omega power half. This is the only possibility from dimensional analysis for the distance. It has to go as nu by omega power half which means that if I say that this distance is equal to some delta times the radius itself, delta is a small number.

So, if I say that this distance is equal to (( )) if I say that this distance is equal to some small number times r; number that is small compared to 1 times r; that means, that delta has got to be equal to nu by r square omega power half which is equal to R e omega power minus half.

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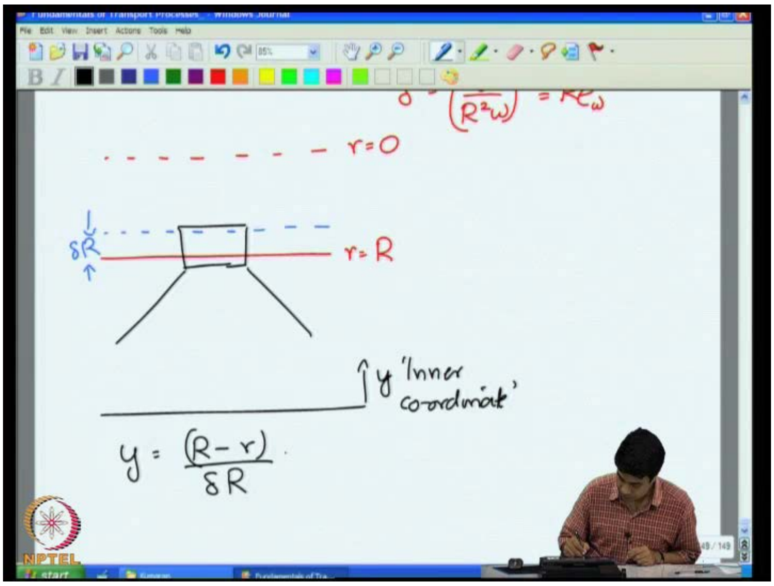

So, thickness over which momentum diffuses in the limit of high Reynolds number is R e omega power minus half times the radius of the pipe itself from the wall. So, if I have to scale my radius by this distance, then I should be able to get a balance between inertia and viscosity within that region near the wall. So, we look at the mathematical way to go about to doing this.

So, first of all, we have the wall of the pipe. This is the center of the pipe. So, this is r is equal to 0, r is equal to capital R, and we said that the momentum diffusion takes place only over a thin region of thickness delta. This thickness is delta times r. So, what I will do is I will focus on this region. I will focus on this particular region where I expect balance between inertia and viscosity.

So, I will magnify this region. And define a coordinate which is a coordinate from the wall. I will define a coordinate which is the coordinate from the wall; the distance from the wall. And I expect this distance to be of order delta times r; that means, that I can define an inner coordinate, this is y; I will call it as inner coordinate. y is defined as r minus r divided by delta times r, where delta some small number. I told you physically, we expect delta to be R e omega power minus half. Mathematically we will see how it comes about.

So, delta is a small number, and I will define delta in such a way that there is a balance between inertia and viscosity within a region of thickness order delta in the limit is R e omega goes to 0. So, delta is going to scale a sum power of R e omega. In the limit is R e omega goes to infinity, delta will scale as sum power of R e omega. That power will be determined from the requirement that within a region of thickness delta times r, you require that inertial and viscous terms are of the magnitude even as R e omega become larger and larger.

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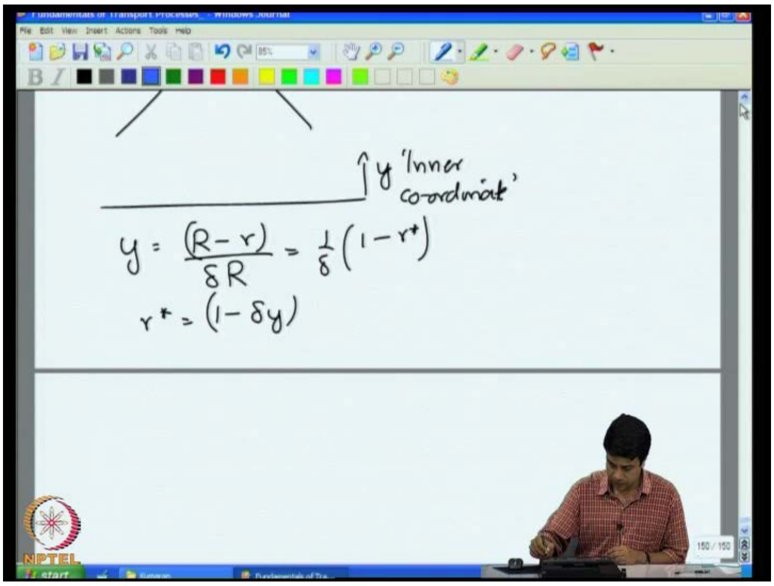

So, this I can write it as 1 over delta times 1 minus r star, where r star is equal to r by capital R. So, therefore, r star will be equal to 1 minus delta y, and this thing; I will put into my original equation, this I will put into my original equation which I had here

(Refer Slide Time: 12:22) and then determine delta from the condition that inertia and viscosity have to be of the same magnitude in this equation.

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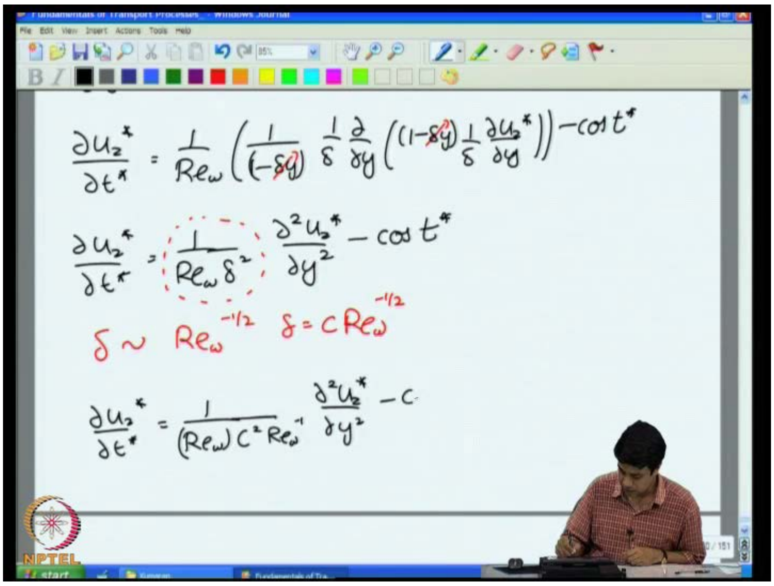

So, my governing equation is partial u z by partial t is equal to 1 over R e omega into 1 by r d by d r minus cos t. So, I substitute r star is equal to 1 minus delta y, and I will get d u z by d t is equal to into 1 by 1 minus delta y minus cos t. And you know that delta is a small number, y is a scale coordinate, y is order one in this magnified region. I defined y in such a way that it remains of order one in the limit as R e omega goes to infinity. And since delta is a small number, I can neglect delta y in comparison one over here. And I finally end up with an equation of the form d u z by d t is equal to 1 over R e omega delta square minus cos t. So, that is the equation that I end up.

This term basically gives me the ratio of the inertial and viscous effects within a region of thickness delta times r, where r is the pipe diameter at the wall of the pipe. And if inertial and viscous terms continue to be of importance of equal importance within a region of thickness delta, even as R e omega goes to infinity, then I require that delta has to be go as R e omega power minus half.

Precisely the same condition that I had got for you based upon physical insight earlier, based upon physical insight from the condition that only nu and omega the only parameters. I had postulated that deltas R e omega power minus half and that comes out mathematically over here. So, we know that delta is proportional to R e power minus half. What is the proportionality constant?

It turns out that the proportionality constant does not really matter. It will change; the dependence of u z on y, but once I express y back in terms of r, the solution will end up being independent of the value of the constant that is used. So, therefore, the simplest choice to use is that delta is actually equal to R e omega power minus half; however, in order to illustrate that the solution does not depend upon the choice, I will actually use a solution; that is, delta is equal to some constant times R e omega power minus half. Solve it with respect to this constant, and then show you that the final solution when expressed in terms of the scaled radius r is independence of the choice of the constant.

So, let us substitute that and we will get d u z by d t is equal to 1 by R e omega into c square R e omega inverse d square u z by d y square minus cos t. So, this thing is just equal to 1 over c square d u z by d y square minus cos t.

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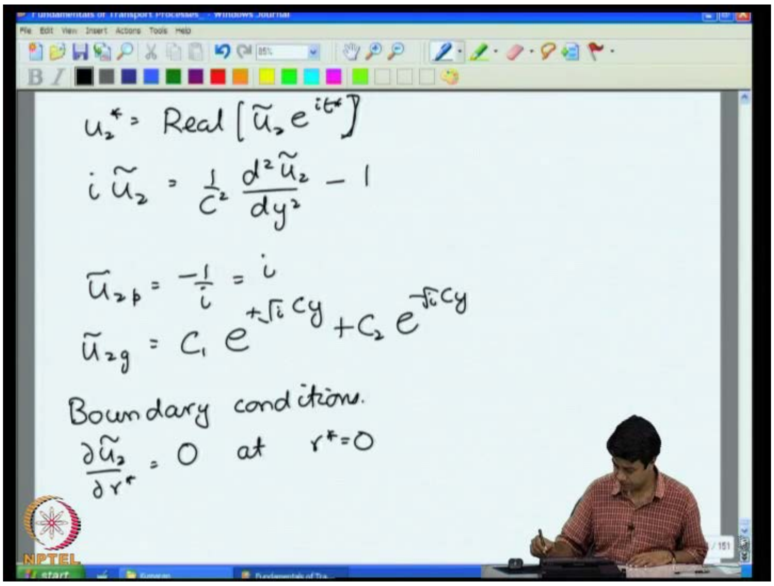

So, now, I will use the usual procedure for solving this equation. I define u z star is equal I am sorry is equal to the real part of u z tilde e power i t star. And once I express the equation in terms of this, the equation becomes i u z tilde is equal to 1 over c square d square u z tilde d y square minus 1.

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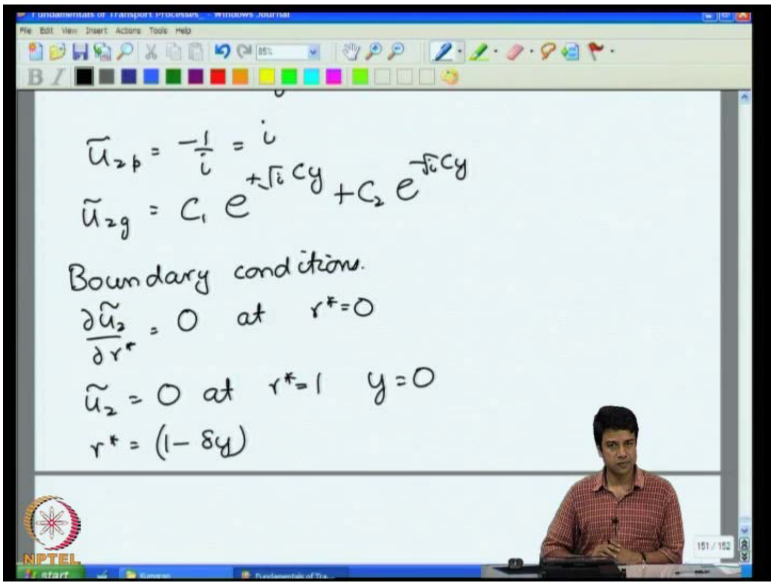

So, this is the equation that I get, and now I can go ahead and solve this equation. This equation once again has two solutions; one is a general solution and a particular solution. The particular solution is just u z particular is just a constant; minus 1 by i is equal to I itself. And the general solution is an exponential; is equal to c 1 times e power square root of i times c y plus c 2 e power minus root i c y . So, the particular solution the general solution is just an exponentially increasing and decreasing function. The particular solution just i, and I have boundary conditions to satisfy.

The boundary conditions are d u z by (( )) is equal to 0 at r star is equal to 0, and u z tilde is equal to 0 at r star is equal to 1. Note that r is equal to 1 minus delta y. So, therefore, r is equal to 1 is equivalent to y is equal to 0. r is equal to 1 is equivalent to y is equal to 0. r is equal to 0 is equivalent to y is equal to 1 over delta; delta goes as R e omega power minus half, In the limit as R e omega goes to infinity, delta goes as R e omega power minus half. Therefore, delta goes as goes to 0. Therefore, 1 by delta goes to infinity.

Therefore, I could as well formulate this boundary condition as d u z by d r is equal to 0 as y goes to infinity because I am considering the limit of R e omega going to 0, I am R e omega going to infinity. As R e omega goes to infinity, delta goes to 0 and therefore, 1 over delta goes to infinity. So, therefore, I could also formulate this as y going to infinity, the derivative has to be equal 0.

Clearly from this, if the derivative has to be equal to 0 as y goes to infinity, this constant c 1 has to be equals to be 0, because this function that it is multiplying is an exponentially increasing function. The function that this is multiplying is exponentially increasing function. So, therefore, if it goes to 0 as y goes to infinity; that means, that the constant c 1 has to go 0 because the particular solutions the constant. Its derivative is zero.

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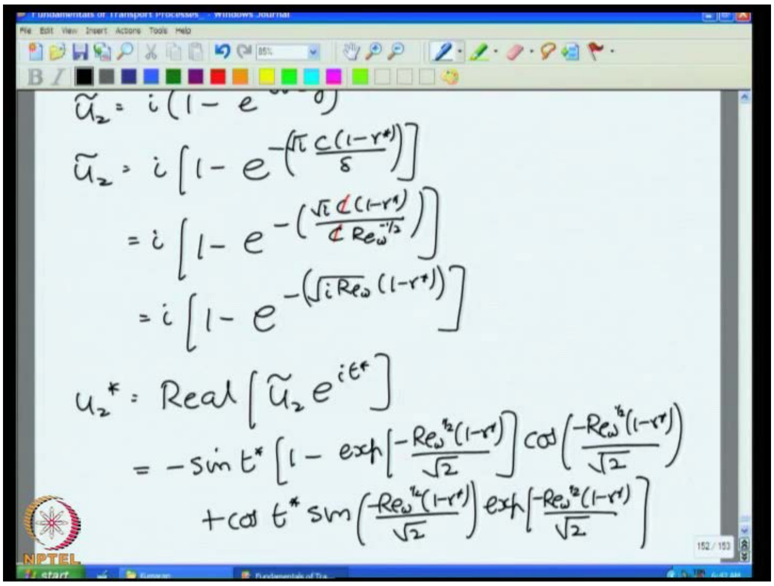

This the term multiplying c 2 goes to 0 as y goes to infinity because it is exponentially decreasing; that means, if the coefficient c 1 has to be equal to 0, the coefficient c 2 is determined from the condition that u z is equal to 0 at R equal to 1. And the final solution that I get for u z tilde is of the form i into 1 minus e power minus root i c times y. So, that is the final solution for the velocity field, and then I can get back the actual velocity. So, this is the complex velocity; is equal to real of e power i t star.

But before we do that, let us go back and look once again at this velocity field in terms of the coordinate r; the actual radius from… the distance from the access of the pipe rather than the scaled coordinate y which basically is scaled distance from the wall of the pipe, where viscous forces are important.

So, if I re-express u z tilde in terms of r, I will get u z tilde is equal to i into 1 minus e power minus root i c into y is 1 minus r star by delta. And delta we know is c times R e power minus half. Therefore, this will be equal to 1 minus e power minus root i c 1 minus r star by c R e omega power minus half. And you can see here that the constant c now cancels out, and therefore, I will get i into 1 minus e power minus square root of i R e omega 2 1 minus r star. So, therefore, even though the factor c was present when I had expressed the solution in terms y, y was the scaled coordinate. Once I expressed it back in terms of r star, the final solution is independent of the constant c.

Therefore, in this conversion, I can; without loss of generality, set the constant c equal to 1. There is no loss of generality when you set the constant c equal to 1. The constant c of course, does affect this equation which is expressed in terms of the scaled coordinate y, but; however, when I converted back into the original coordinate, I get back the exact same solution that I had before. So, therefore, c is just a scaling factor. It is basically tells me how much I magnify this region in order to look more closely at the region where there is a balance between inertial and viscous effects. So, that magnification basically will change, as I change c.

However, the final solution is the actual solution of the physical problem. The actual solution of the physical problem is not going to depend upon the magnification with which you look at it. It is the actual solution and therefore, that will not depend upon how I scale the inner coordinate or what value I use for c. The only requirement is that c has to be a constant so that the delta decreases proportional to R e power minus half as R e goes to infinity so that inertial and viscosity are of the same magnitude within the region that I have just magnified. So, the exact value c that I use does not really affect the final solution.

So, now, my velocity u z is going to be equal to real part of u z tilde times e power i t star and this is going to be equal to minus sin t star into 1 minus exponent of into cos of plus cos t star. (No audio from 40:45 to 41:22)

So, that is the final solution that I get. Note that my original solution that I had, was just minus sin t, which is just this term here; this first term here. This is the solution that I got for most of the channel. When I neglected the viscous term completely, I got a solution just as minus sin t ; however, as I get very close to the wall of the channel, when minus when this term; this factor minus R e omega power half times 1 minus r star is order 1 or the distance from the wall is proportional to R e omega power minus half; it is small compared to 1.

Then there is a correction and this correction basically makes the wall velocity go to 0 at the wall itself. You can verify that when r is equal to 0, the wall velocity, the velocity at the wall is identically equal to 0. And that correction is only in in a region of thickness R e omega power minus half near the wall of the channel. So, this gives me an oscillatory profile, for most of the channel, the flows is the solution is just constant; it is a plug flow, but; however, the viscous effects do become important when one reach the thickness of order nu by omega power half near the wall of the channel.

Over this region, viscous effects are important and there is the viscous and inertial terms are of the magnitude; that means that there is diffusion of momentum over this thickness from the wall of the channel. And due to the diffusion of momentum, the velocity at the wall comes down to zero as you approach the wall of the channel, and you are able to satisfy the boundary conditions basically because I have included viscous effects in this very thin region.

This thin region is actually called the boundary layer near the wall of the channel and the approach that I had used here is what is called (No audio from 43:40 to 43:55) singular perturbation expansion. (No audio from 43:57 to 44:11) Basically a singular perturbation expansion is used whenever I have a differential equation in which the small parameter is multiplying the highest derivative. The equation has a small parameter multiplying the highest derivative. In this case, the second derivative. So, simplistically, if I have to try to solve the problem, I would say look there is a small parameter there. Therefore, I can completely neglected that one, and therefore, tried to solve the rest of the equation to get a solution.

However, because there is a small parameter multiplying the highest derivative, the original equation had boundary conditions equal to the order of the highest derivative. In this case, the original equation had two boundary conditions because this was second order differential equation in the r coordinate; however, when I neglected that terms because there was a small parameter multiplying it, I have neglected the highest derivative and therefore, the equation has reduced to and ordinary equation in r and because of that, I am not able to satisfy boundary conditions.

So, the way to solve this problem is to realize that even though I have neglected the highest derivative, there is going to be thin reason where that derivative is important. The derivative basically represents the gradient of the velocity with respective to distance from some surface. And when I neglected that, I implicitly made the assumption that the length scale for the variation of the velocity was the pipe radius itself.

So, when I scaled r by capital R, I was making the implicitly assumption that the length scale for the variation of the velocity was the pipe radius itself. Because of that, it turned out there is the same parameter multiplying this equation. However, if the length scale for the variation of the velocity is not r, but some smaller value, the length scale is small; that means, the derivative is large; the gradient is large. So, if the derivative is small in such a way that the gradient is large, I could still have balance between that highest derivative and the other terms in the equation, if the variation is only over a small distance compared to what I had originally assumed as the radius of the pipe.

In that case, I can still get a balance between the highest derivative and all the other terms. And therefore, I have to rescale coordinate within that region where there is a balance because the distance is small and gradient is large. So, that was the inner scaling that I have talked about. I rescale the coordinate within that small region. Once I have done that, all the terms in the equation are of equal magnitude, I can solve the problem there.

Physically, the reason that we were not able to solve the problem was because we have neglected momentum diffusion. And if you neglect momentum diffusion completely, then there is no stress exerted by the fluid on the surface. And when there is no stress exerted, the fluid velocity cannot reduce to 0 at the wall of the pipe. So, because of that, I was not able to satisfy the boundary condition.

And as I explained to you, what the Reynolds number R e the Reynolds number greater than 1 limit was basically saying was that the length scale for momentum diffusion or the time scale for the momentum diffusion was small; was much larger than the period of oscillation. nu by r square or rather r square by nu was at time scale required for momentum to diffuse over a distance comparable to r. That was large compared to the time period of the oscillation; that means, that over a period of the oscillation, the momentum diffuses to over a distance which is small or much smaller than the radius of the pipe, and because it diffuses over a distance smaller than the radius of the pipe, the gradient; the velocity is very large near that near the surface of the pipe and because of this, the gradient is much larger than I anticipated earlier.

Earlier I had anticipated that the gradient would go as u z by r because I scaled my length by r, but; however, because the momentum has not diffused very far from the wall of the pipe over a time period of the oscillation, the distance is smaller; that means, gradient is the larger; that means, the viscous term is much larger than I what I anticipated in my simplistic argument.

So, I have to rescale my distance from the wall of the pipe by defining a scaled coordinate over which there is a balance between diffusion and inertia, between inertia and viscosity, in the limit as R e becomes large. As R e becomes larger and larger, you would expect this distance becomes smaller and smaller, but this distance, this momentum diffusion at the wall still exist, if the distance becomes smaller, but there is still diffusion at the wall and just by simple scaling, we saw that this distance goes as R e omega power minus half.

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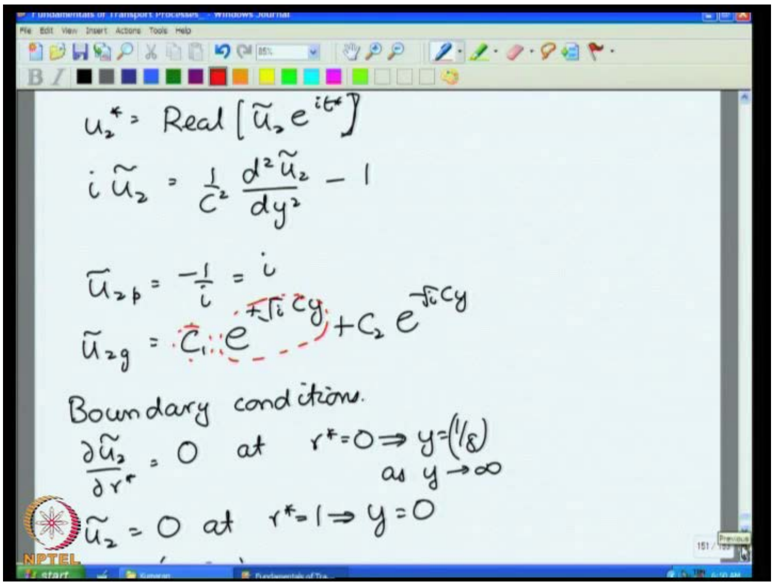

And once we have scaled it in terms of R e omega power minus half, we ended up with an equation in which all the terms were of equal magnitude.

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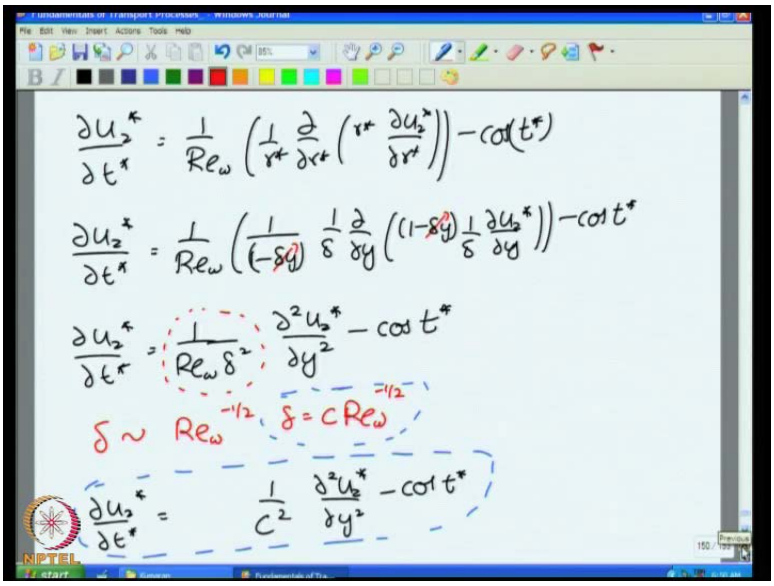

We ended up with this equation in which all the terms were of equal magnitude and this equation was quite is easy to solve. This equation had undetermined constant here c which basically gave me the ratio of delta and R e omega power minus half. As I said, delta is a boundary layer thickness which I insert into the problem in order to analyze it is the extent of magnification of the region near the wall that I am carrying out in order to see the velocity gradient near the wall.

So, this is thickness that is inserted in order for the purpose of analysis into the problem. And once I solve the problem, I showed you at the end that the final solution that I get actually does not depend upon this thickness because this constant actually cancels out in this equation. So, therefore, without lose of generality, I could just have used c is equal to 1 in the beginning, and that is the default option that we will use when we look at boundary layer problems. Just that c is equal to 1, go ahead and solve the problem; it is the most general solution.

And a final comment to make about this problem. (Refer Slide Time: 49:53) If you look at this equation, if you look this equation, this looks remarkably similar to the unsteady diffusion from a flat plate except for this inhomogeneous term here. Except for this inhomogeneous term here, this looks exactly like the equation that we had for the unsteady diffusion from a flat plate. In that case as well, I had d u z by d t is equal to the kinematic viscosity times d square u z by d z square. So, this, apart from the inhomogeneous term, this looks exactly like the solution for the flat plate.

So, if I solve the flat plate problem, I would get exactly the same solution except that I would not have this contribution due to this inhomogeneous term here fine, which means that I would not have this part, (Refer Slide Time: 35:42) I would not have this part which is due to the inhomogeneous term. Other than that, the rest of the solution would look exactly the same.

So, this is identical to the solution that I have for a flat plate except that I have this inhomogeneous forcing term. Apart from that, it looks exactly the same and the reason is as follows.

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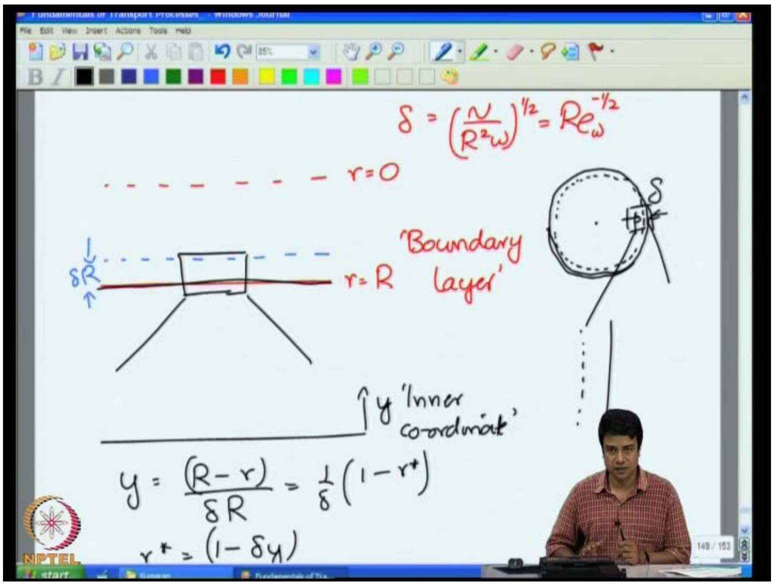

So, if I look at this boundary layer a solution here. So, what I am doing is I am concentrating on a thin region near the wall of thickness delta R e inverse. If I look at the pipe from this side, this is my pipe, I am looking at a thin region near the wall of the pipe. This has thickness delta. This has thickness delta. (No audio from 53:07 to 53:20)

If the thickness delta is small, if the thickness delta is a small compared to the radius, this region basically if I blow this up, this region up, this basically looks very much like a flat plate because the radius of the curvature is large compared to the thickness of the boundary layer. So, basically when this thickness is small compared to the radius, this region near the wall actually looks very much like the flow faster flat plate.

Instead of having flat plate that is oscillating, you have the flow in the bulk of the pipe that is oscillating where the flat plate itself is the stationary. So, therefore, the boundary layer solution that you get will be identical to the boundary layer solution for the flow faster flat plate. This is true for any configuration; no matter what the shape of the geometry is, if you go very close to the surface such that the radius of curvature is large compared to the thickness near the wall, the system will always look very much like a flat plate, and you can get the solution of the diffusion equation in a manner similar to the solution near of flat plate. And that the reason that these two solutions look remarkably similar.

So, this is my first introduction to advanced asymptotic techniques, regular perturbation analysis, singular perturbation analysis. These things; these techniques form the work horse of the analysis that we will do later in the limit, where convection is dominant as well as in the limit where diffusion is dominant. So, before we proceed to look at the conservation equation themselves, I still have a little bit more to do in unidirectional flows and that is to look at systems in spherical coordinate systems. This is another example of curvilinear coordinates, where the coordinate planes as I told you are not flat.

This is a useful for systems with the spherical symmetry such as spherical catalyst particle. If I want to analyze the diffusion from the surface, I prefer to have a coordinate system where the surface of the particle itself is one coordinate; is constant value of one coordinate; that is spherical coordinate system.

We will briefly look at that before we look at general conservation equations. So, this completes our discussion on the cylindrical coordinate systems with unidirectional transport. We will start spherical coordinate system in the next lecture. We will see you then. Thanks.