Fundamentals of Transport Processes
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Module No. # 04
Unidirectional Transport Cylindrical co operation - IV Steady Flow in a Pipe
Lecture No. # 19

So, welcome to lecture number 19, Fundamentals of Transport Processes, where we were looking at transport in curvilinear coordinates, where the coordinates are not straight parallel lines.

Initially, in the first part of our discussion on unidirectional flows, we had considered problems in Cartesian coordinates system, where the x, y and z axis are all perpendicular to each other. The unit vectors are constant; they are independent of position. And, which solved the equations for the flow fast flat plate, the temperature gradient due to a heated plate, both the steady conditions, unsteady conditions, with and without velocity fields and so on.

The problem with that is that, it is applicable only when you have surfaces that are planes, so that you can put a boundary at a coordinate, say z equal to 0. Very often, we come across systems where the surfaces, the bounding surfaces, are not planes. For example, the flow in a pipe, which we will see a little later, the surfaces is a cylinder. And we try to express the surface of a cylinder in Cartesian coordinates, we get an equation of the kind x square plus y square is equal to r square, where r is the radius of the pipe. And, it is very difficult to enforce boundary conditions on those surfaces, because they are not at constant values of the coordinates. Therefore, we look at other kinds of coordinates systems, where the bounding surfaces are at constant values of that coordinates.

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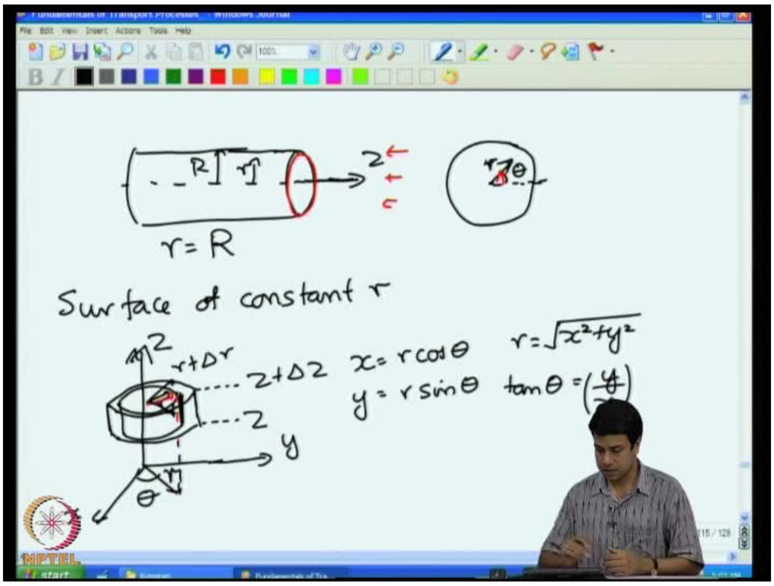

In the particular case of the flow around through a pipe, for example, you would consider a surface, where the surface of the pipe is at a fixed value of radius. In this particular case, this surface of the pipe is at r is equal to capital R, at a fixed value of the radius. So, I will take a coordinate system, where I have an axial coordinate, which is basically the z coordinate in this figure, which is the distance along the length of the pipe, that is one coordinate. The second coordinate is the distance from the axis of the pipe, that is, the r coordinate.

So, r basically tells you, how far a point within the pipe is from the axis, the perpendicular distance. And of course, there is a third coordinate, you need three coordinates to specify any position in three-dimensional space and that third coordinate is the angle that the radius vector makes with any particular axis, usually it is taken as the x axis.

The angle that the radius makes with the x axis is given by the coordinate theta. Note that, theta is dimension-less, r has dimensions of length, z has dimensions of the length. So, how do I relate r, theta and z to x, y and z? It is clear that along the x y plane, right, the distance made by any position to the x axis is given by the coordinate, is given by the radial coordinate. So, therefore, if I had some position on the surface, at a distance r from the origin, I just take the projection of that on to the x y plane.

The distance of that projection is r, the angle it makes with the x axis is theta. So, it is obvious that x is equal to r times cos theta, y is equal to r times sin theta. And, z is of course, is the same. The axis of the cylindrical coordinate system is identical to the z coordinate in a Cartesian coordinate system. The other way, r is the distance from the z axis, which is square root of x square plus y square. And, tan theta is equal to y by x. So, that is the relation between r, theta and x y. So, in order to analyze transport in a cylindrical coordinate system, we have to take a differential volume which is bounded by surfaces of constant coordinate.

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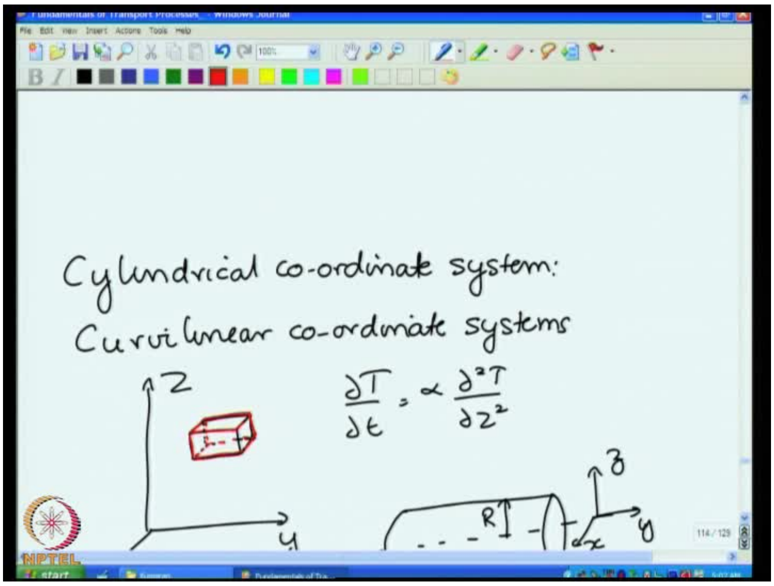

If you recall, when we did the analysis in a Cartesian coordinate system, we took surfaces to be planes, we took planar surfaces, we took planar surfaces. So, the front and the back were at constant values of x, the left and right were the constant values of y, the top and bottom were at constant values of z. In a similar case, in the Cartesian, in the cylindrical coordinate system we take 2 surfaces at constant r, r and r plus delta r.

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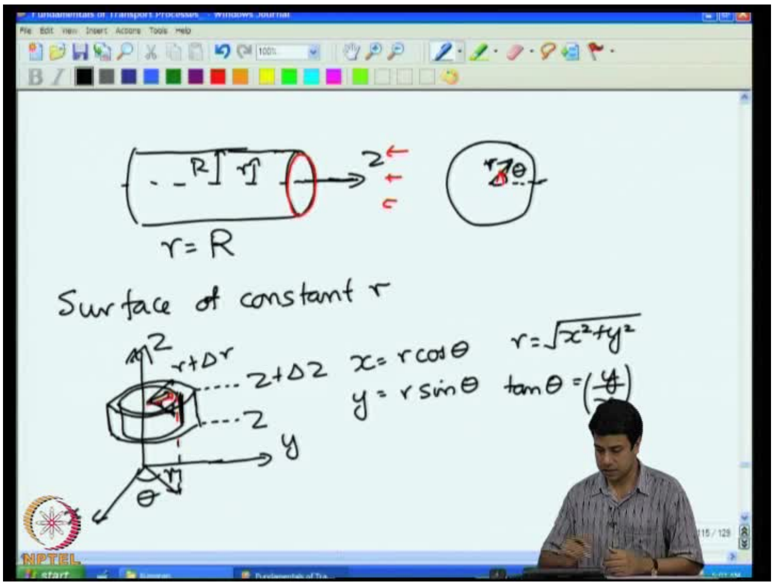

These are, of course, cylinders, cylindrical cross sections, one has radius r and other has the radius r plus delta r. And then, there are 2 surfaces at constant z, one at z, the other at z plus delta z. Therefore, in this case, the differential volume is an annulus of thickness delta r and height delta z.

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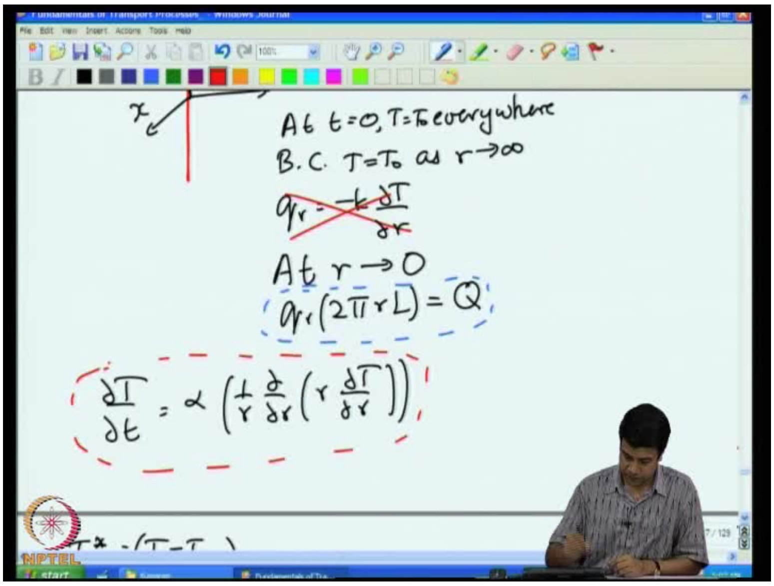

In this particular case, we have assumed that, there is no variation with respect to theta. Everything is the same as you go around the z axis, so long as your distance from the z axis remains the same. So, there is no variation with respect to theta and that is why, we could choose surfaces at just r and r plus delta r and z and z plus delta z. And once you do that, you get a differential equation for the conservation, for conservation of mass momentum and energy that has this particular form. The differential equation that you end up with has this particular form. This is, of course, in the absence of flow and not that this is different from what we got for a Cartesian coordinate system.

In that case, we got on the right hand side alpha times d square t by d z square. In this case, it is more complicated and as I explained in the previous class, the reason is because the surface area changing with radius. I have cylindrical surfaces.

Obviously, the surface area of that surface is going to depend upon the radius of the cylinder. That changes as I go from r to r plus delta r. So, even though the flux will be a constant, if the area changes, there is going to be change in the total amount of heat or mass flow. And that accounts, for this more complicated form of the differential equation.

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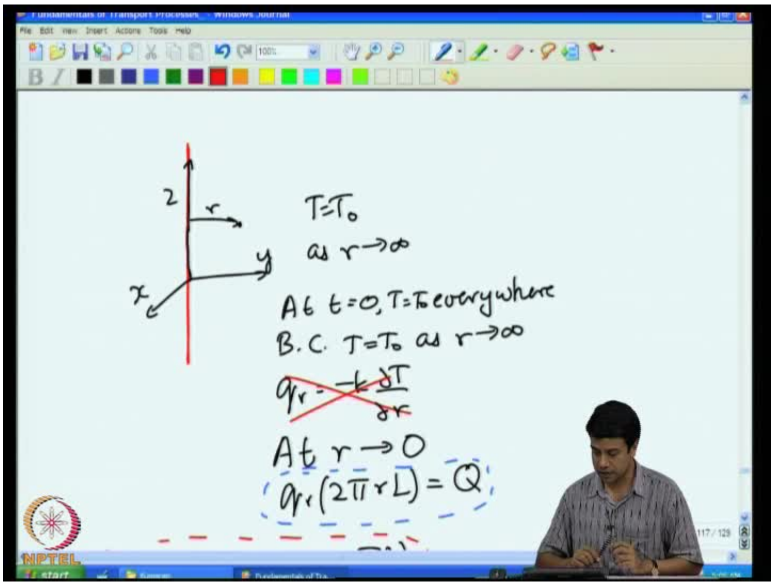

So, last 2 lectures, we looked at problems involving similarity solutions and separation of variables in cylindrical coordinate system. In a cylindrical coordinate system, we, we found that, for the separation of variables, we cannot impose a boundary condition on a surface. We have to impose it on a wire of infinitesimal thickness. You cannot enforce a temperature boundary condition on this. The boundary condition has to be of the form, total heat coming out of this wire is given by a particular value Q in this case, or the total mass coming out is given by a particular value, in the case of a mass transfer problem.

If the total heat coming out is a constant value, the wire is of infinitesimal thickness and therefore, the flux has to go to infinity, as the radius of the wire goes to 0. The flux goes to infinity, the temperature also goes to infinity, but the total amount of the heat coming out is a constant. So, that is the feature of these problems. This is called a line source in three dimensions. In, in the case of these lines source, you cannot specify the temperature at the source of the heat flux at the source, you have to specify total heat coming out of the source. And if you do that, you can then get a similarity solution. And the similarity solution has this form.

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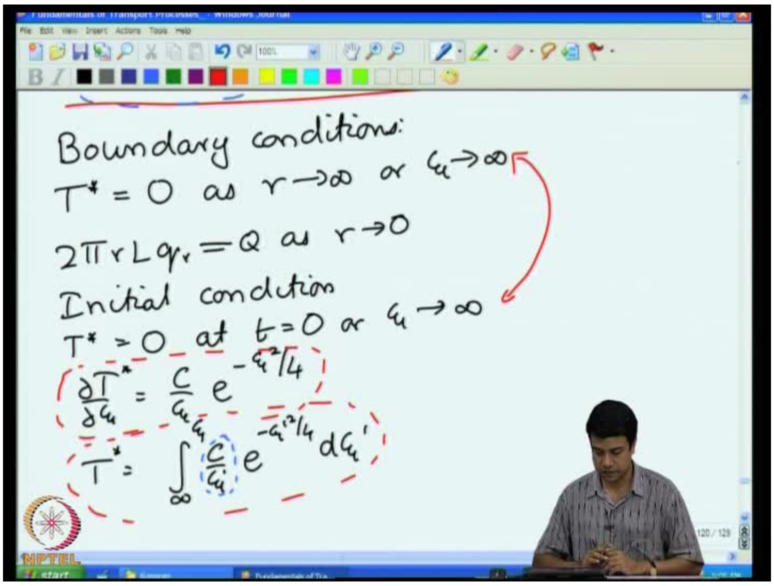

…where there is a constant in that, the constant c, we had determined from the condition the total amount of heat coming out of the surface. Note that, the temperature goes to infinity in this case, because I have a factor c by psi prime. Psi prime is the denominator. As psi prime goes to 0, I am integrating over a diversion quantity. So, the temperature actually goes to infinity, but never the less, the total amount of heat coming out is of course, a constant.

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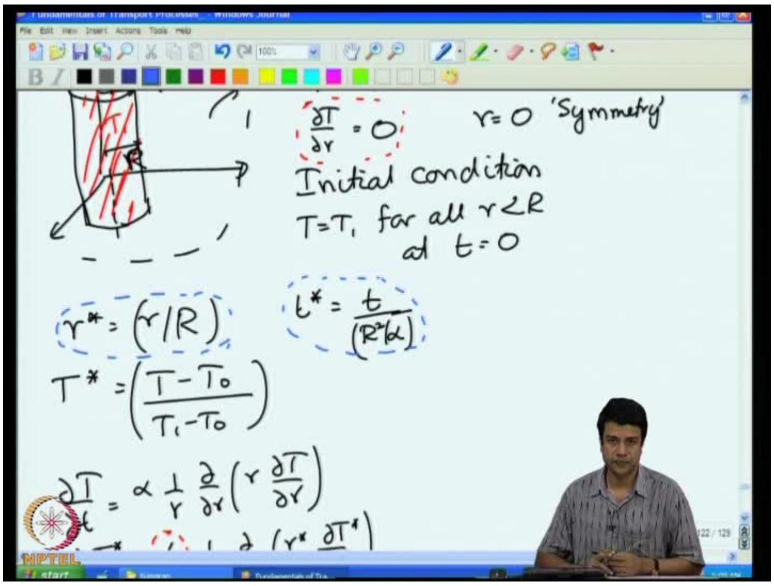

So, this is a limiting case, where the thickness of the wire is small compared to the other length scales in the problem. In that case, I can consider the wire as a line source. And then, we had looked at the heat transfer to finite cylinder. We had done this pretty quickly in the last lecture. So, I will briefly review this, before I go into the topic for the present lecture. What it is, is a cylinder. We will assume for the moment that, it is of infinite length in the z direction. It is finite only in the radial direction. The cylinder of temperature T 1 is dipped into a fluid which is originally at the temperature T naught. The fluid can be considered as an infinite path, so that, the temperature of the fluid does not change appreciably as the cylinder is cooling. We can consider that the mass of the, of the, of the heat bath is large compared to the mass of the cylinder, so, the decrease in temperature of the cylinder does not appreciably increase the temperature of the bath.

So, I have to solve this problem, the heat transfer problem, within the cylinder to see how it cools, as a function of time. Initially T is equal to T 1 everywhere within the cylinder. At time t is equal to 0, the cylinder is dipped into the bath. Therefore, at time t is equal to 0, the temperature on the surface of the cylinder is equal to T 0 and as time progresses, the cylinder cools.

So, in this case, since we have a cylinder of finite radius, one can scale the radius r, write the radius of the cylinder itself. And if you scale the radius r by the radius of the cylinder itself, the scaling for the temperature is, what we had got previously, I am sorry, the scaling for time is what we had got previously. The scaling for time is basically the time it takes for the diffusion to take place over distance of r, where alpha is the thermal diffusibility. One could equally well solve this problem for mass diffusion.

In that case, we assume that the concentration within the cylinder c 1 at time t is equal to 0, it is dipped into a bath with concentration c naught, at long times because of the diffusion from the cylinder to the bath, there is going to be a constant concentration everywhere.

So, we had got the thermal balance equation written in terms of the scale temperature, the scaled time and the scaled distance. And the boundary conditions, note that, this is second order differential equation in r, it is first order in time. So, you need 2 boundary conditions, 1 initial condition. This cylinder however, has only 1 boundary. That boundary is at r is equal to capital r. At that point, we require that the temperature is 0. The scale temperature has to be equal to 0 because, T is equal to T naught at r is equal to capital r. How about the other boundary? I have ((taught)) you based on physical considerations, ok… that the other boundary condition should be that, r star.

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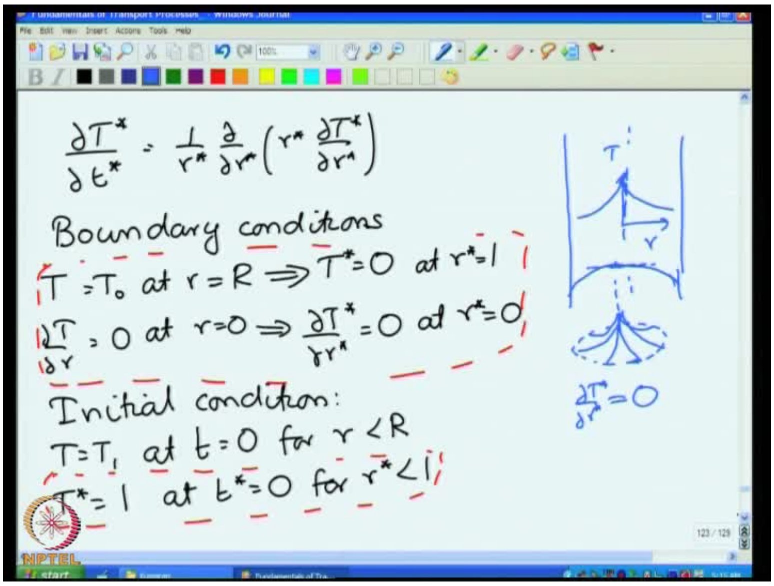

The other boundary condition should be that (()) r is equal to 0, d t by d r has to be equal to 0. Let us briefly review that argument. This is my cylinder. This is the centre and at functional distance r, if I plot the temperature. If I plot the temperature as a function of r… Let us assume that the temperature gradient is not 0 at the, along the axis, that means, that if I plot the temperature as a function of r, the temperature profile will look something like this. It could be, it could have a negative slope, it could have a positive slope. It depends on the situation, but let us just assume that, the slope is not 0.

Note that, there is nothing changing in the theta direction. So, as you go around the cylinder there should be no change in the temperature. So, that means, if I go on the left hand side, the temperature profile should be an exact mirror image of this one. It should look something like that. But, a temperature profile like this does not have a continuous slope at the axis itself because, from the left hand side, it comes up like this, the slope in this case is positive, from the right side it comes up like this, the slope is negative.

Similarly, as I go all the way around the cylinder, I will get different slopes at different locations. As I approach the origin from different positions around, if the field is axis-symmetric, I will get different slopes from each direction, different derivatives. D t by d r will be different for each location that I calculate the slope for. The only way that, the derivative can be single valued is, if it actually comes to 0 at the center. If it actually comes to 0 at the center, the slope is 0, no matter where I approach it from, ok.

So, the symmetry of the configuration itself dictates that, this d t by d r is equal to 0. Because, if it is not 0, you are getting different slopes from different directions, and the derivatives are different, that means, that the heat flux is equal to k times d t by d r. So, the flux is actually coming out different in different directions from the center in different locations. If I had a heat source at the origin of course, I could have a non-zero slope at the origin. Because, I could have heat flux coming out in all directions, but this is just a cylinder with no source in it. Therefore, the slope has to be 0 at the origin. So, that is, that is a symmetry boundary conditions.

It emerges from the fact that, the cylindrical coordinate system has, is axis symmetric, about, around the axis. As you go around the axis at different locations, you should get exactly the same temperature field. That implies the d t by d r has to be equal to 0, exactly at the axis. And, we had an initial condition that T is equal to T 1 at time t is equal to 0, for all values of r. That means T star is 1, at t, t is equal to, at the initial time t is equal to 0.

So, as I said, by, automatically, in this particular case, you get homogenous boundary conditions on both boundaries.

T star is equal to 0 at r is equal to 1, and the derivative is 0 at r is equal to 0. The only forcing is at initial time t star is equal to 0, when capital T star is equal to 1. That is the requirement, for our separation of variables procedure. As I told you, in separation of variables, in order to get discrete Eigen values, those were, Eigen values for beta n equal to n times phi, in the previous problem, to get discrete Eigen values, you require that the boundary conditions be homogenous and that then, there is a forcing at initial time. That, those properties are already satisfied by this, the solution of this equation.

So, we did the separation of variables of procedure, separated it out into a function of r, a function of T and we got differential equations for r and for T. So, this is the differential equation for r where beta has to be a constant. We took the constant as negative, because we require that theta has to go to 0 as T goes to infinity.

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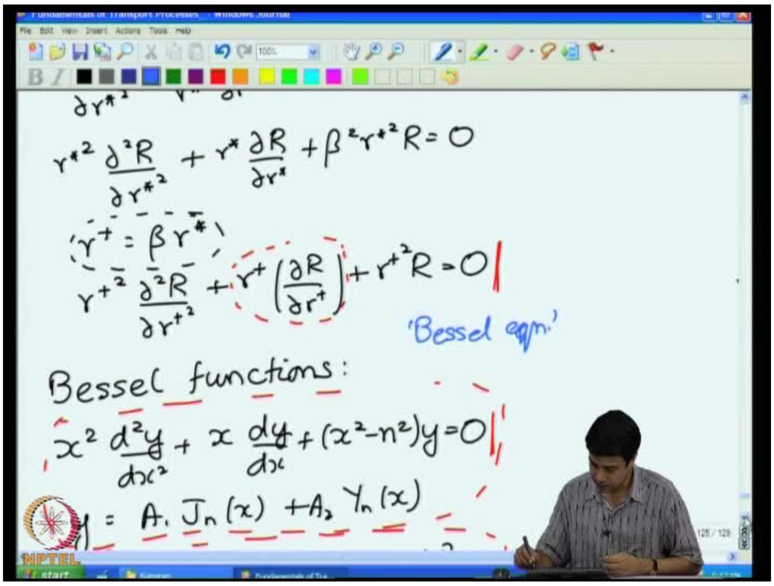

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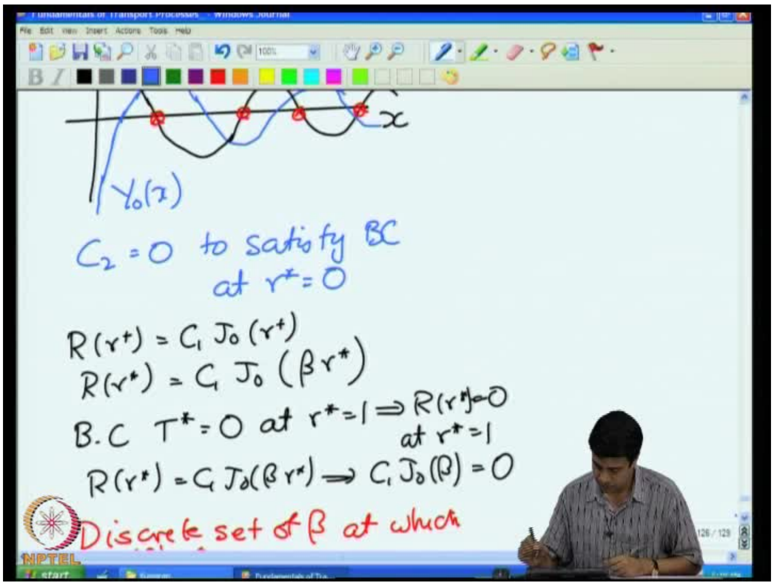

Only if the constant is negative, will theta decrease exponentially as t goes to infinity. And that negative constant was inserted into the equation for r and we got a differential equation, whose solution, this differential equation is called the Bessel equation, is called the Bessel equation and the solution of these are Bessel functions, J n and Y n. And in this particular case, for this particular equation, n is equal to 0. The solution for equation that we got are c 1 J naught of r and c 2 Y naught of beta, I am sorry, c 1 J naught of beta times of r and c 2 Y naught of beta times r. As I told you, these are similar to the sine and cosine functions. They are also oscillatory in nature, but they decay as x power minus half, as the argument becomes large, ok.

And at the, at r is equal to 0 itself, J naught is finite, Y naught goes to minus infinity. Therefore, the temperature has to be finite with 0 slope. The coefficients c 2 has to be equal to 0 and we just got the solution as c 1 times J naught of r plus. And then, we have the boundary conditions T star is equal to 0 at r is equal to 1. The J naught of course, satisfied the 0 slope condition, right at the origin. J naught satisfies the condition that the slope has to be equal to 0 right, at r is equal to 0, but we have another boundary condition which is that, r is equal to 0 at r star equals to 1 or c 1 times J naught of beta has to be equal to 0, ok.

So, c 1 times J naught of beta has to be equal to 0, that means, that beta has to have, it cannot have any value. It has to have a particular value in such a way that, this function J naught is equal to 0. There is a function, I am sorry, this is Y naught. So, this is the function J naught.

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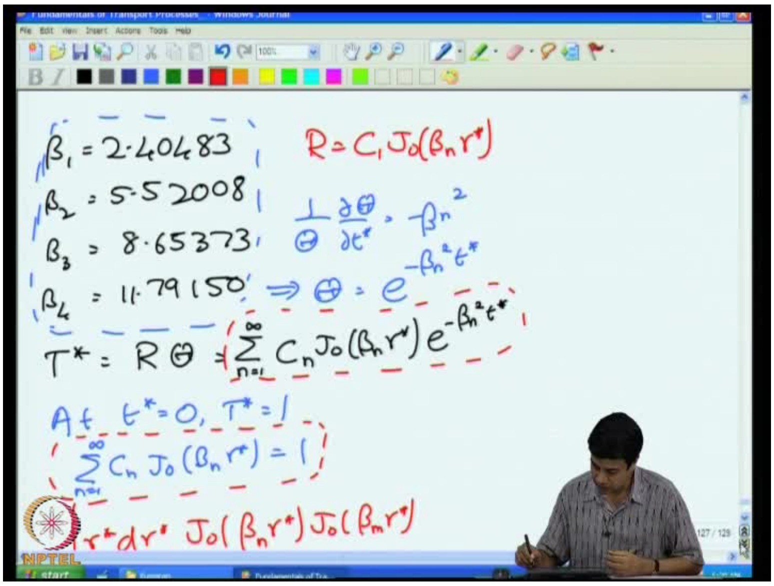

This function J naught has to be equal to 0. So, J naught of beta has to be equal to 0, that means, that beta has to have a set of discrete values. This is the analog of the values n phi that we had got for the flow past, for the transfer from a flat plate, when we did separation of variables in a channel between two plates separated by a distance edge distance edge. In that case, we got the Eigen values as n times phi. These discrete Eigen values are analogs of that, they are not exactly n times of phi and they are not exactly separated by equal distances, but they are still discrete values that satisfy J naught of x is equal to 0. And therefore, you can write the solution, I gave you the first few set of these solutions, beta 1, beta 2, beta 3, beta 4 and therefore, general solution can be written as the summation, over all of these discrete solutions of J naught of beta n times r times e power minus beta n squared times t.

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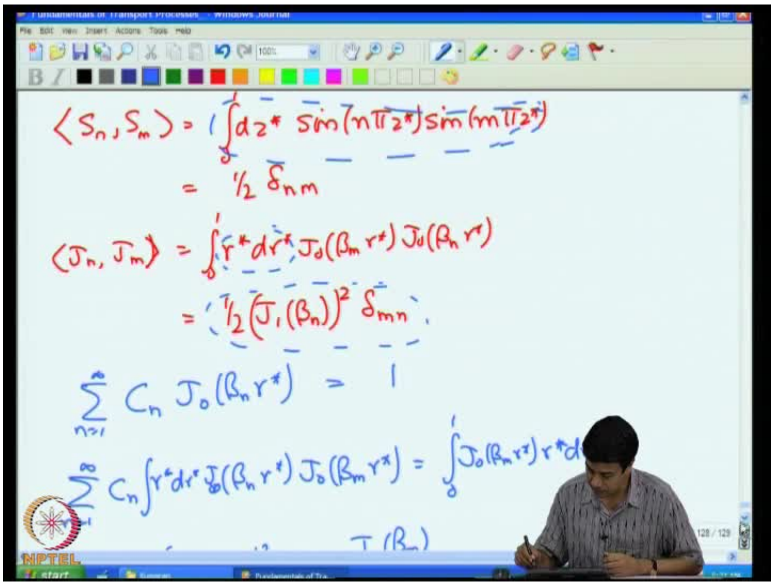

Note that, you have the same exponential decay that we had in the transport from the flat. The coefficient C n of course, has to be determined from the orthogonality conditions. The orthogonality conditions for the Bessel functions are slightly different from the orthogonality conditions for the Cartesian coordinate system, for the sine and cosine functions. In the case of sine and cosine functions, we had integral dz of the 2 functions which is equal to delta m n. In the case of the Bessel functions, you have integral of r dr times J naught of beta m r J naught beta n r. This is integral of r d r is once again a delta function. It has a non- zero amplitude, in this case, it is called J 1 where Bessel function of order one of beta n the whole square, ok.

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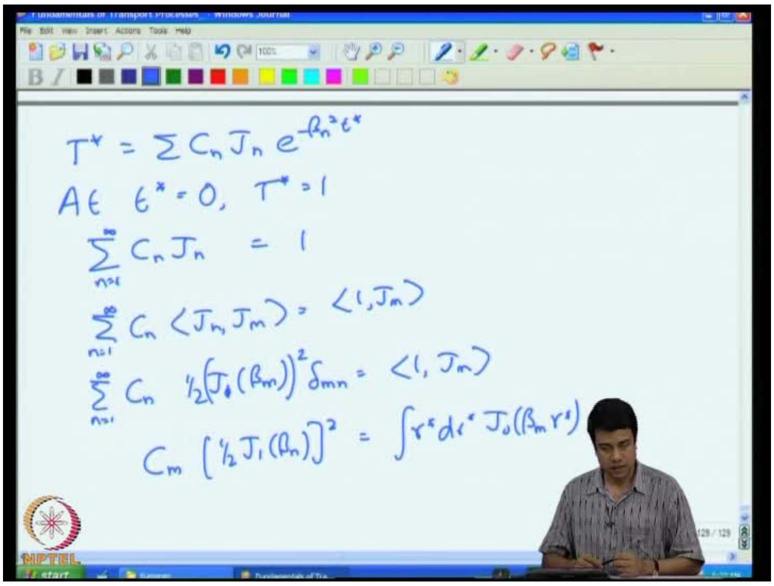

So, using this you can get back the solution, you can calculate each and every constant C n. So, I have the requirement that at t star is equal to 0, the temperature has to be 1. Therefore, summation C n times J naught of beta n r has to be equal to 1 and I enforce the orthogonality condition the usual way, multiplied by J naught of beta n r times J naught of beta m r and integrate from 0 to 1. This is equivalent to, in our previous case, we wrote that T star is equal to summation C n J n e power minus n minus beta n squared t star. At t star equal to 0, T star is equal to 1, which means the summation C n J n is equal to 1. How do I get the constant, I take the inner product. J n comma J m is equal to inner product of one comma J m. Therefore, this summation is equal to infinity. C n times of J 1of beta m the whole squared times delta m n is equal to 1 comma J m.

So, therefore, C n times of delta m n, delta m n is non-zero, only when n is equal to m. So, summation C n times delta m n, this is going to give me C n. So, I will get C m times half J 1of beta n whole squared is equal to this, which is integral r d r J naught of beta m r. From that, I can determine each and everyone of these coefficients. There is an infinite number of such coefficients, but it should be noted that the contributions to the higher order terms decays.

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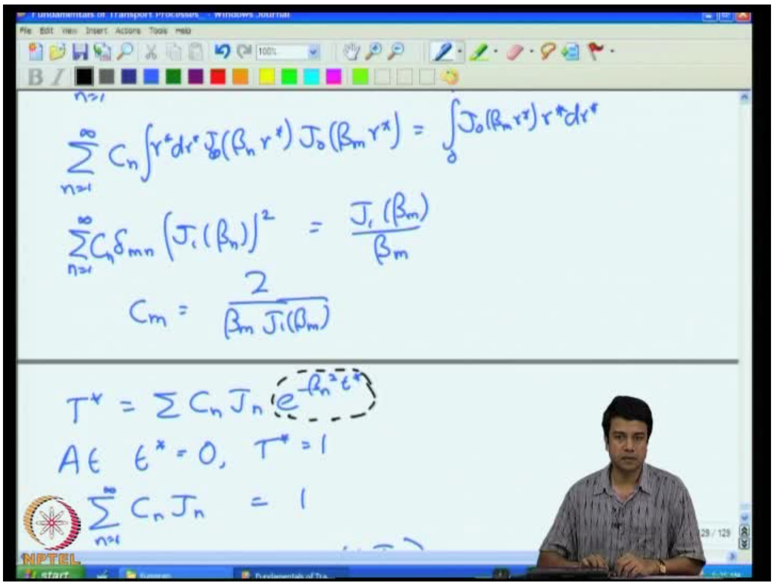

The contribution due to the higher order terms actually decays exponentially. Therefore, beta n increases, this decays exponentially in time. Therefore, like in the case of heat transfer from a flat plate, I can take a finite set of such coefficients, truncated at a particular value. That value will of course, depend upon the accuracy that I want for the series. The infinite series gives you the exact solution. You can get a good numerical approximation by truncating this solution at a finite set of Bessels functions.

So, the procedure is exactly parallel to the procedure for the flow past of flat plate, except at the differential operator is slightly, a more complicated, instead of d square by d z square that I had over there, I have one by r d by d r of r partial of capital r by partial r, and because of that, the solutions are not simple sine and cosine function, they tend out to be Bessel functions.

Never the less, the Bessel functions have all the orthogonality properties and other properties that we require, in order to beget, to, to obtain all of the coefficients in this exponential. So, this summarizes the solution of separation of variables in cylindrical coordinates. Many problems in cylindrical coordinates can be solved this way. They always have the operator 1 over r t by d r of r times d t by d r is equal to the ((unsteady term)) and because of that you will always get the solutions of the type of Bessel functions.

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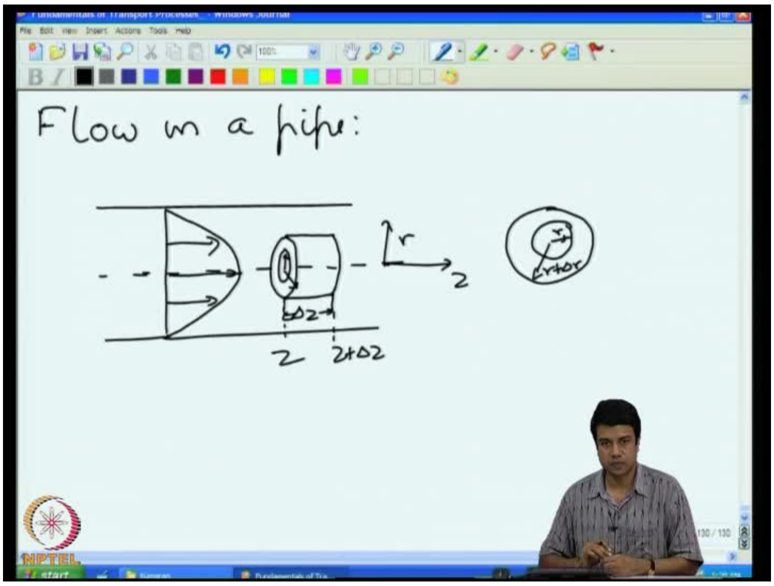

So, let us go on to the next problem, which is a technologically important problem, the flow in a pipe. So, in this case, if you have a cylindrical pipe, this is the axis of the pipe. You apply a pressure difference across the ends of the pipe. So, let us just call this as the z axis, this as the r axis, the radial axis. There is a pressure difference applied across the ends of the pipe and that results in a pressure gradient along the pipe. At each distance z, as you go downstream the pressure keeps decreasing.

And, that pressure gradient is responsible for the flow field. So, as you probably know, the velocity profile in the pipe is a parabolic velocity profile. This is the parabolic flow and our task is to use momentum balances to calculate this parabolic velocity profile. So, this is a problem in momentum transfer, pressure difference applied across the pipe generates a flow through the pipe.

So, how do we solve this problem? As usual, we take a cylindrical shell, a cylindrical shell of inner radius r and outer radius r plus delta r. So, if I look at it from the side, I will see something like this. This is the inner radius r, this is the outer radius r plus delta r, and of course, this distance delta z here, between z and z plus delta z. Delta z is the distance along the axis, delta r is a difference between… So, this is delta z, the distance along the axis between 2 locations z and z plus delta z and this is the annulus between 2 locations, r and r plus delta r.

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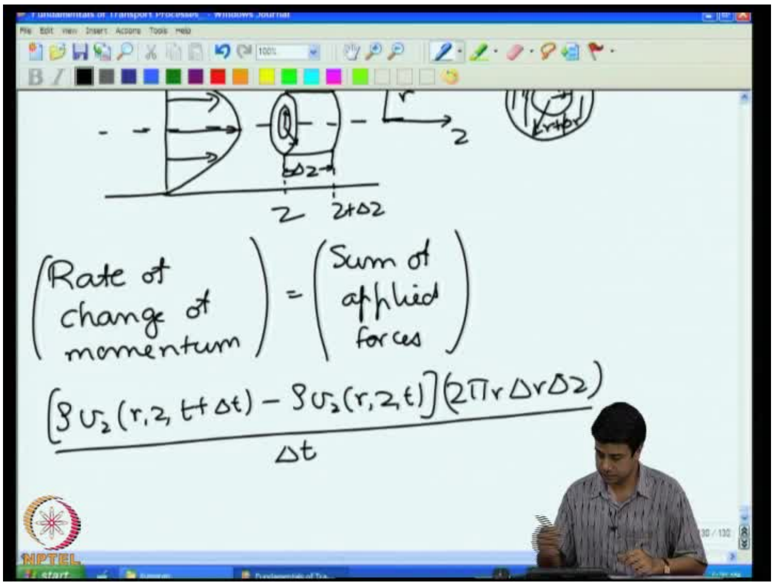

So, we will write the momentum balance equation for the cylindrical shell. The velocity is only in the z direction. There is no velocity perpendicular to the wall of the pipe. So, the velocity in this case, is only along the z direction. So, we will write the momentum balance equation for the velocity in the z direction. In other words, we are writing momentum balance equation for the z component of the momentum. Momentum balance. Rate of change of momentum is equal to sum of applied forces. How to say rate of change of momentum within a time interval delta t? Momentum in this volume is going to be equal to the density times the velocity times the volume itself. Mass times volume. Mass is equal to density times the volume itself.

So, therefore, the rate of change of momentum is going to be equal to rho times v z at r z t plus delta t minus rho v z r z t times the volume, volume in this case is the volume of the cylindrical shell. So, this is equal to this cylindrical cross section area times the distance, 2 pi r delta r delta z. So, this is the change in the momentum, within a time interval delta t. The rate of change of momentum is this whole thing divided by delta t itself. So, that is the rate of change of momentum.

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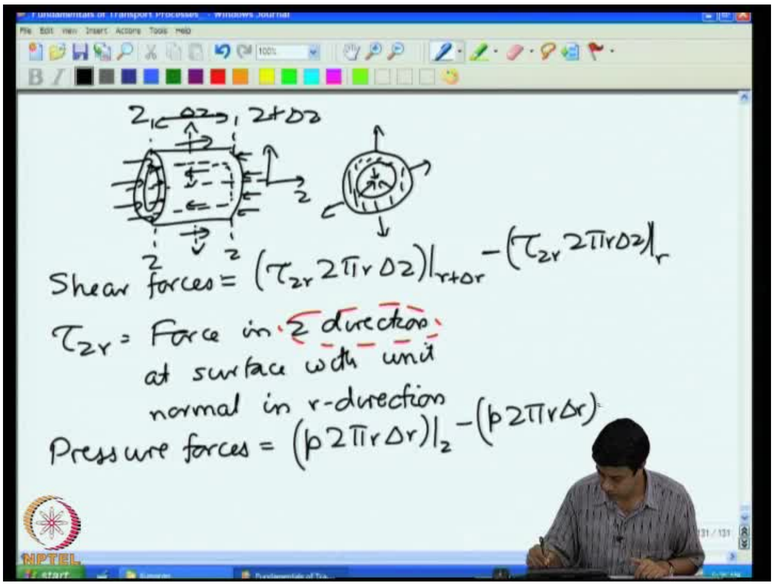

What about the applied forces? Let us go back to looking at the cylindrical volume. This is the axis of the cylindrical volume. There are applied forces, there are shear stresses acting on these surfaces. There are shear stresses acting on these surfaces. They act not only on the outer surface, but also on the inner surface. They act not only on the outer surface but, also on the inner surface. There are shear stresses acting on these 2 surfaces.

So, therefore, the shear forces is going to be equal to the shear stress times the area of the cylindrical surface. Shear stress is, I defined tou z r is equal to force in z direction at surface with unit normal in r direction. So, ((there is)) the component of the shear stress tou z r. I am writing the momentum balance equation for the z component of the momentum. Therefore, I have to consider the z component of the force, force in z direction. The cylindrical surfaces have unit normal in the r direction, the normal to the surface is in this direction. The outer surface, the outward unit normal is in the plus r direction, it goes outward. At the inner surface, it is in the minus r direction. If I look at this 2 cylindrical surfaces like this, side on, the outward unit normal is outward, the, for the inner surface, the outward unit normal is directly towards the center.

So, therefore, the shear force at the outer surface is going to be equal to the shear stress times the area, area in this case is, 2 pi r delta z, 2 pi r is the circumference, delta z is the thickness. So, that is the area. This, at the location r plus delta r. So, that is the shear force acting on the outward, outer surface. As I said, on the outer surface, the unit normal is directed outwards. At the inner surface, the unit normal is directed inwards. Therefore, the force acting along the inner r direction is going to be equal to minus tou z r into 2 pi r delta z at r.

So, this is the force acting on the surfaces, the outer and the inner cylindrical surfaces, for this differential volume. However, I also have a pressure gradient within the flow. That means that, the pressure acting on this surface is not the same as the pressure that is acting on this surface. That is, because there is a gradient in the pressure, as you go downstream. Therefore, I also have to include the pressure forces. There is a pressure force that is acting along the plus z direction at the surface at z. This is at z, this is at z plus delta z and this is the thickness delta z.

So, there is a pressure force acting in the plus z direction at the location z. This is given by the pressure times the area, area in this case is the annular area where the pressure force is acting between the two cylinders. This has to be 2 pi r times delta r at the location z. So, there is a pressure force acting in the plus z direction at the surface at z. At the other surface, there is a pressure force acting in the opposite direction, because the pressure always acts inward on a volume. The pressure is acting in the minus z direction. The net force exerted by that pressure, acting in the minus z direction is minus p times 2 pi r delta r at z plus delta z.

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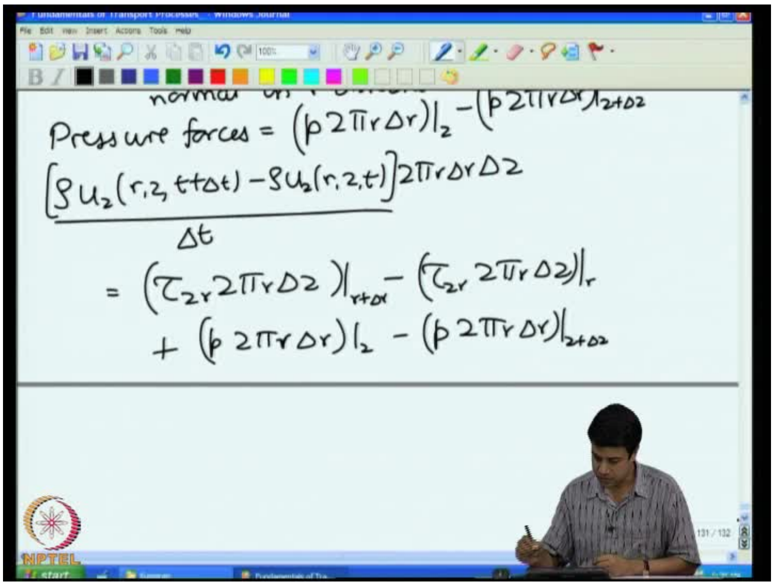

So, these are the 2 forces, the shear forces acting on the cylindrical surfaces parallel to the surfaces, pressure forces acting on the annular surfaces, perpendicular to the surfaces. So, these two together account for the rate of change of momentum in this differential volume. So, putting all that together rho times u z at r z t plus delta t minus rho u z at r z t times 2 pi r delta r delta z divided by delta t. Rate of change of momentum is equal to the sum of the applied forces, is equal to tou z r into 2 pi r delta z at r plus delta r minus tou z r 2 pi r delta z at r plus p 2 pi r delta r at the location z minus p 2 pi r delta r at the location z plus delta z.

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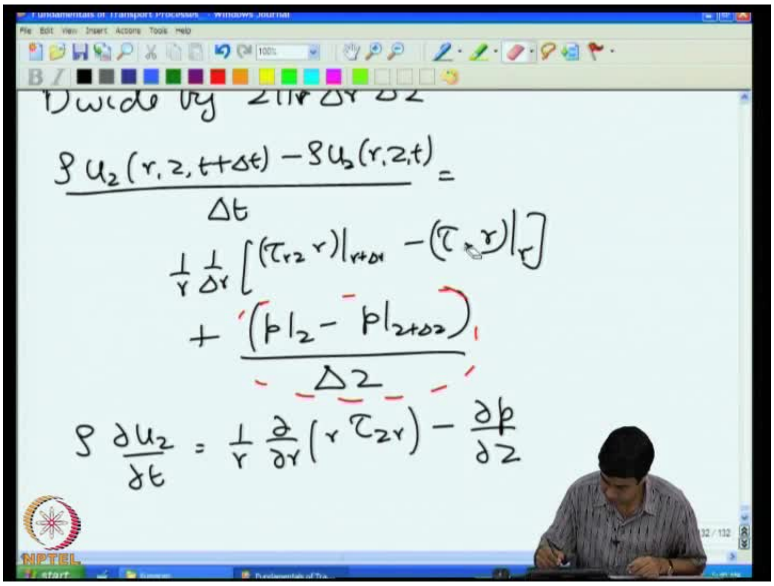

So, that is the final momentum balance equation and as usual, we divide throughout by volume, divide by 2 pi r delta r delta z, divide by the total volume. So, once I do that, I will get rho u z at r z t plus delta t minus rho u z at r z t by delta t is equal to… In this case, now, one has to be careful. I have r here at r plus delta r and r here at r itself. These two, at these two locations, the radius is not the same. Therefore, I cannot divide throughout by r because, the radius here at r plus delta r is different from the radius here at r. Therefore, if I divide throughout by 2 pi r delta r delta z, what I will get is, 1 by r 1 by delta r into tau r z into r at r plus delta r minus tau r z into r at r, and plus the second term, it is calculated at the average radius, so, that is the same for the inner and the outer surfaces. So, here I will get p at z minus p at z plus delta z by delta z. And, if I take the limit delta t, delta z delta r going to 0, these differential, difference equations reduce to differential equations. So, this becomes rho times partial u z by partial t is equal to 1 by r d by d r of r tau z r and this term here, p at z minus p at z plus delta z by delta z… P at z plus delta z minus p at z is d p d z. But, I have p at z minus p at z plus delta z. So, this is equal to minus partial p by partial z. Let us make a small difference.

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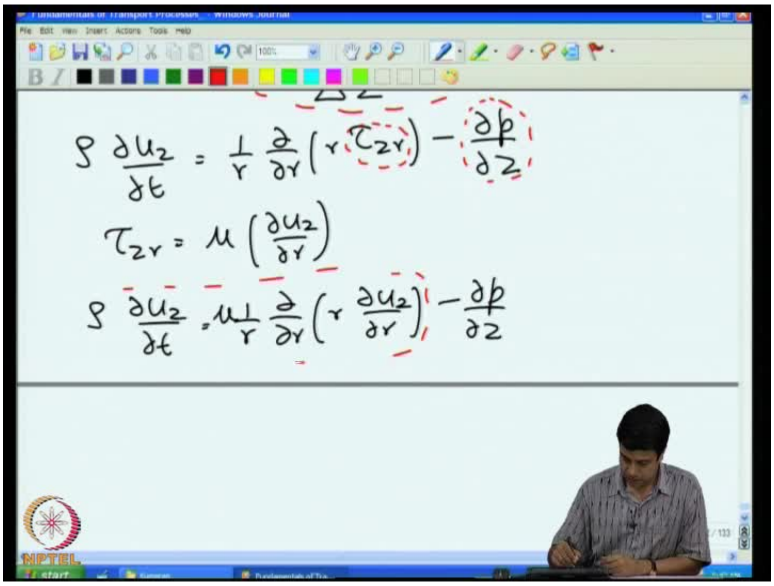

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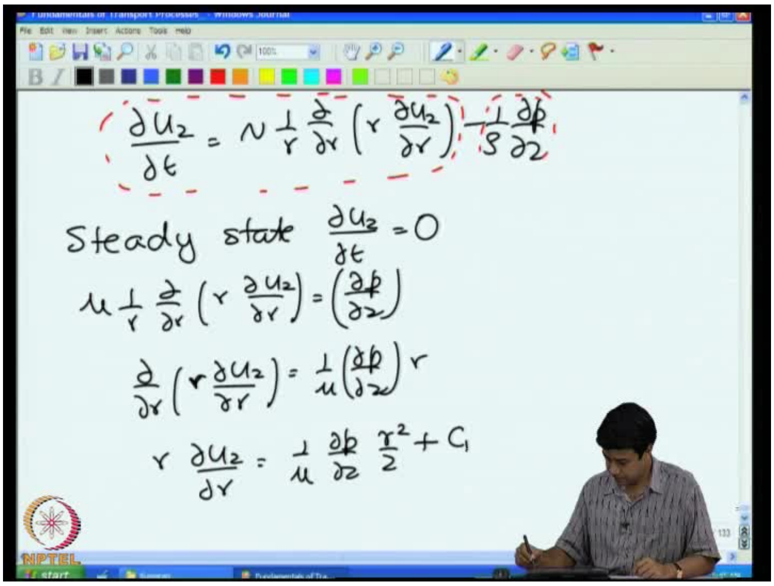

So, this is my equation. This is the analog of the, this is the momentum flux. This is the analog of the heat and mass flux, similar to what we had in the case of the heat transfer and the mass transfer problem. However, I have an additional term here, which is the pressure gradient. This is special only to fluid mechanics. There is no analog for the pressure gradient in heat and mass transfer. In order to get a differential equation for the velocity, I need to write down the shear stress in terms of the velocity gradient. Newton's law of viscosity… is equal to mu times partial u z by partial r. And with that, I will get rho times d u z by d t is equal to 1 by r d by d r of r d u z by d r minus d p by d z times the viscosity mu. Note that, this part of it is exactly identical to what we had in the case of heat transfer. If I divide throughout by the density, I can write this in terms of kinematic viscosity d u z by d t is equal to mu into 1 by r d by d r of r d u z by d r minus 1 by rho d p by d z.

So, this part of it is exactly identical to what we had for the temperature field, except that I had substituted u z for the temperature and nu, kinematic viscosity instead of thermal diffusibility, alpha. However, this pressure gradient is now different. This, this is special only to fluid mechanics. So, this is the differential equation for a flow with a pressure gradient in it. How do I solve it? Simplest case first. Steady state d u z by d t is equal to 0 and therefore, I will have mu 1 over r d by d r of r d u z by d r is equal to d p by d z. Dp by dz is the pressure gradient along the axial direction.

This equation can be integrated to find out, what is the velocity profile, from the consideration that the pressure is independent of the radius. That is, as I go across the tube, there is no net change in pressure across the tube, in the radial direction. I will show you later that, that is an exact result. For that you have to write down the momentum equation in the radial direction. That momentum equation in the radial direction will contain similar terms. There will be terms that depend upon the velocity; there will be one term which contains d p by d r. However, the velocity u r is identically equal to 0. The velocity u r is identically equal to 0 at all points. We have velocity only in the z direction. Therefore, d p d r has to be equal to 0, because it is not balancing any viscous stresses. You do not have a pressure variation in the radial direction. It has to be balancing some other stress, and there is no other stress to balance. Therefore, the pressure is a constant across the radius of the tube.

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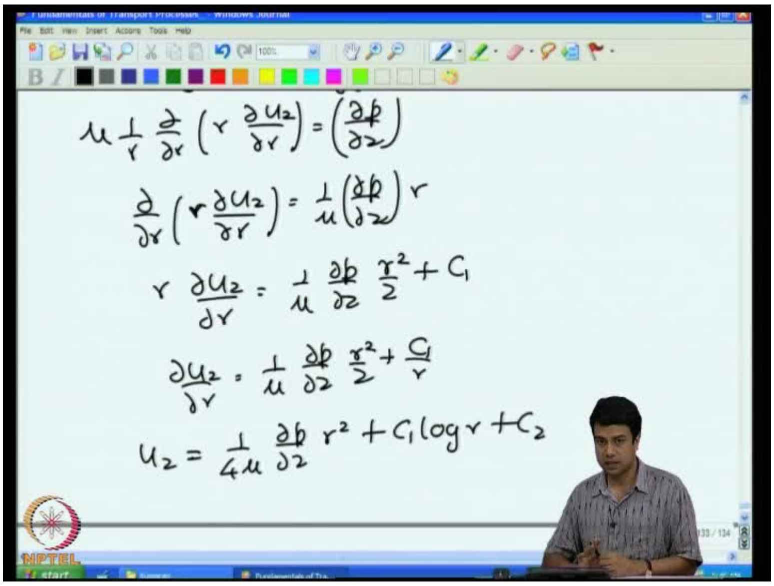

So, with that I can integrate this. I will get d by d r of r d u z by d r is equal to 1 by mu d p by d z into r. Therefore, r d u z by d r is equal to 1 by mu d p by d z r squared by 2 plus the constant C 1. And, d u z by d r is equal to plus C 1 by r. I integrated a second time to get u z is equal to 1 by 4 mu d p by d z r squared plus C 1 log r plus C 2, where the constants C 1 and C 2 are to be determined from the boundary conditions. Let us look at what are the boundary conditions for this problem.

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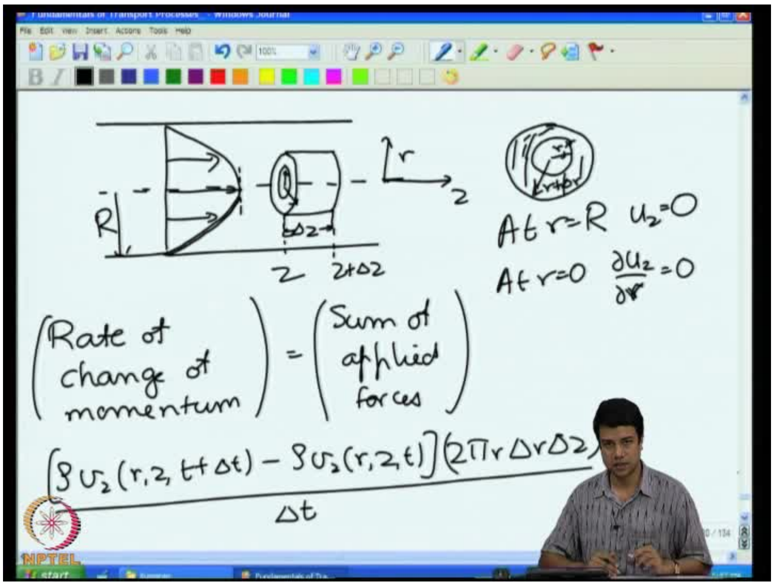

One condition, of course, is that, at the wall of the tube, at r is equal to R, I require that the velocity has to be 0. The wall is stationary and the fluid velocity has to be equal to the velocity of the surface at the wall, you require that u z has to be equal to 0. So, that is the boundary condition at the surface at r is equal to capital r. How about at the axis itself, at r is equal to 0. Just I back you to you earlier, that, symmetry requires that d t d r is equal to 0 at the axis. In this case itself, at r is equal to 0, symmetry will require the d u z by d r has to be 0. Whereas the slope of the velocity, at the origin itself, has to be 0, otherwise, as you come from different sides, you will get different slopes for the velocity, as you approach the center. It is possible to have that, provided there is net stress exerted by some object at the center.

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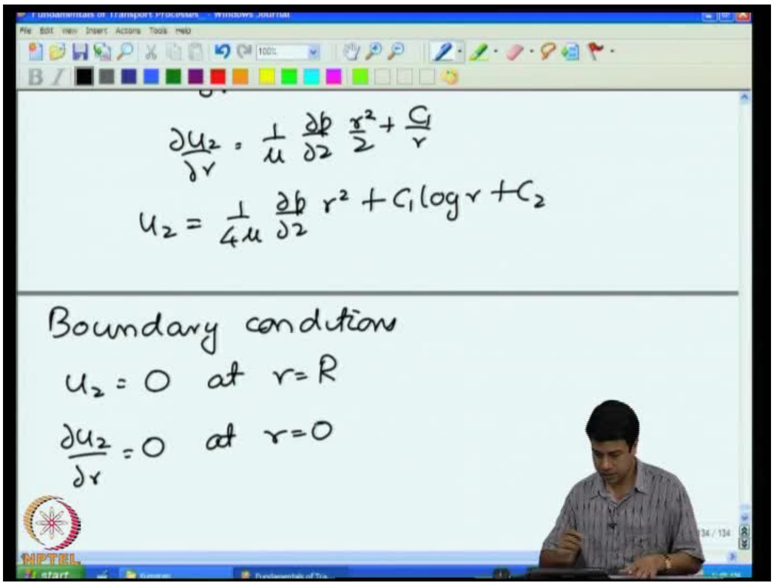

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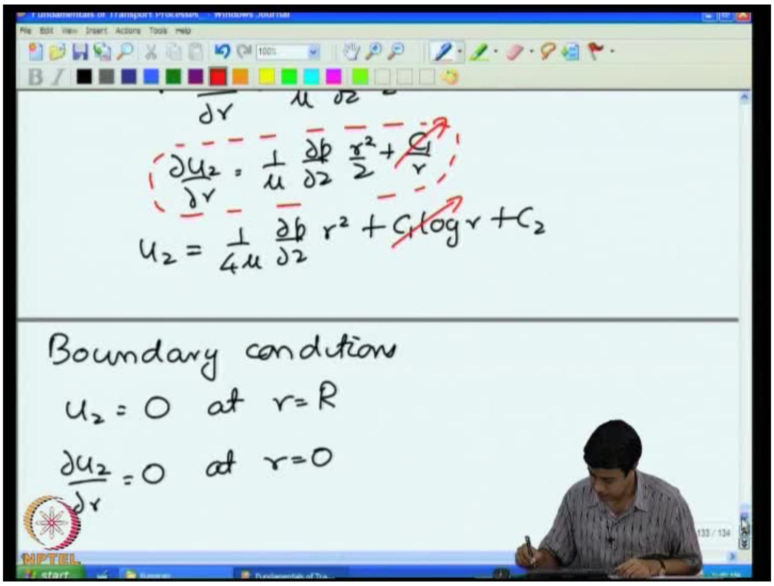

In this case, we have no object at the center. It is just a pipe, there is no surface at the center. Therefore, the velocity delivered there has to go to 0 at the origin. So, the boundary conditions u z is equal to 0 at r is equal to capital r and d u z by d r is equal to 0 at r is equal to 0. From this equation, you can easily see that, the d u z by d r will be 0, only if C 1 is equal to 0, otherwise, it goes to infinity. So, the symmetry condition requires that C 1 is equal to 0 at r is equal to (()). Otherwise, the velocity will be infinite at r is equal to 0, at the axis of the tube.

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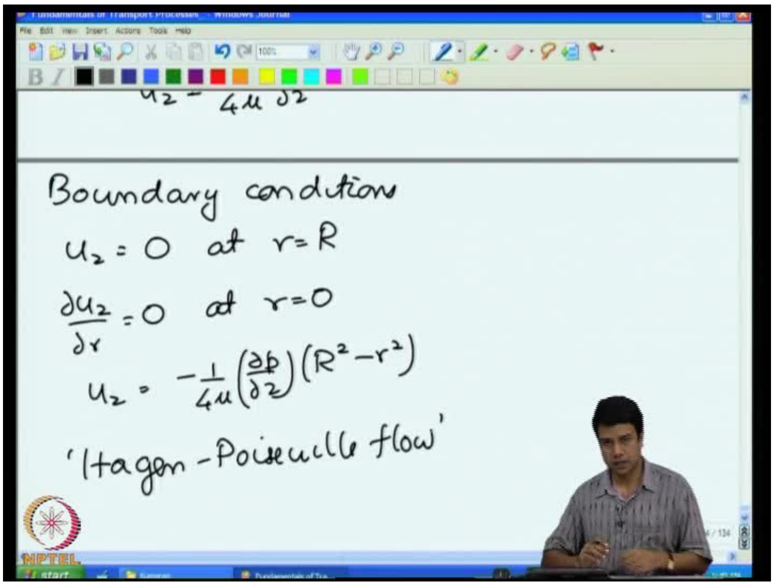

So C 1 is equal to 0 and C 2 can be determined from the condition that, u z is equal to 0 at r is equal to capital r. And, my final equation for the velocity becomes u z is equal to minus 1 over 4 mu d p by d z into R squared minus r squared.

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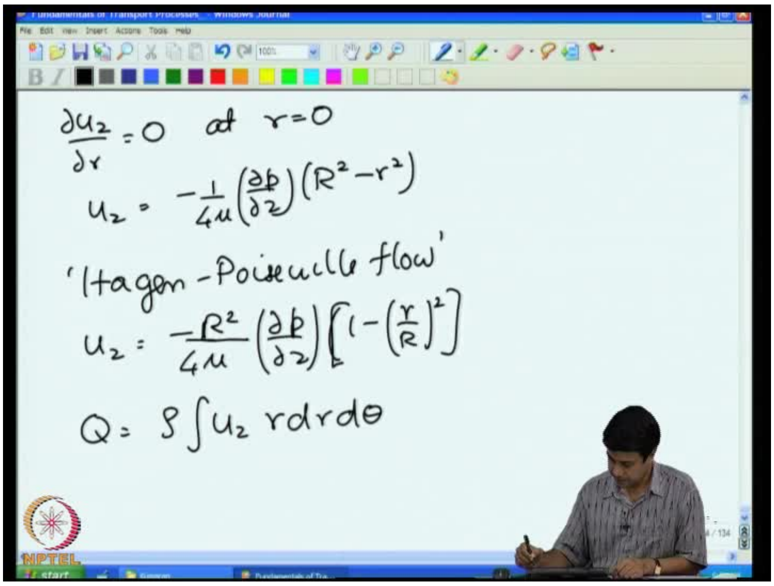

So, this is the final expression for the velocity field for the flow in a pipe. This is called the Hagen- Poiseuclle flow for the flow in a pipe and I can also write that as u z is equal to minus 1 over four mu R square, I am sorry, 4 mu d p by d z into 1 minus R by r the whole squared. Note that, the velocity is positive when the pressure gradient is negative. In other words, d p by d z has to be negative, pressure has to decrease along the length of the tube for the velocity to be positive.

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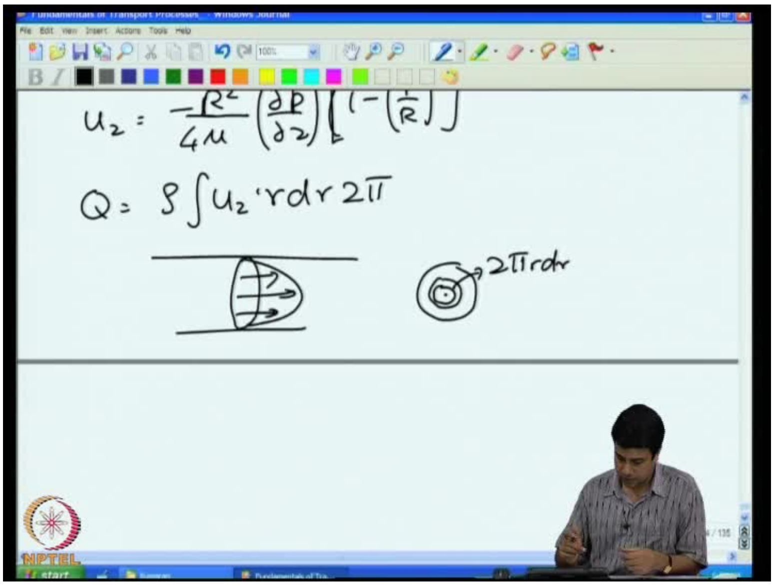

And therefore, the pressure gradient and velocity have opposite signs, because I need to have a higher pressure at the inlet and lower pressure at outlet in order for the velocity to be positive. Now, one can calculate the total flow, heat flow, total mass flow due to this velocity profile. The total mass flow Q is equal to rho times integral of u z times r d r. So, basically what I have is, a pipe of the certain cross section and I am calculating it has a certain velocity profile and I am calculating the integral of u z times the differential area. Because, if I look in a cylindrical coordinate system, the differential area between two annulus, is going to be to 2 pi r times dr.

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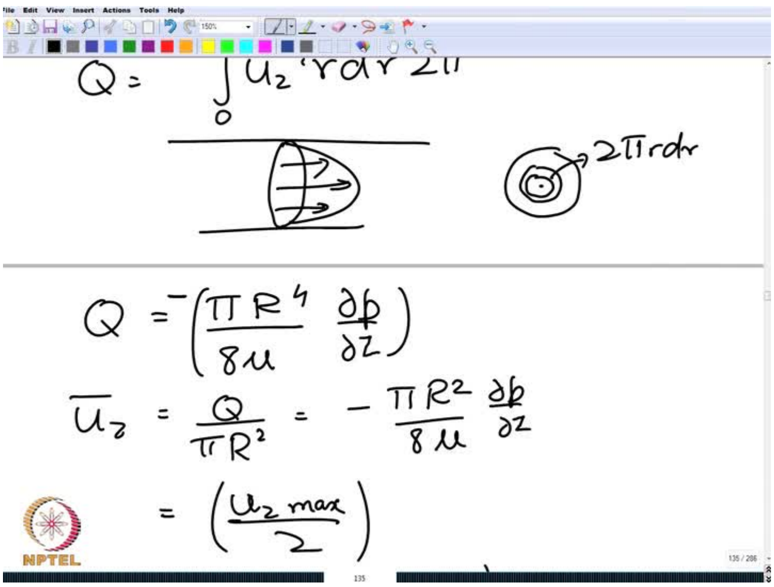

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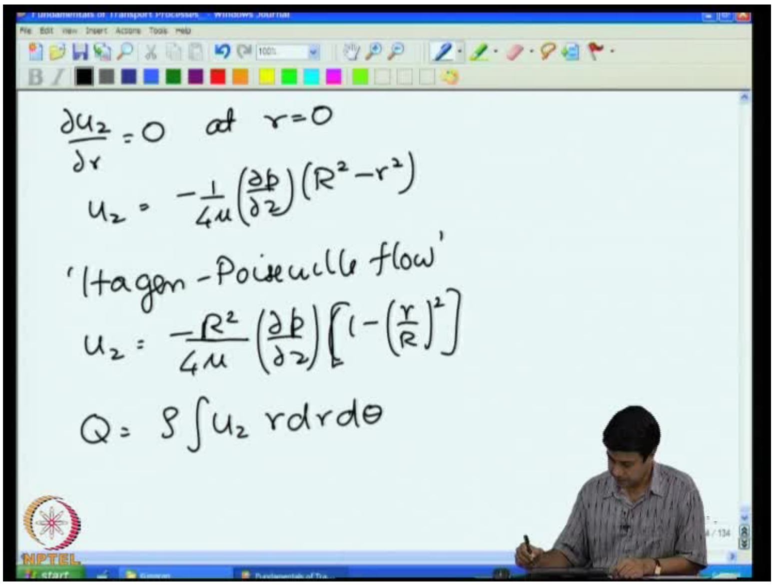

So, integrate 2 pi r d r into 2 pi. Second, calculate, that to get heat flux, mass flux which Q is equal to pi R power 4 by 8 mu into d p by d z with the negative sign. Once again, because d p by d z is positive, flow is positive in this direction. And, I can calculate the average velocity from this. The average velocity u z bar is equal to Q by pi R squared is equal to minus pi R square by 8 into d p by d z. Go back to this expression, the maximum velocity is at the centre where r is equal to 0.

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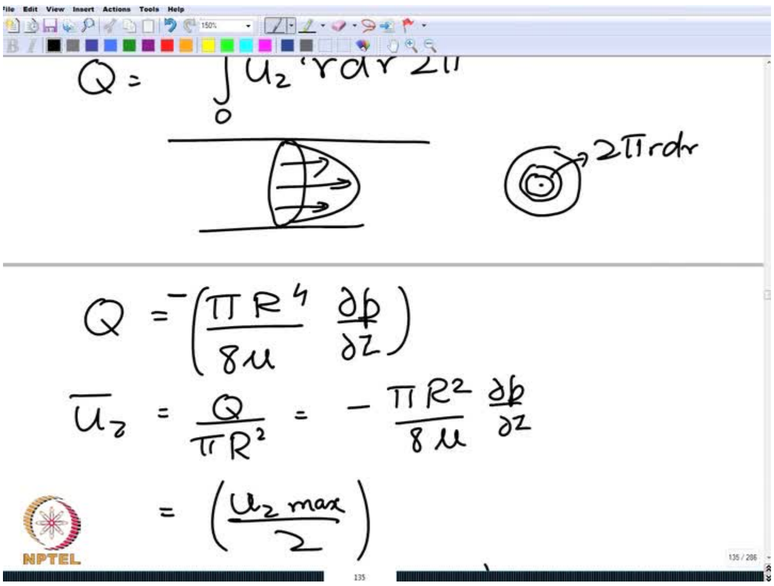

The maximum velocity is minus R square by 4 mu times d p by d z. The average velocity is minus R square by 8 mu d p by d z. Therefore, this is equal to u z maximum by 2. The average velocity is half the maximum velocity for the flow through a pipe. Next step, we will determine the relationship between the friction factor and the Reynolds number from the solution that we have just calculated. We will get our result that f r e is equal to 16, that f is equal to 16 by r e for this particular case. So, since we are out of time, we will continue that in the next class and then we will look at a slightly more complicated situation that is an oscillatory flow in a pipe. So, we will continue this in the next lecture, since we are out of time right now.

Thank you.