Fundamentals of Transport Processes

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Lecture No. # 18
Unidirectional Transport Cylindrical co-ordinates - III Separation of Variables

So, welcome to this lecture number 18 in our course on fundamentals of transport processes. We were discussing transport in cylindrical coordinate system. As I explained in some detail in the last lecture, we used curvilinear coordinate systems when the symmetry is of the configuration we are analysing, have a cylindrical symmetry. So, when we do that the surfaces of constant coordinate now become cylinders constant r, and the advantages that now we are applying boundary conditions on surfaces of constant coordinate. So, you have chosen the coordinate system in such a way that the surface of the cylinder has a constant radius r. The disadvantage of that is that the differential equation has a slightly more complicated form, because the surface area is changing as the coordinate changes. So, I as I change the radius of the cylinder the surface area changes, this is in contrast to the cartesian coordinate system where at every plane within my cubic volume the surface area was exactly the same.

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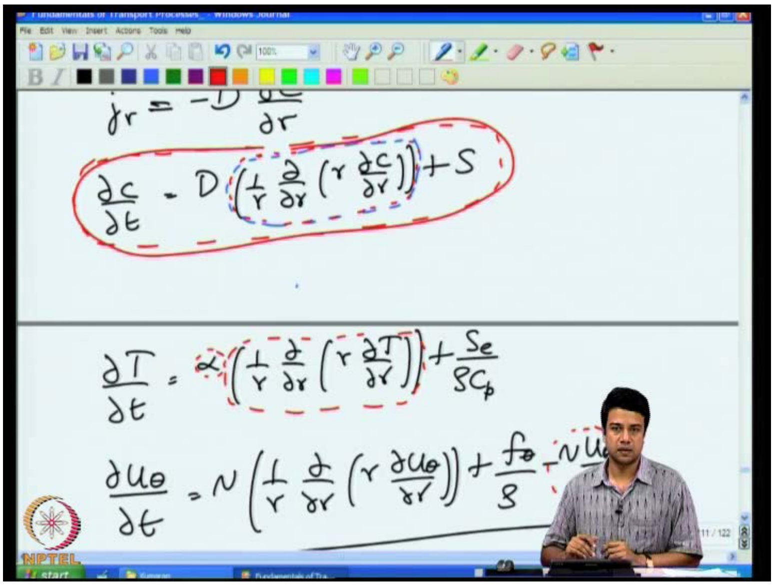

So, last class we looked at the differential equation that results from this. So, these differential equations have a slightly more complicated form the concentration, the momentum, and the temperature equations the form of the differential operator is more complicated simply, because the surface area is changing as the distance changes. And because of that if you solve for a steady flow between two cylindrical surfaces.

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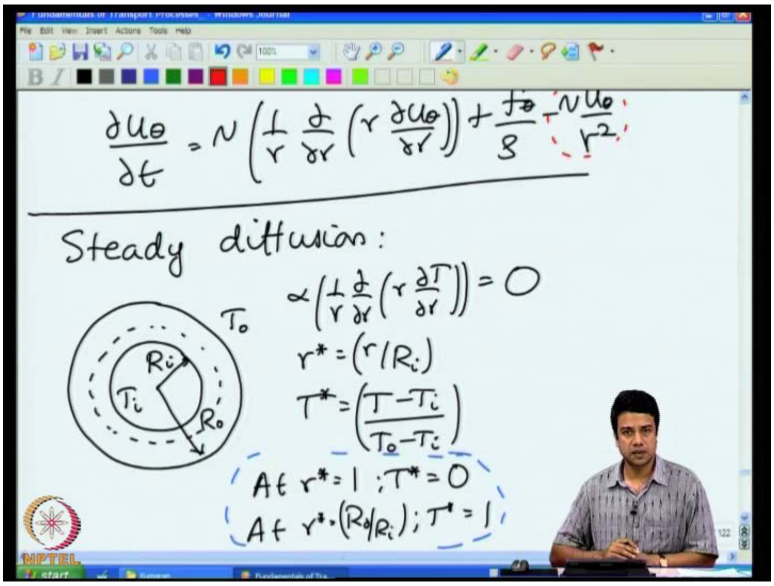

We solve for the steady flow between 2 cylindrical surfaces in the last lecture where you have 2 surfaces the 2 walls of a tube there is an inner surface for this wall and the outer surface, and for these 2 surfaces you fix 2 temperatures the temperature in between does not vary linearly with position rather it has a logarithmic variation with position.

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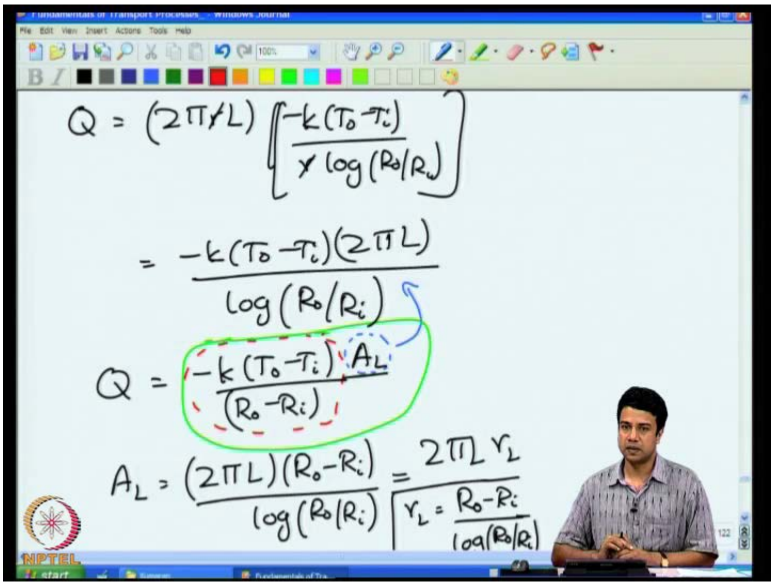

So, this was the equation the solution that we got for the temperature profile in between the 2 and we had seen how we got the log law for the heat flux the heat flux is equal to the temperature times the difference in temperatures times the heat conductivity divided by the thickness times and area which is based upon the logarithmic mean radius r naught minus R i by log of R0 by R i.

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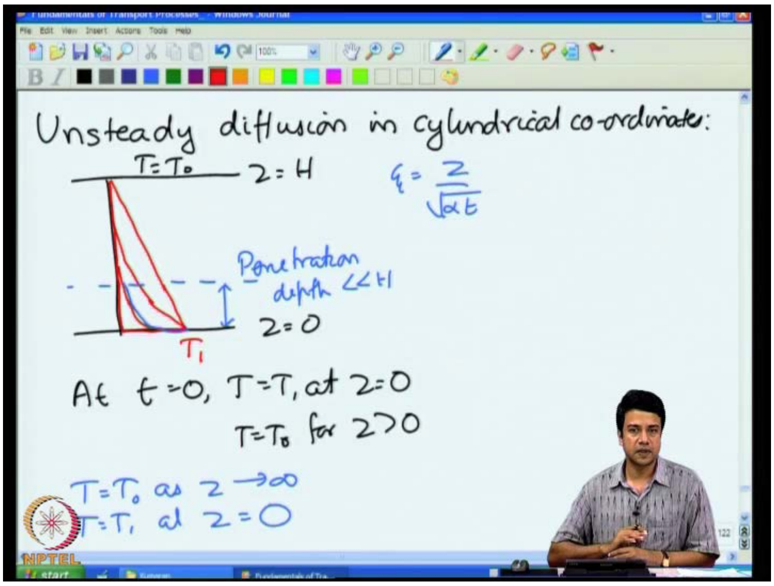

And then we solved the problem of separation of variables in a cylindrical coordinate system and I explained to you that this is different from Cartesian coordinates if I have a Cartesian coordinate system, I have a surface where the temperature is instantaneously increased and if I consider diffusion from this surface into an infinite medium there are no length or time scales in the problem and therefore, I can transfer in terms of a similarity variable. In the case of a cylindrical coordinate system even if I have a cylinder in an infinite medium fine, and I consider a temperature difference between the surface of the cylinder and the temperature far away the temperature in infinite medium cannot be solved using similarity solution, because the cylinder radius is still a parameter in the problem. So, I can still define a non dimensional radius as the ratio between the distance and the radius of the cylinder and therefore, for transfer from a cylinder of finite radius you cannot use the similarity solution for obtaining a solution.

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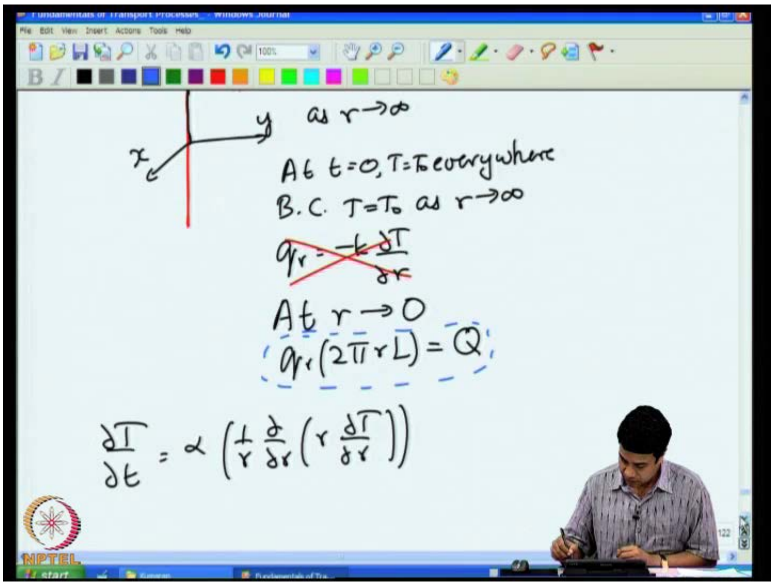

However one can consider a cylinder of infinitesimal thickness a wire whose radius is very very small, and for this wire since the radius of the wire goes to 0 one can still use a similarity solution for this equation. However, as the radius goes to 0 if the heat flux or the temperature is fixed at the surface, or if I fix the temperature gradient at the surface the heat flux is equal to the thermal conductivity times the temperature gradient. If the temperature is finite the heat flux is finite. The surface area of a wire of infinitesimal thickness goes to 0 as r goes to 0. Therefore, I cannot specify a temperature boundary condition at the surface; because what it would mean is that is the temperature is finite the total heat coming out goes to 0, because this wire has infinitesimal thickness.

What I need to specify is actually the total heat coming out per unit time Q which is equal to qr times 2 pi r L in the limit as r goes to 0. So, as I approach very close to the surface of the wire the radius decreases, the heat coming out will be per unit time will be finite only if qr is proportional to one over r in this limit. So, this kind of a boundary condition can be satisfied only if my solution of the equation is such that qr goes as one over r in the limit as r goes to 0.

So, we do not really have a temperature scale at the surface of the wire itself. All we know is that how much heat is coming out per unit time from the surface of the wire. So, that is all we know at the wire itself far away the temperatures are constant and at time t is equal to 0 the temperature is a constant everywhere. So, this was the differential equation for the temperature field and we had scaled it. And specified the boundary at initial conditions I pointed out that there is additional term here which is not present in the Cartesian coordinate system, and that comes about because the radius changes and the surface area changes as you go out in a cylindrical coordinate system.

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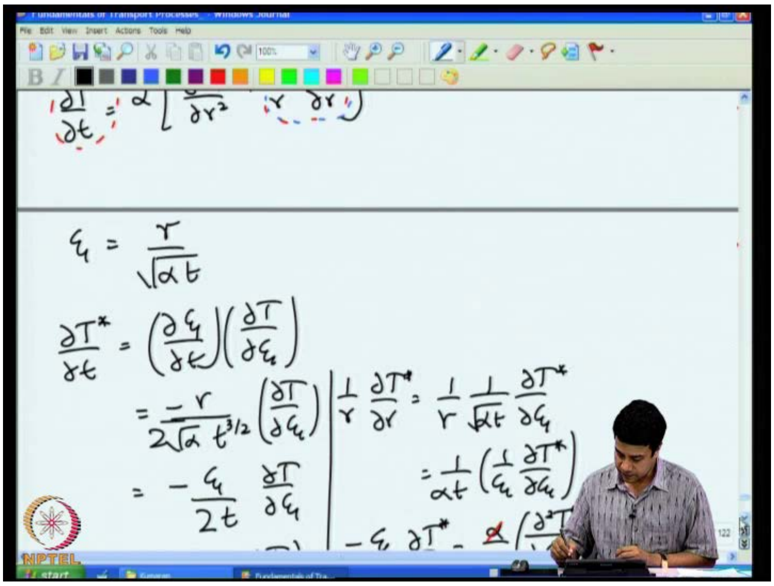

And we had defined the similarity variable as r over square root of alpha t and express the mass the heat conservation equation in terms of this similarity variable. And we got an equation of the kind d square t by d psi square plus one over psi plus psi by 2 dt by dpsi is equal to 0.

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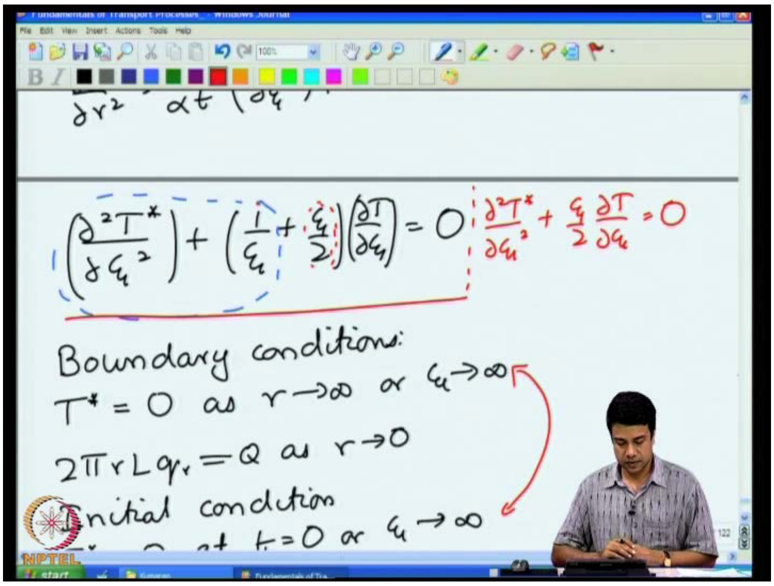

Recall that when we solved the problem in Cartesian coordinate system, the equation was of the form d square t by d psi square plus psi by 2 dt by d psi is equal to 0 that was the equation in a Cartesian coordinate system in a cylindrical coordinate system.

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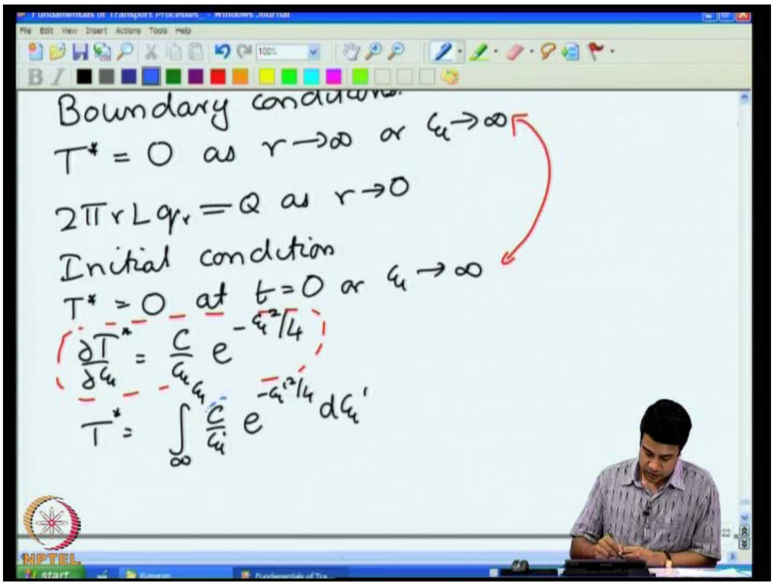

I have this additional term here one over psi times d psi by dt and that is because once again the surface area is changing as the radius changes. And we have 2 boundary conditions one is that t is 0 as r goes to infinity or psi goes to infinity and this 2 pi r L times qr is equal to the heat coming out per unit time from the wire. This in the limit as r goes to 0 at the surface of the wire itself, what is being specified is the total heat coming out per unit length of the wire not the temperature or the heat flux, because the heat flux cannot be specified because the surface area is going to 0. So, the heat flux is finite the heat coming out will go to 0 and we solved this quiet easily this term was not present in the Cartesian coordinate system once again and therefore, we were able to specify the temperature, but when this term is present t star actually goes to infinity as you approach the surface despite t star going to infinity.

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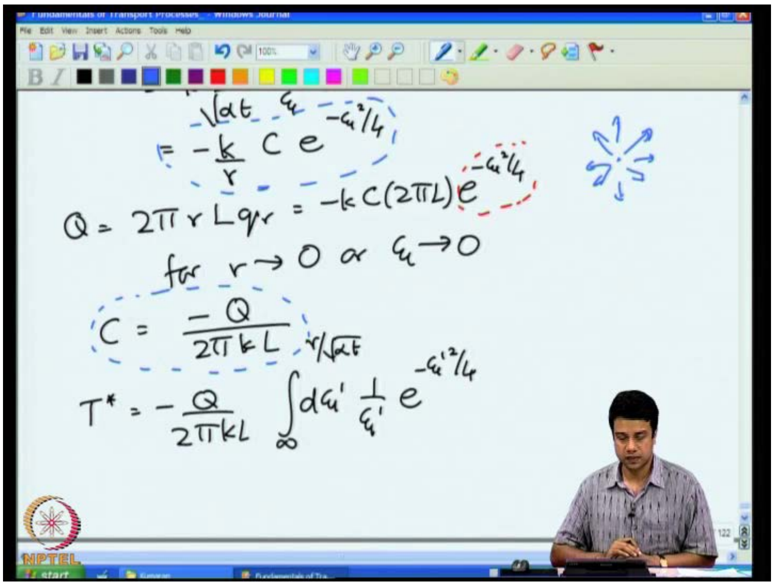

We were still able to find the constant C in the equation. And the reason we found the constant C was because once we put the temperature into the equation for the heat flux. Once we have put the temperature for the equation into the heat flux we got a temperature field and that is going as one over r in the limit as r goes to 0 yes I am sorry we were getting a heat flux that was going as one over r in the limit as r goes to 0 and because the heat flux goes as one over r surface area is proportional to r therefore, the total heat coming out is finite in the limit as r goes to 0 and we manage to get a solution for the total temperature field as well as the total heat coming out at the surface of the wire itself the heat is finite the heat coming out is finite even though the flux goes to infinity and the temperature also goes to infinity.

So, this as I said is the temperature field for a line source of heat it is a line source in three dimensions. If I just consider the plane perpendicular to the wire itself what it looks like is a point source of heat from which heat is coming out symmetrically in all directions. So, this is a point source in 2 dimensions or a line source in or a line source in 3 dimensions we will see the point source in 3 dimensions when we do a spherical coordinate system a little later. Now let us look at heat conduction from a finite cylinder. So, the problem is as follows.

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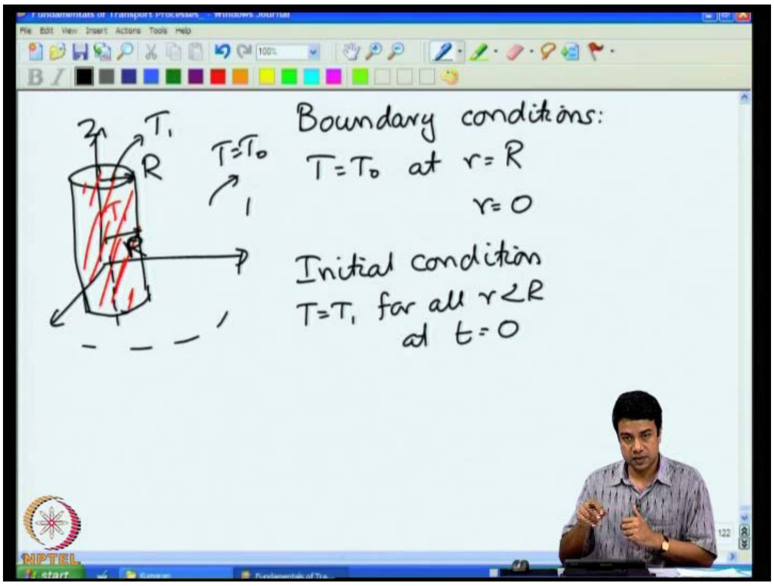

I have a cylinder we have considered this to be of infinite extent in the vertical direction. And the temperature within the cylinder is initially at some value T1 this is immersed into a bath where T is equals to T naught. So, temperature t is equal to t naught everywhere in the bath the cylinder temperature is T1 it is immersed in this bath and as time progresses the temperature of the cylinder decreases with time in the final state of course, the temperature will be T naught everywhere. So, at the final state temperature is just going to be T naught everywhere.

However initially at time t is equal to 0 the temperature everywhere within the cylinder is going to be equal to T1. And only at the outer surface as soon as it is immersed the temperature is going to come down to t naught think of immersing let us say a cylindrical cup at a high temperature into a vessel with cold water at low temperature the vessel is large enough that the heat conducted from the cylinder does not change the temperature of the vessel appreciably. So, that is the kind of problem of course, if you solve the problem for a cup the temperature is going to be a function of height as well it is going to be a function of the Z coordinate as well not just the distance r from the centre or from the central axis. So, this is a simplification of that problem where I have a cylinder with at temperature T1 at time t is equal to 0 I immerse it in a bath.

So, what are the initial and boundary conditions for this problem we have already seen what is the heat conduction equation for this problem what about the initial and boundary conditions. As the equation is second order in space and first order in time I require 2 boundary conditions and one initial condition. So, the boundary conditions T is equal to T naught at r is equal to R, if R is the radius of the cylinder. If R is the radius of the cylinder, then I require that since this cylinder is immersed in a bath at temperature T naught the temperature at the surface of the cylinder also has to be at the same temperature T naught at r is equal to R. Initial condition I can think of one quite easily T is equal to T1 for all r less than r at t is equal to 0. What this says is that the moment I have immersed the cylinder the temperature everywhere within the cylinder is still equal to T1 which was the initial temperature only the surface has come into contact with this bath whose temperature is at t naught.

What about the other boundary condition I have a second order differential equation in the radial coordinate. So, I should have another boundary in the radial coordinate. This is going to be a common feature when we deal with curvilinear coordinate systems. Because the other boundaries should be at r is equal to 0 at the central axis itself. So, if I have a cylinder with the central axis, I have one boundary condition on the surface of the cylinder r is equal to R, but how about the boundary condition at the central axis of the cylinder this is not really a boundary there is there is no boundary between the domain and some external region at r is equal to 0 it is entirely within the fluid itself the this r the axis is entirely within the fluid itself.

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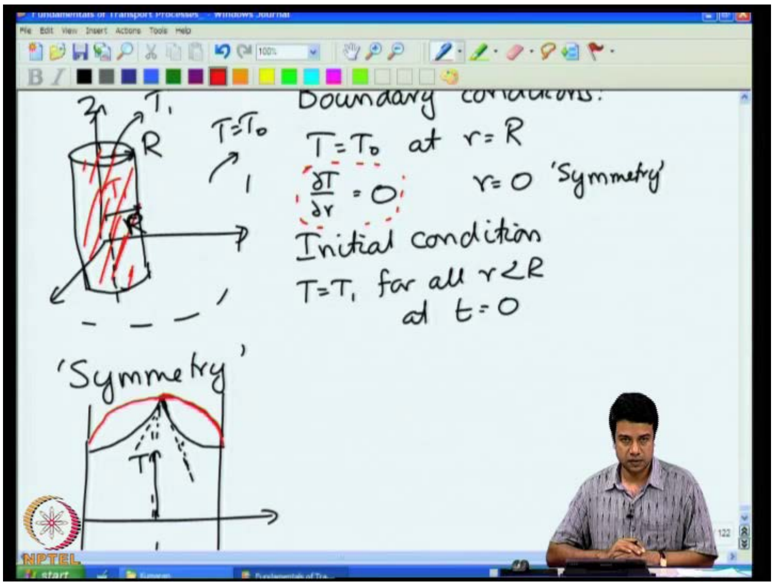

At this so, it is at this boundary it is not really a boundary, but at this boundary in the r coordinate we enforce what are called symmetry conditions. Let me explain that a little bit. Now there is no source or sink of heat along the central boundary unlike in the previous problem where we had a wire and that wire was emitting heat in this particular case it is just a fluid with an axis going through the centre and there is nothing there that requires that at r is equal to 0 the derivative of the temperature has to be equal to 0 why is that this is a symmetric condition. Let us assume for the sake of argument that the derivative is not 0 at the centre.

So, this is my cylinder this is my central axis. So, this is my r coordinate and I will plot the temperature the temperature as a function of the distance the radius and let us assume for the moment that the derivative of the temperature with respect to radius is not 0 at the centre. So, let us assume for the moment that we have a non 0 derivative at the centre the derivative of the wall could be 0 or non 0 depending upon the boundary condition there, but we will assume for the moment there is the derivative at the centre is not equal to 0. So, the slope is not zero.

However, note that there is no change in the temperature field as you go all the way around the axis we have assumed that the system is axisymmetric. So, there is no change in the temperature field as you go around the axis. So, if Ii go all the way around the axis and come to the other side the temperature field on this side has to look something like this exactly a reflection of the temperature field on the right side because the temperature field is independent of location on the axis. So, if I go all the way around the temperature field on the left side has to be a reflection of the temperature field on the right side. Note that therefore, the temperature field on this side has to be at this. So, just symmetry requires that if the derivative of the temperature is non zero; that means, that the derivative of the temperature is different at different as you approach the centre from different locations.

If you approach it from the right the temperature decreases as I go as r increases if I approach it from the left the temperature decreases towards the left. So, the derivative itself is not uniquely specified at the centre itself. The only way that the derivative will be uniquely specified at the centre itself is if the derivative is 0 at the centre if I have a derivative going to 0 at the centre, if I turn this all the way around and come back on the left side the derivative will still be 0 at the centre. So, the derivative now is continuous as I go from the right or from the left.

So, this symmetry basically requires that the derivative of the temperature has to be 0 as you approach the centre if it is not since the field is axisymmetric the temperature derivative will be will not be uniquely specified it will the derivative will be positive on the right side I am sorry negative on the right side positive on the left side as you approach from the two sides it goes to two different values it is not the single valued function at the centre itself. The only way that the derivative will be a single valued function at the centre is if it is exactly the same it is if it is exactly equal to 0 on both sides. So, that is the physical region for this boundary condition the symmetric condition that we have here.

This will be 0 provided there are no sources or sinks at the centre as I explained to you in the previous case if you have a wire at the centre there is a source of heat in that case the derivative need not be 0. In fact, the temperature gradient the heat flux and therefore, the temperature gradient have to go as one over r for the total heat generated to be a constant in this particular case there is no source or sink. So, the temperature derivative has to be equal to 0 and this symmetric condition is something that we will encounter often in curvilinear coordinates both cylindrical and spherical coordinate system. And this comes out of the fact that one boundary is located along a line or a point within the domain and within that domain everything has to vary smoothly. So, that is the reason for this. Define dimensionless temperatures and radius once again scale radius and temperature.

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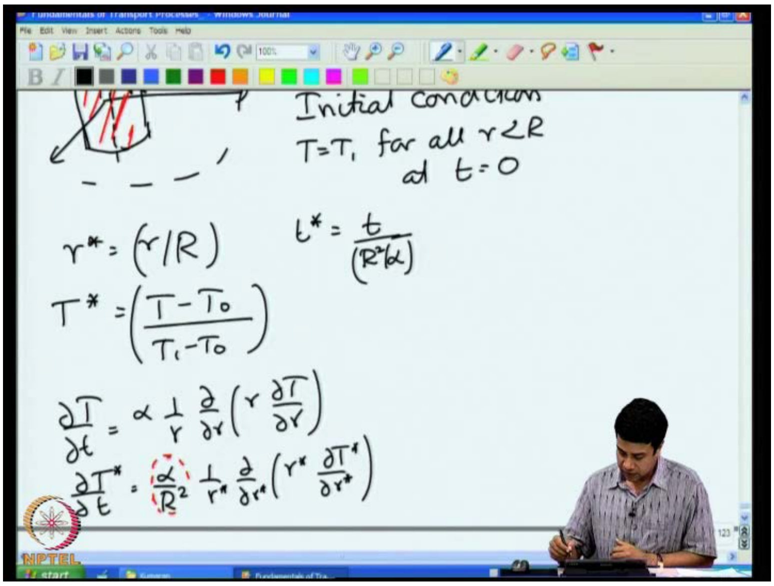

let me just remove the symmetry argument radius is easy r star is equal to r by R. Temperature I can define T star is equal to T minus T naught by T1 minus T naught, as usual. My original governing equation was dT by dt is equal to alpha times 1 over r d by dr of r dT by dr expressed in terms of t star and r star

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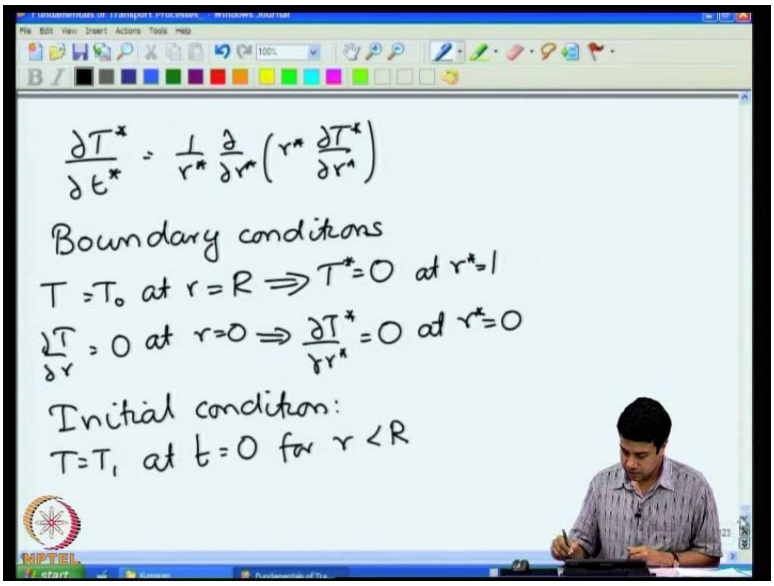

You can easily check that this becomes dT star by dt is equal to alpha by r square 1 over r star d by dr star of r star dT star by dr. So, this is a scaled equation except I have not scaled time yet.

However, the scaling of time is now obvious from this equation in case the scaling of time is just given by this. So, therefore, I can define a scaled time variable as t star is equal to t by R square by alpha. Physically, this is the time it takes to diffuse over a distance comparable to R. So, I am scaling I scaled the distance by R. The time is being scaled by the time it takes for diffusion to take place over a distance comparable to R. So, if I scale it in this way my equation now becomes dT star by dt is equal to 1 over r d by dr of r dT by dr what are the initial and boundary conditions. Boundary conditions first one was that T star is equal to T naught at r star is equal to at sorry at r is equal to R, which is equivalent to writing T star is equal to 0 because T star is equal to T minus T naught by T1 minus T naught. So, t star is equal to 0 at r star is equal to1.

The other condition was the symmetry condition dT by dr is equal to 0 at r is equal to 0 equivalent to writing dT star by dr star is equal to 0 at r star is equal to 0. And then I have my Initial condition and that was that T is equal to T1 at r at t is equal to 0 for r less than R. So, T is equal to T1 implies, that t star is equal to 0 at I am sorry t star is equal to one at r at t star is equal to 0 for r star less than 1. So, these are the boundary and initial conditions this equation this problem cannot be solved by a similarity transform, because there is a length scale in the problem there is the radius R. The way to solve this equation is to use what is called separation of variables and we had seen in detail how to do separation of variables in a cartesian coordinate system.

I will only point out what the complications are when we deal with a cylindrical coordinate system. First things first when we do separation of variables we have to make sure that we have homogeneous boundary conditions in all the spatial coordinates. We had taken care to see that when we have looked at the steady state heat conduction between 2 surfaces you make sure that they were homogeneous boundary conditions in all the spatial coordinates. In this particular case you can see that we already have homogeneous boundary conditions in all the spatial coordinates t star is equal to 0 on one surface the derivative of t star is equal to 0 on the symmetry axis. It is only at the initial time that t star is not equal to 0 it is only at the initial time that t star is not equal to zero.

Therefore we already have a well posed problem there is already homogeneous boundary conditions and there is an initial condition that is non homogeneous. So, therefore, we can solve it using a separation of variables procedure.

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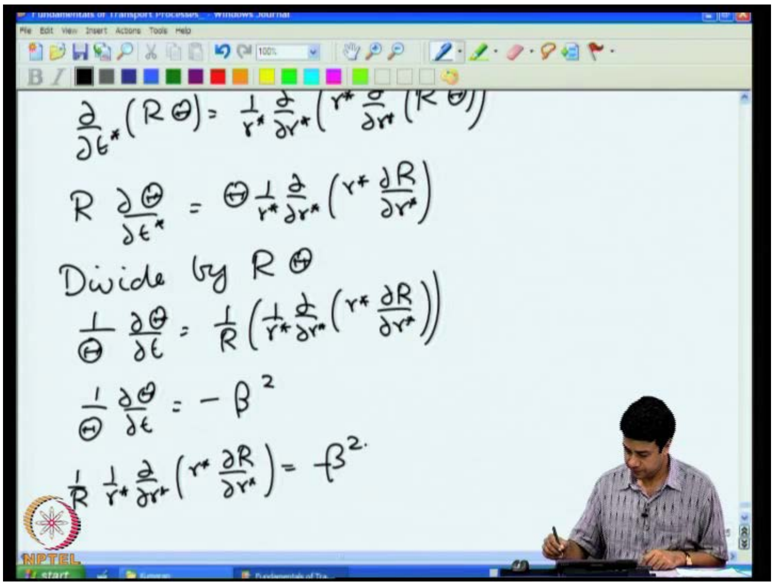

So, let us look at how we would do the separation of variables as in the case of a Cartesian coordinate system. We will separate T star is equal to some function of r star times some function of time. So, this is the usual procedure that is used for all separation of variables problems. And I insert this into the conservation equation and I will get d by dt of R theta is equal to 1 over r d by dr of r d by dr of R times theta. Now since R is only a function of r and theta is only a function of time I can take out the the factors R on left side and theta on the right side.

So, R d theta by dt is equal to theta times one over r and Divide throughout by R times theta to get one over theta d theta by dt is equal to 1 over r into one over r star. So, this now is the differential equation that I have to solve and you can see that the left hand side depends only upon t the right hand side depends only upon r . So, therefore, both sides have to be equal to constants if they were not constants then I could change t and keep r a constant the left side would change the right side would not and that would destroy the equality.

So, the very fact that the left side depends upon only one independent variable the right side depends only upon the other independent variable, means that both sides have to be identically equal to constants. Does that constant have to be positive or negative? We saw from our experience with the separation of variables in cartesian coordinates that this constant has to be negative. In particular I have an equation of the type one by theta d theta by dt has got to be equal to some constant if this constant is positive theta will increase exponentially in time, if this constant is negative theta will decrease exponentially in time. When we solve the unsteady problem we require that as time t goes to infinity theta has to decrease exponentially. So, the temperature goes to 0 everywhere physically, we would expect that I have taken a cylinder of temperature T1 immersed it in a bath with temperature t naught as time t goes to infinity the temperature throughout the cylinder has to be come down to t naught; that means, that t star has to decrease to zero.

That will happen only if this constant is negative. So, this constant has to be negative. Let me call it as beta so, good. So, we have the constant negative how we evaluate the value of this constant we put it inside it into the equation on the left hand side 1 over R one by r star d by dr r for r dR by dr is equal to minus beta square.

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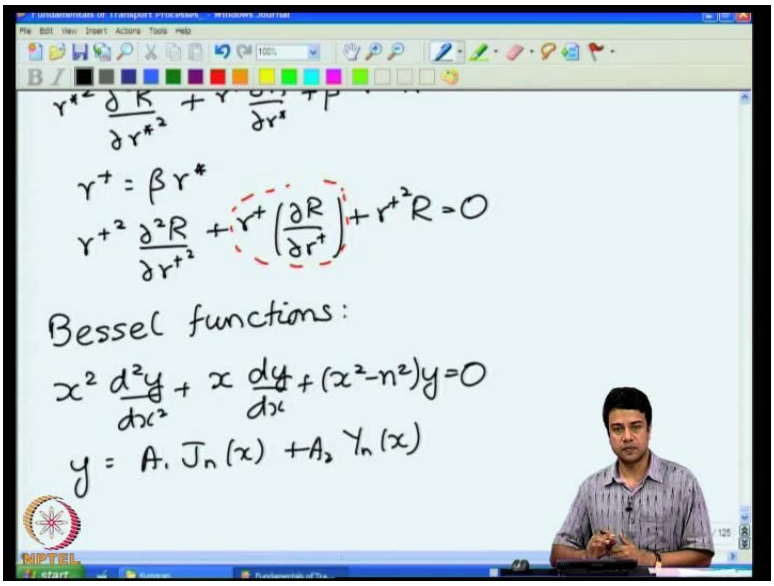

So, therefore, if I expand out this equation I will get d square R by dr star square plus 1 over r dR by dr star plus is equal to 0.

That is the differential equation for r star, I can multiply throughout by r square to get r square d square R by dr square plus r dR by dr plus beta square R is equal to 0. I can reduce this a little bit I am sorry beta square multiply throughout by r star square.

Now, I can reduce I can remove this parameter beta in this equation note that the first two terms on the left hand side are equidimensional in r, they have r square times d square r by dr square and r times dR by dr. So, they are equidimensional in r. So, if I multiply r by any constant, these first two terms do not change, because they are equidimensional in r. Therefore, I can make this substitution r plus is equal to beta times r star, and then I will equation get an equation in terms of r plus as r plus square d square R by dr plus square plus this into dR by dr plus r square R is equal to 0 .

So, this is the differential equation we have to solve. If you recall when we did the same thing in Cartesian coordinates this term was not there and therefore, we managed to get a solution just in terms of sine and cosine functions. In this case there is this addition term present which means that the solutions are not sine and cosine functions they are what are called they are a class of special functions called bessel functions. Special functions might seem a little difficult to comprehend at first, but they are they are no they really no different from trigonometric functions, if I were to give you sign of some angle and ask you what the value is you'd have to go look up some table.

You know the broad features of sine functions or cos functions where they pass through 0 where the derivative is 0 and how the derivatives are related to each other and so on. Those broad features you know, but to get an exact value you if you are asked you would have to go and look in some tables. Same thing with Bessel functions they have and the features the characteristic features of these Bessel functions are known, but if you actually wanted to know the value you will have to go and find it in some tables.

So, they there they are similar to sine and cosine functions except they are little more complicated they do not for example, have a constant amplitude as x goes to infinity they decrease as x power minus half they do have an oscillatory form in that sense they are similar to sine functions, but the some Bessel functions diverge at 0 some diverge at infinity and. So, on the general Bessel functions or solutions of the Bessel equations which are equations of the form x square d square by dx square plus x dy by dx plus x square minus n square y is equal to 0.

So, this is the general Bessel equation and the Bessel function solutions for these are written as y is equal to A1 times Jn of x plus A2 times Yn of x. So, these are the general solutions for the Bessel equation the 2 Bessel functions

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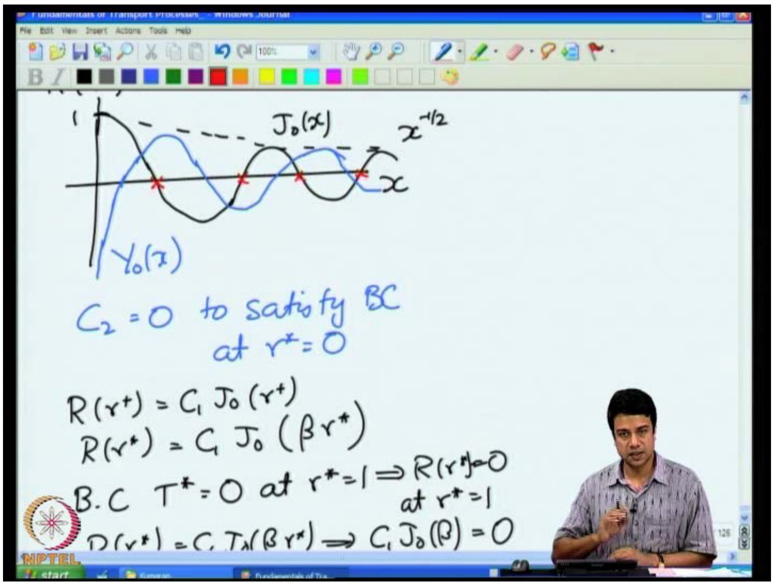

These are linearly independent solutions of the Bessel equation in a similar manner to how the solutions for d square y by dx square plus y is equal to 0 are solutions of the form y is equal to A sine x plus B cos x. Exactly analogous to this solution for sine and cosine functions these are the solutions for the Bessel equation the Bessel equation has these 2 solutions.

You can easily see that this equation is identical to this one except that n is equal to 0 in the equation on top. So, n is equal to 0 for this equation and therefore, the solution for R of r plus is equal to C1 J0 of r plus C2 times Y0 of r plus. These functions J0 and Y0 you can find it in standard texts in or in standard mathematical special functions books. I will just sketch probably how they look as a function of x J naught of x you can plot as a function of x. It starts off at 1 and then it has an oscillatory form. This amplitude actually this amplitude of the peaks actually decreases as x power minus half as x becomes large and the frequency tends to a finite value. So, this is the function J naught of x.

The other function y naught of x actually starts at minus infinity. And it has a decay similar to J naught of x. So, this is Y naught of x. So, the general solution for the Bessel equation is some combination of J naught and Y naught. Now first thing we can say is that just from the form of these 2 solutions C2 has to be equal to 0. The reason is because Y naught goes to minus infinity at r star is equal to r plus is equal to zero. However, my boundary condition requires that the derivative of the temperature has to go to 0 at r is equal to 0. So, the derivative of the scale temperature has to go to 0 at r is equal to 0. If the solution had any contribution proportional to Y naught then that contribution would go to minus infinity as r star goes to 0. So, just the symmetry condition alone tells me that because the solution Y naught goes to minus infinity at r star is equal to 0 the constant C 2 has to decrease to 0. So, I require just from the form of these solutions that C 2 is equal to 0 to satisfy B C boundary condition at r star is equal to 0.

Therefore the solution that I have is of the form R of r plus is equal to C1 J naught of r plus or r of r star is equal to C naught C1 J naught of alpha r star, no not alpha beta and this beta is, because I have defined r plus is equal to beta times r star when I did the reduction. If you recall when we did the problem of separation of variables in a Cartesian coordinate system, we had a similar solution except it was a sine function, with beta times r star in it that function beta in that case was determined from the condition that we had homogeneous boundary conditions at z star is equal to one and therefore, beta is equal to n pi.

In this particular case our Boundary Condition is that is that T star is equal to 0 at r star is equal to one. This fixes the value of beta, because this if t star is equal to 0 at r star is equal to one for all times this implies that r of r star is equal to 0 at r star is equal to one. It implies that R is equal to 0 at r star is equal to 1. Since r of r star is equal to C1 J naught of beta r star this will imply that C1 J naught of beta is equal to 0 this implies that C1 of times J naught of beta has to be equal to 0. C1 J naught of beta has to be equal to 0, means that beta has to be one of these values, where this curve crosses through 0 it has to be one of those values where the curve crosses to 0, because if I require J naught of beta is equal to 0; that means, that beta has to have a set of discrete values.

In Cartesian coordinate system we had similarly got discrete values, because the temperature at z star is equal to one had to be 0 and therefore, beta was equal to n pi in this particular case it is not n pi, but it is a set of values at which the Bessel function goes through 0 a set of discrete values discrete solutions of the equation J naught of beta is equal to 0. So, therefore, the solution beta has to be such that discrete set of beta at which j naught of beta is equal to 0.

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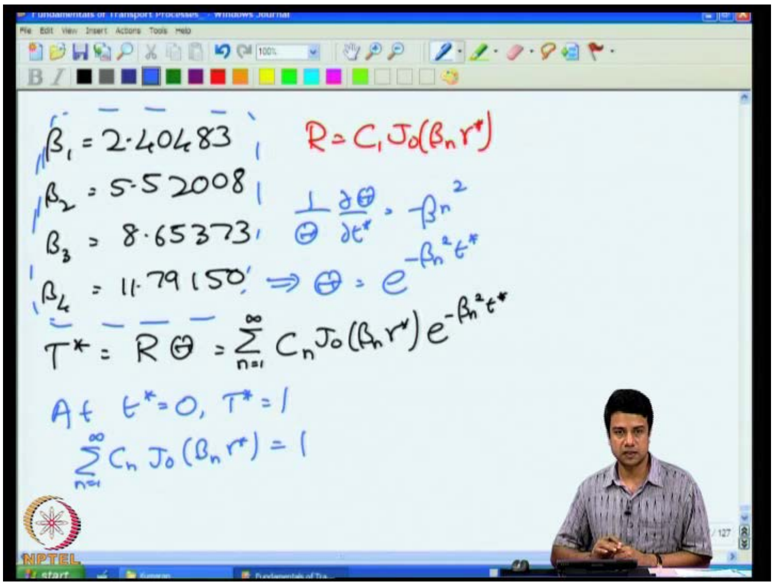

And I have the numerical values of the first few such discrete solutions can easily be obtained from standard books for example, I will have beta 1 will be equal to 2.40483 beta 2 is equal to 5.5208 beta 3 is equal to 8.65373 150 and. so on. You have a discrete set all the way up to infinity. So that means, that beta can have only discrete values at which this Bessel function goes through 0. So, this is the discrete values at which the Bessel function goes through 0.

So, that is the solution for r. R is equal to C1 times J naught of beta n r star. Where beta n is the set of discrete values how about the solution for time the equation for time is 1 over theta d theta by dt star is equal to minus beta n square, which would imply that theta is equal to e power minus beta n square times t star. So, that is the temporal part of the solution. So, the total solution T star, T star is equal to R times theta which is equal to C times J naught of beta n r star times minus beta n square t star. So, this solution for any value of n satisfies the equation this solution for any value of n satisfies the equation. That means that the most general solution is one which is the summation of all of these discrete solutions for all values of beta n that I have shown you over here.

For all these values of beta n going from one all the way to infinity the sum of all these solutions is the most general solution that we can get. So, this has constant C and n in it. How do we evaluate the constant Cn? If you recall when we did separation of variables for Cartesian coordinates we had defined an inner product an orthogonality relation which basically say that the product of 2 different basic functions integrated has to be equal to 0. So, the initial condition that we have here is that At t star is equal to 0 this is equal to one; that means, that summation n is equal to 1 to infinity of Cn J naught of beta n r star is equal to1.

How do we solve this? We have an orthogonality condition for the Bessel functions as well this slightly different from the orthogonality condition for the sine and cosine functions.

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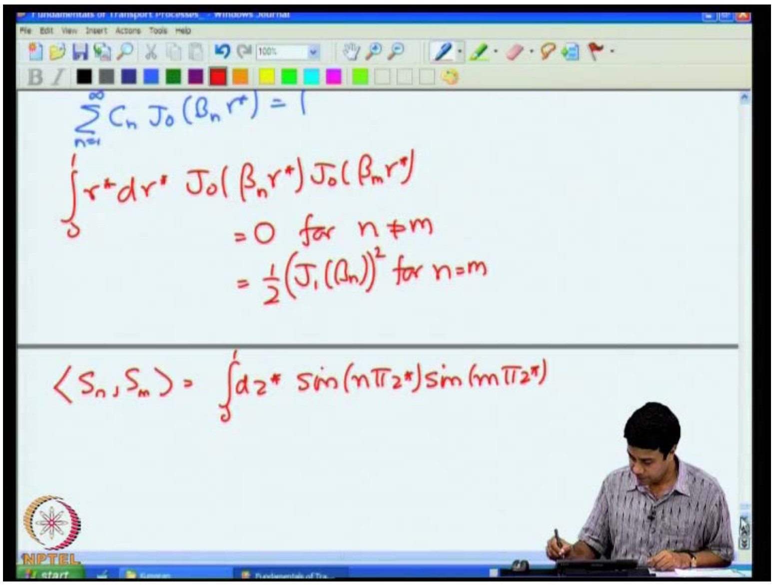

What it says is that integral 0 to 1 r star dr star. So, this r times dr that is because cylindrical surface element is r times dr times J naught of alpha n r star J naught of alpha m times r star, where alpha n and alpha m I should use betas here beta n times r star beta m times r star where beta n and beta m are 2 members of this discrete list of solutions. This is equal to 0 for n not equal to m and this is equal to half J 1 of beta n the whole square for n is equal to m.

If you recall our discussion of orthogonality relations in a Cartesian coordinate system. We had written that in short notation as Sn comma Sm is equal to integral 0 to 1 d z star sine of n pi z star sine of m pi z star.

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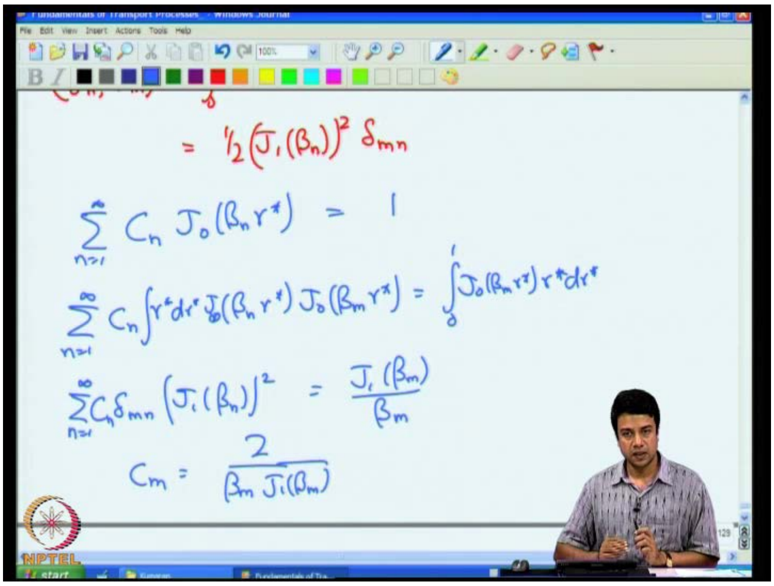

In a similar manner we can define J n comma J m for the Bessel functions in a product as integral 0 to one, note r star dr star J naught of beta n r star J naught of beta mr star. This was equal to one half only if n was equal to m and for a 0 otherwise. So, we assume this as half delta n delta nm.

In a similar manner this is equal to 0 if n is not equal to m and it is equal to half J 1 of beta n whole square times delta mn. So, this orthogonality relation is the exact analogue of the orthogonality relation for the case of Cartesian coordinate system. The basis functions instead of sine and cosine functions are Bessel functions in this case that is the only difference. The basis functions they form a complete set any function can be expanded in these basis functions these basis functions have orthogonality relations and these can be evaluated used to evaluate the constants.

So, this was the boundary condition the initial condition that we need to evaluate. So, therefore, I have the initial condition as summation of Cn J naught of beta nr star is equal to 1 multiply both sides by J m n is equal to 1 to infinity summation n is equal to 1 to infinity Cn times J naught of beta nr star j naught of beta mr star and then I integrate this over r dr .This will be equal to integral J naught of beta m r star r star dr star from 0 to 1. The left hand side is of delta m n J1 of alpha m the whole square the right hand side can be evaluated. You can do that by looking at standard tables or the standard special functions books the solution it turns out is equal to J1 of alpha n divided by alpha n J1 is the Bessel function of order 1.

On the left hand side I have delta m n times J1 of alpha m the whole square from m is equal to infinity this is non 0 only when n is equal to m I should write the right hand side as m the right hand side is an integral . So, therefore, I get C m is equal to 1 by beta m you know I am sorry I should get 2 by beta m J1 of beta m. So, this is the final expression for the constants obtained using the same orthogonality relations that we had used earlier.

So, once again, in this case the solution in the spatial coordinates is not quite sine and cosine functions it is slightly more complicated than a sine and cosine functions. Yet the entire machinery that we had developed in a product orthogonality relation base is set expansion in that base is set all of that the formalism that we had developed for the Cartesian coordinate system still works here except that I have to define the inner products slightly differently. I have to define the inner product based upon integral of r star times dr star because that is equivalent to integral in area identical in cylindrical coordinates except for that I can use exactly the same procedure to solve for this heat conduction problem in a cylindrical coordinate system as well.

We get discrete Eigen values in this case as well there is a discrete set of beta n's the values at which the Bessel function passes through 0. So, that discrete set still carries over for this system as well the origin is the same except that the operator for heat conduction is slightly more complicated is 1 over r d by dr of r times dt by dr and that results in a slightly different form of the basis function, but we the same formalism that we used the inner products the orthogonality relations can still be used in this case as well. So, this completes the separation of variables part of the analysis in cylindrical coordinates next we will look at a problem of practical interest that is the flow in a pipe I said we can get friction factor versus Reynolds number relations for that case we will derive that in the next class after briefly reviewing what we have done so far. We will see you in the next lecture.