Chemical Reaction Engineering

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Lecture No. # 36
Stability Analysis – Basics

Friends, let us continue our discussion on stability of chemical reactor. But before we go to this example little background if you recall in out last session we saw that when we are looking at the behavior of an adiabatic stirred tank reactor or even a stirred tank reactor with a jacketed vessel in which exothermic reaction is taking place. There is a possibility that there will be more than one steady state depending upon the operating condition that we choose.

We also saw that there were three steady states for example, in a case of adiabatic reactor. Now out of these two steady states are inherently stable as we called them last time and one was inherently unstable. So what we will do in todays class is to look at some formal definition of stability and look at the condition, which needs to be satisfied for a steady state to be stable or steady state is unstable.

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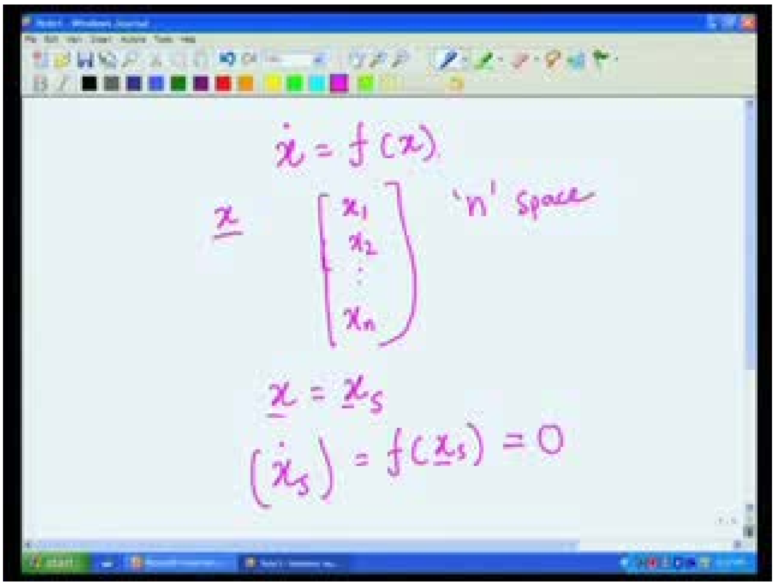

So let us start this discussion by looking at a system whose dynamics is given by x dot equal to f of x. Note here we assume that this is the dynamics which describe how system which is represented by state vector x. So this is a vector in n-dimensional space so it has component for example, x 1 x 2 up to up to x n. So this is a n-dimensional space that we are we are looking at. Now we would like to know the steady state, which satisfies this dynamics so let us say that that steady state is x of x of s again a vector and by very definition of a steady state it implies that x s dot when x is equal to x s which is f of x s, which is zero that is steady state implies that dynamics has disappear and we have reached a steady state which is invariant with time and hence the time derivative is derivative is zero.

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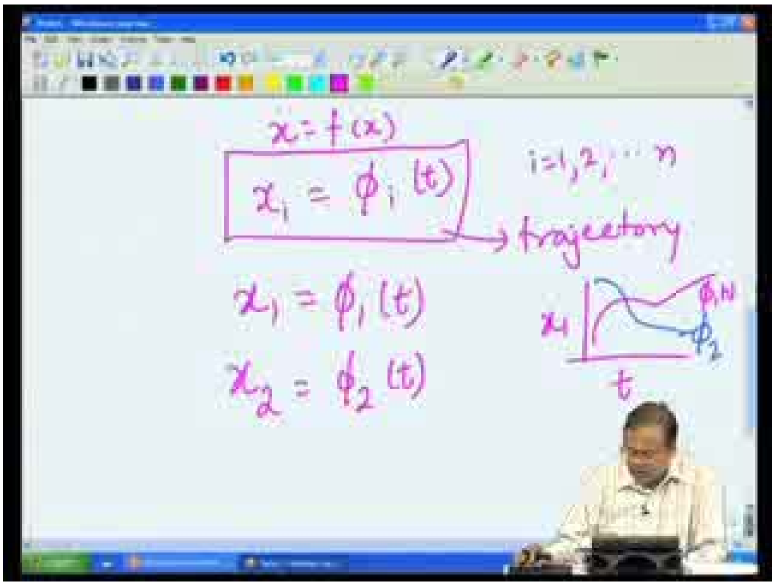

So we would like to know whether this steady state x s is stable or otherwise. This is the question that we are trying to answer in todays session. First thing that comes to mind is what is meant by stable? So, what is a proper definition of a state to be stable steady state to be stable? So let us try to look at that by considering our dynamics let us say x dot equal to f of x and let us say that it is solution or it is trajectories are given by some function phi of t that is we do not know what this solution is but let us say that that solution is represented as some function phi of t where i is 1, 2 up to n our n-dimensional space. So what this trajectory or what these solutions imply is a moment in n-dimensional space of how x will change as the time progresses.

This x i as phi of t is refer to as trajectory of x i that is how i changes with changes with time and we are now interested for example, in looking at let us say we have two-dimensional space for our simplicity will keep the discussion to two-dimensional. But the same applies idea applies to higher dimension problem also. We have one solution x 1 which is phi 1 of t and let us say x 2 is phi 2 of t. So if i were to plot x 1 versus time let us say this is that this is that solution and this correspond to x 2 versus time some function. So this is my phi 2 and the first function is let us say phi 1 of t.

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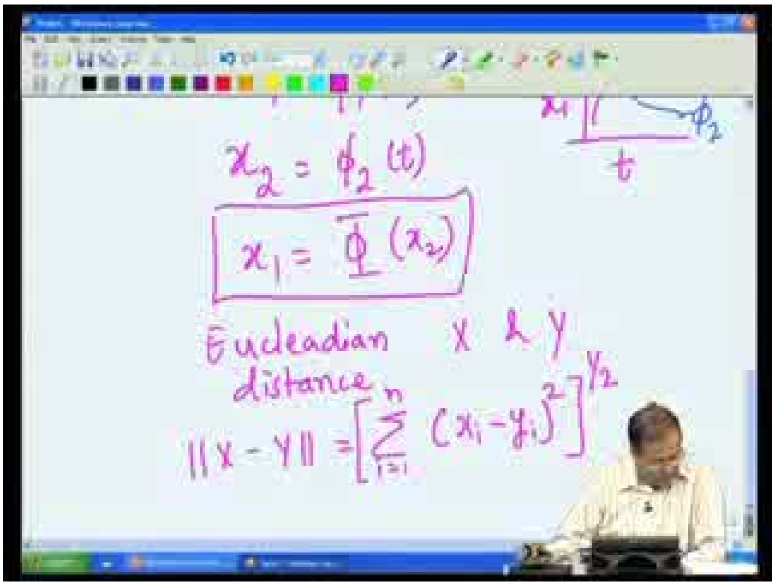

We now say that given this solution we may not know that solution but let us assume for sake of sake of argument that we know this solution. Then I can invert these relationships that is I know x 1 as a function of time, I know x 2 as a as a function of time. So I can always generate x 1 as some function of x 2 and this is what is called as face space that is representing one solution in terms of another variable. Few definitions before us before we actually look at look at the definition of stability itself. So all we have done in writing in this x 1 as a function of x 2 is having known that x 1 is phi 1 of t x 2 is phi 2 of t, I have just eliminated time and expressed x 1 as function of some function of function of x 2.

Let us let us introduce a concept of what we call Eucleadian distance between points x and y in general and we will make use of this Eucleadian distant in describing what is meant by stability and so on. So Eucleadian distant between point x and y is defined as summation x i minus y i the whole square i going from one to n components. So we have two point x and y and the Eucleadian distance between this two is what we call norm of x minus y and that is summation x i minus y i the whole square and this summation is over all the components n components of our space state vector and raise to raise to power half.

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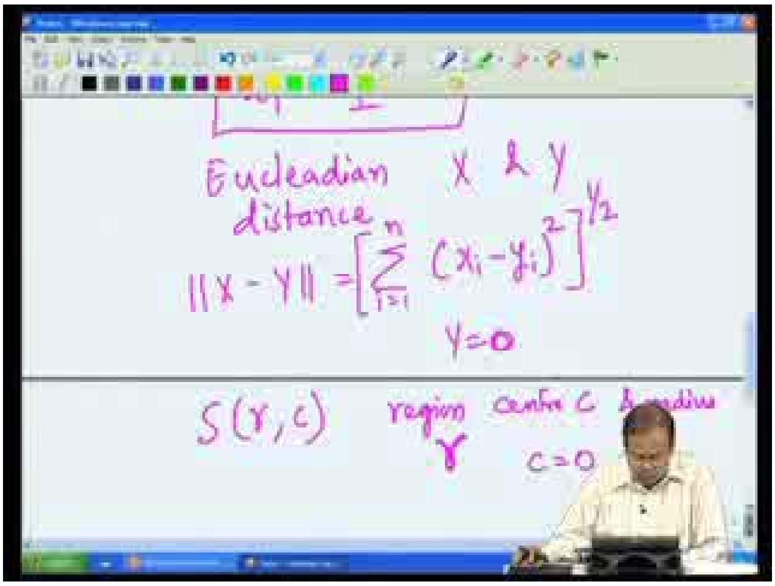

One more one more idea before we go to the definition of definition of stability. Let us define a region s of r and c which is denotes a region for a two-dimensional space it will be a spherical region of centre c centre c and radius r and c is said to be the critical point. Now why we do say c is the critical point with c equal to equal to zero. In our in our considerations of stability of a steady state what we are going to what we are going to assume is y is zero and it will become clear why y is y is zero when we actually look at look at one particular system. So if y is zero than we can call the steady state solution as x equal to zero and this will become clear when we actually look at look at the problem on hand namely stability of a steady state.

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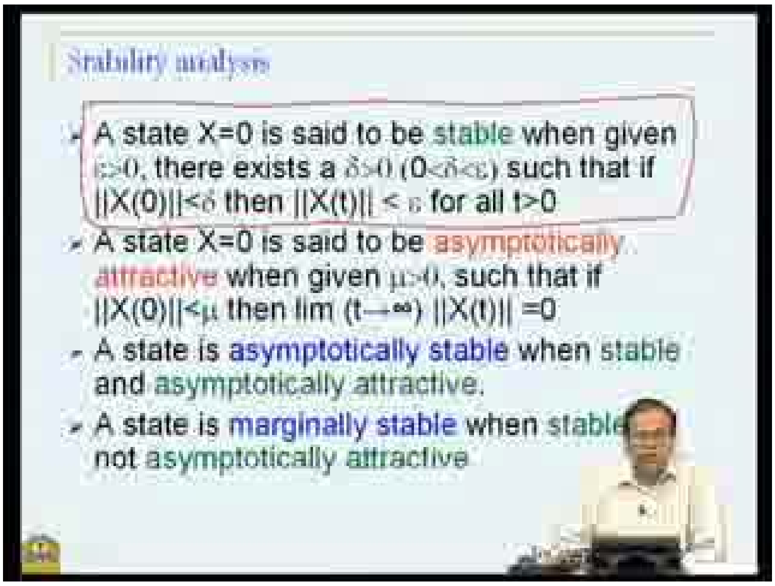

We will start with the first definition and that pertains to this particular part and I will explain what it what it what it means in with an example but let me just read it out to begin with a state x equal to zero and we just now saw that by putting y equal to zero our steady state has become x equal to zero. A state x equal to zero is said to be stable when given some epsilon which is greater than zero there exist a delta greater than zero and delta between zero and epsilon such that the norm of if norm of x zero is less then delta than the norm of x of t. We have defined this now just now is less than epsilon for all time t greater than greater than zero.

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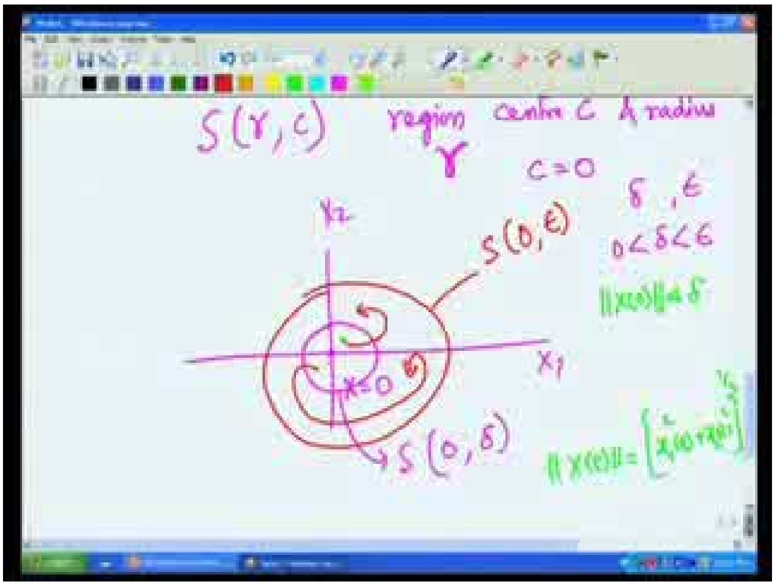

Now this will simplify this for a two-dimensional problem to see what is actually the definition of definition of stability, so let us go back to two-dimensional system and first of all first of all let us let us say that we have a steady state x equal to zero and we are trying to look at whether that steady state is stable or not. So let us say that in a two-dimensional space let us say we have only x 1 x 1 and x 2. Let us say this represents my steady state that is x equal to zero. Now I called this steady state stable for the case where I can find some delta and some epsilon such that zero less than delta less than epsilon such that let us say that circle is s of centre zero and radius delta so this radius is delta.

Now around this steady state point x equal to zero I have a circle of radius delta and if I can say that if I can find for all the points in this circle wherever I start if my norm of so for all x of zero less than delta. Remember this norm of x of zero with this definition when y is zero this is nothing but for will be nothing but x 1 at zero value square plus x 2 zero square raise to half. So this being less than delta which for two-dimensional system now are represented by this by this point delta. All my x of t now what is my x of t norm of x of t is nothing but x 1 of t the whole square plus x 2 of t the whole square because our other point is zero raise to half remains within this circle of radius epsilon.

So let me say that this represents my s of zero and epsilon then starting with this x of zero the green point which I said, my system is said to be stable if solution as a function of time remains in this circle of radius epsilon for all times from time t equal to zero to time t equal to t equal to infinity. This is what I called a stable steady state. Once again the idea is idea is simple I want to know whether my x of zero is stable or not that stable definition formal definition says that for some region in the neighborhood of x equal to zero, if my initial starting point lies in this small neighborhood.

Then my dynamics requires that if the solutions for all times x of t t greater than zero remains within this neighborhood of epsilon, then I say that this point is point is stable. So what it means is suppose I start over here the solution may look different but for all times if I remain in this in this region I would call such steady state x equal to zero as a stable steady state.

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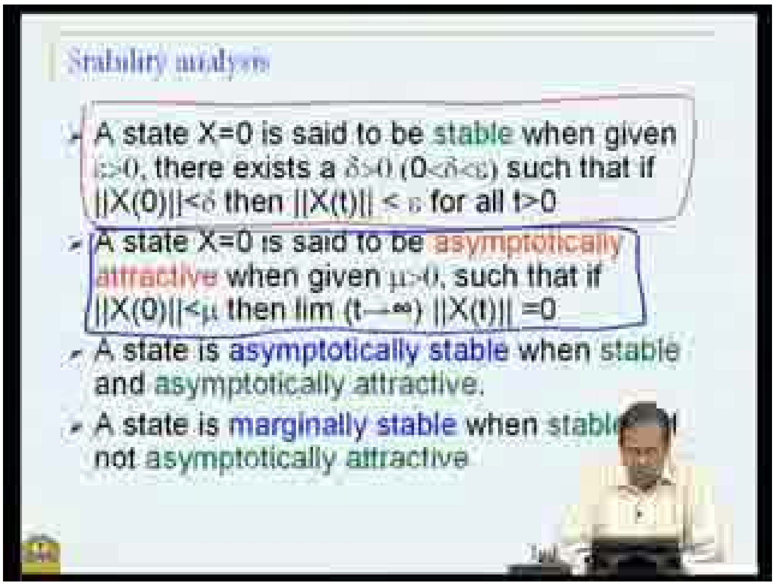

Now the second definition which talks about whether the steady state is attractive and this is this particular portion once again i will read it out to you, a x equal to zero is said to be asymptotically attractive when given mu greater than zero such that for all x that is norm of x less than mu the limit as t goes to infinity the norm of x goes to goes to zero, this is when we say that this system is asymptotically attractive.

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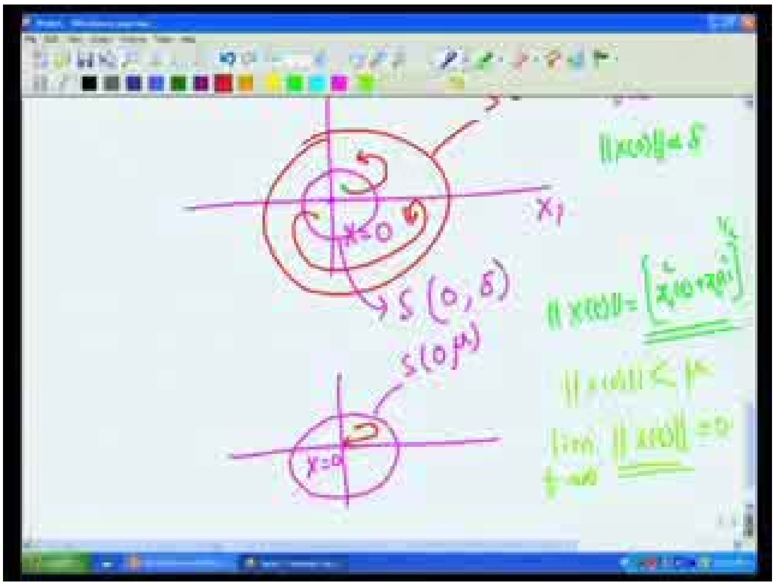

Now let us go back and see what is what is meant by asymptotically attractive state. This is what this is what we said about asymptotically stable. Now for asymptotically attractive state what we are saying once again for x equal to zero if we can find some region as denoted by s of zero and radius mu and the starting point somewhere over here then for all for all x of zero less than mu limit as t goes to infinity norm of x of t is zero if we can satisfy that condition than we say that our state is asymptotically attractive. Now what is the meaning of norm of x of t going to going to zero?

If we look at our definition of norm this can happen only when x 1 and x 2 individually goes to zero because norm of x of t is x 1 square plus x 2 square and summation raise to square root that is power half. So for norm of x if t to zero at long times both x 1 has to go to zero, x 2 has to go to zero that means that is our steady state. What is our steady state, x equal to zero that means x one equal to zero x two equal to zero. So what this attractive definition says is that at this starting from here at long times we actually go to the go to the state x equal to zero.

The definition of stability alone said that all such starting points we remained in the close neighborhood of close neighborhood of x equal to zero. The definition of attractive state asymptotically attractive state says that for all such starting points we actually go to steady state x equal to zero as time progresses.

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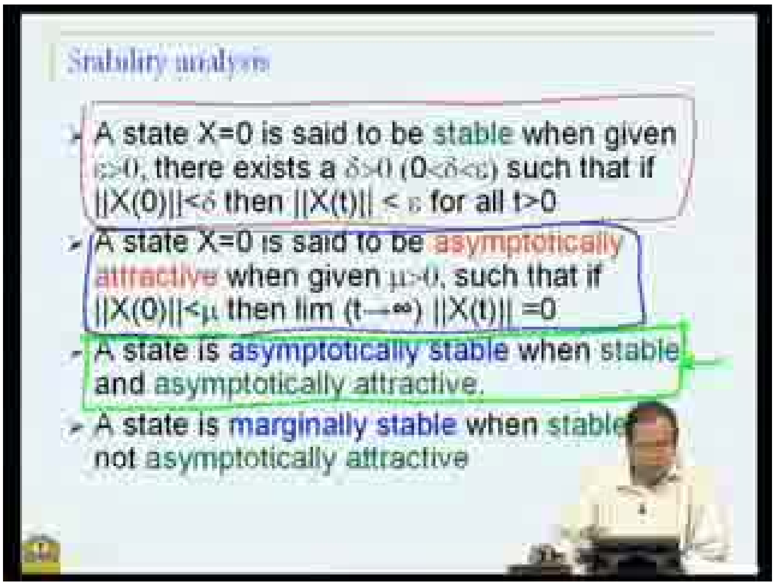

Now comes the definition of our asymptotically stable steady state which says that a steady state a steady state is said to be asymptotically stable and when it is stable and asymptotically attractive. Now state is said to be marginally stable when it is stable but not asymptotically attractive. Now at first glance this definition means seem to have some redundancy in it saying that it should be stable and asymptotically attractive does not definition of stability indirectly or directly imply attractive and definition of attractive directly or indirectly encompasses the definition of stability. The next set of examples will convince you that that is not necessarily the necessarily the case.

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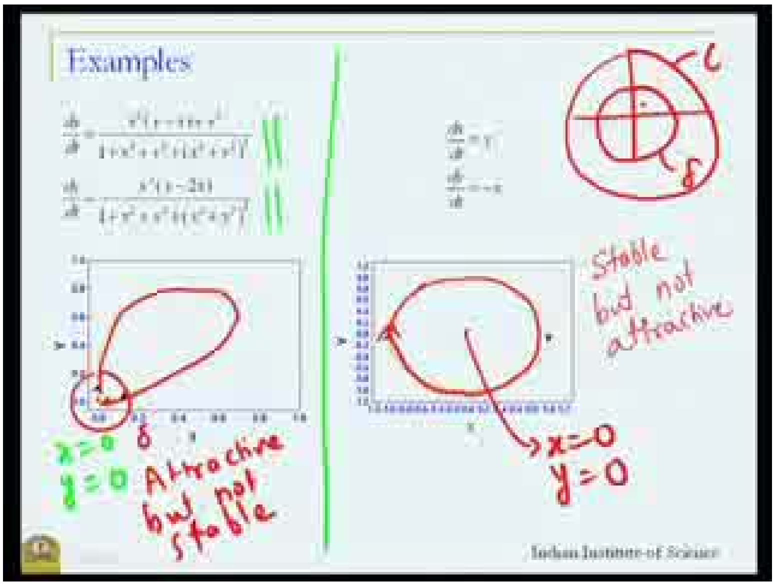

So what we are seeing here is example on the first hand on the first hand an example of whose dynamics are given by this x and y and it is steady state value is when x is equal to zero y is equal to equal to zero. Now if you actually look at the dynamics of this system and try to try to try to look at the trajectories it turns out that you try to take any starting condition as time progresses it actually comes back to zero and zero, so the system is attractive. But from a definition of stability of a system remember, what was the definition of stable system? For any starting point within this region delta our solutions always remained within epsilon.

Now it turns out for this particular example on the on the left hand side it is impossible to find for this given this stating point in some region delta it is impossible to find an epsilon such that the solution is entirely within this within this epsilon region. So it does is a attractive state that is solutions eventually go to x equal to zero but it is not a stable state so this is a example of attractive but not stable. On the right hand side is another example dx dt equal to y and dy dt equal to x whose again steady state solution is x equal to zero and y equal to y equal to zero.

Now if we if we start with some starting point let us say let us say over here. The dynamics of this system keep it on this circle or a limit cycle. So the solution never actually converges so it never goes to goes to zero so that never happens. So what does it mean it means that for this particular example we can always find an epsilon and delta such that the long time solution is within this bounded with in this region epsilon? But so it is a stable steady state but it is not an attractive steady state. Because the solution does not does not eventually go to go to x equal to zero. So this is stable but not attractive.

And hence to have an asymptotically stable solution we need both state has to be stable and asymptotically attractive. So we call a steady state asymptotically stable only when the stable it is stable according to this definition and it is asymptotically attractive according to according to this definition. So that is that is that is how we define an asymptotically stable. So when we are talking about the talking about the stability of a steady state or otherwise, we are looking at asymptotically stable steady states of course, a system is said to unstable when it is not stable I mean that is that is that is understood.

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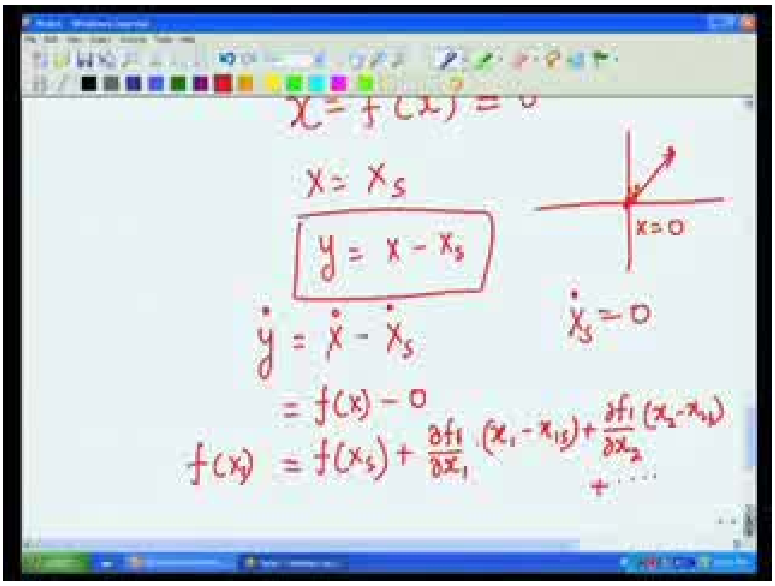

Now let us go back to our dynamic and try to see how can we how can we find out the stability of steady state for which we have a dynamics which is which is given by given by x dot equal to f of x. So we want to find the stability of the steady state that is when x dot equal to zero when x at that time is some steady state x s. So we would like to know how to find the stability of such steady state. Now before we actually find the stability and determine the criteria we are going to do a small variation in the sense that we are going to look at how the behavior of the system is away from the steady state value.

Remember when we talked about the definition we said that this is my this is my steady state this is my steady state and this is my starting point in the close neighborhood of this of this steady state. We of course also said that we are going to call x equal to zero as our as our steady state steady state value. So what are we what are we going to do we are actually going to look at the perturbation of my system because what is what is the meaning that I start from this particular point that means I have perturb my steady state by some quantity in this case by a small quantity so let me enlarge it so a small perturbation.

So I am going to look at what happens when I introduce a small perturbation what happens to what happens to my solution. Now in order to in order to keep the discussion with a with a x equal to zero as my steady state I am going to define the variable y as x minus x s. So if i if i do that then I can write y dot as x dot minus x s x s dot. I am going to further going to write I know x equal to x s is my steady state I know x s dot is equal to zero and I know my x dot is f of x and this quantity is quantity is zero.

So what I am going to do now is I am going to write f of x in the following manner. I am going to expand using tailor series tailor series expansion and this would around the steady state value x of s. So this will be f of x of s plus remember x is a n component quantity. So i will have terms like del f i or let me let me write it f of x 1 to begin with so f of x of s plus I am going to expand this using tailor series tailor series expansion. So this will be del f 1 del x 1 into x 1 minus x 1 s plus del f 1 del x 2 into x 1 minus x 1 minus sorry x 2 minus x 2 s and so on.

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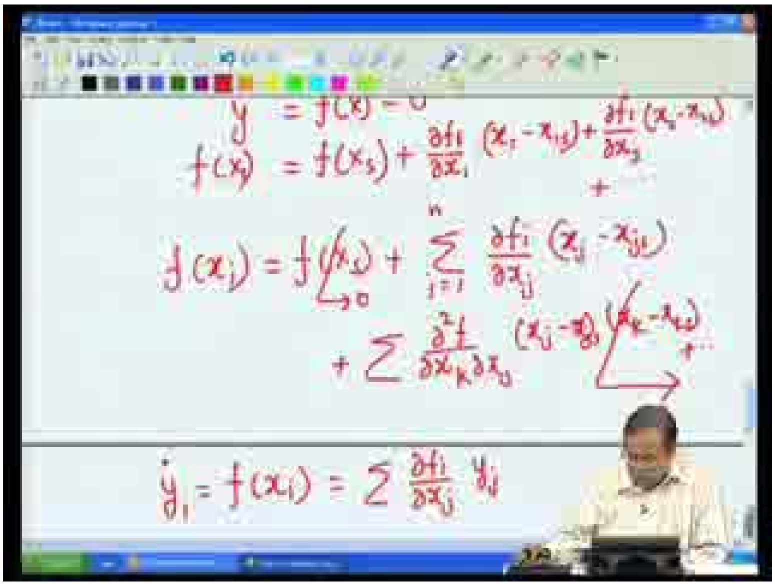

In short if I now generalize this then I can write f of any x i as f of x s plus summation del f i del x j x j minus x j s j going from one to n plus I will of course, have second order term higher order term so i will have terms like del square del x k x j x j minus zero x j minus x j s into x k minus x k s and so on, a simple tailor series tailor series expansion. So now knowing this and making another assumption that I am going to neglect all higher order term and keep only the first order first order first order term that means I am going to neglect all these all these terms and then go back to my y dot equation go back to my y dot equation, which is which is actually nothing but f of x and I am going to put for that so that would be y dot equal to this has this has gone to zero or if i want to write y i dot for example, that would be f of x i which will be summation del f i del x j and x j minus x j is nothing but is nothing but y j.

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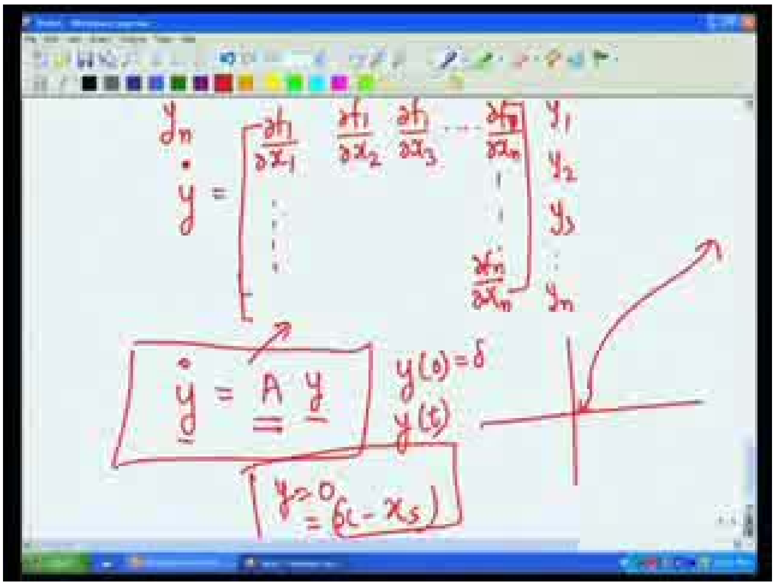

So this if I write for all such i y 1 y 2 y i up to y n and put all these things together I will have the dynamics the perturbation dynamics y dot defined as if I just expand this terms we have y 1 y 2 y n and the first term here will be del f 1 del x 1 del f 1 del x 2 del f 1 del x 3 del f n del x n. So all these similar quantities up to del f 1 del x n del f n del x n. So I will have I will have all these all these terms which now I can write in a more compact form as y dot equal to A of y where A is my Jacobean matrix del f 1 del x 1 del f 1 del x 2 del f 1 del x 3 and so on and so forth up to up to.

So this is my is my dynamics of my perturbation variable y and I am interested in knowing what happens what is the steady state solution of this y equal to zero. Remember our y is y minus x s. So when x reaches the steady state x s my y is y is zero and this will then tell me that if I start with some y of zero having a value delta let us say something like this. What happen to my y of t does it go back here does it go somewhere else and so on and so forth. So that will then tell me what is the solution or what is the status of this steady state f of x of x of s.

Let me let me recap what we what we saw just now before I go on to describing the solution of these of these or how do we how do we find this condition for condition for stability. Let us put things in perspective we have system whose dynamic is given like this whose dynamic is given in this particular manner. What we are trying to trying to find out is if the steady state x equal to x s is the steady state solution of this such that x s dot equal to f of x equal to equal to zero is it is it stable or otherwise that is the question that we have in front of us. In order to make this question little less abstract let us go back let us go back to our problem of reaction engineering and let us try to make this little less abstract so let me let me see if I can.

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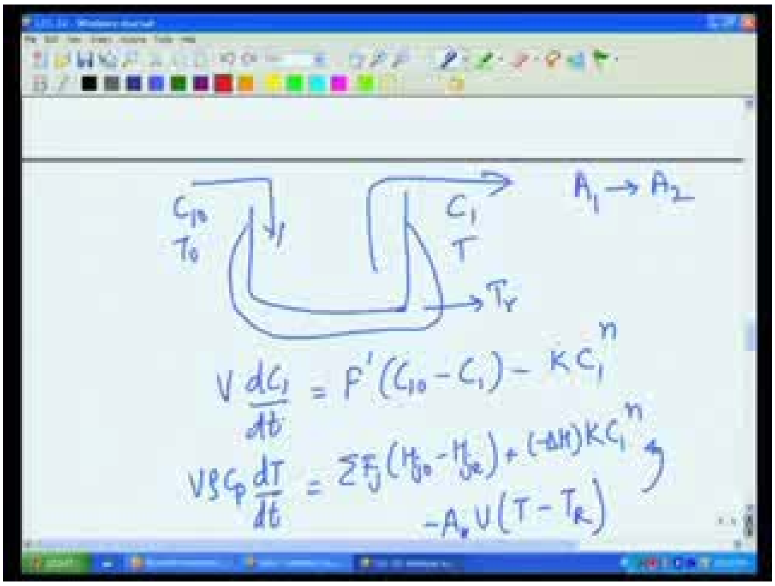

I am looking at stability of a CSTR so from our last session last session we know that this is my CSTR operation for which mass balance equations and energy balance equation are given in this particular manner. dc 1 we are looking at reaction A 1 going to A 2 in this CSTR which for which we are trying to maintain a steady state and by providing a cooling jacket or a jacket in which a coolant is being circulated at a temperature T r and we are trying to find out the steady state operations and what is what is possible or otherwise.

So what we did last time was we define these non dimensional quantities we define these non dimensional quantities we define these non dimensional quantities and converted this dimensional form of mass and energy balance into the non dimensional form of mass and energy balances mass and energy balances. Now we will take this mass and energy balances and represent it in the form of heat generation term and heat removal term as we have done over here. So what I am going to do is I am going to rewrite those mass and energy balances just to just to keep things in a proper perspective.

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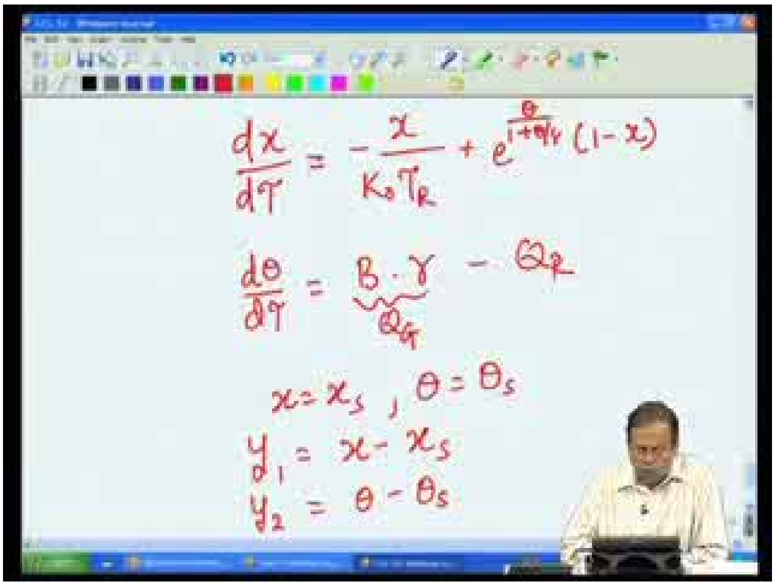

So we can and I am going to take only first order reaction simply for a matter considerably. So we had our dimension less mass balance in terms of dimension concentration as one minus x and we had our dimensionless temperature as B into r minus Q R and we call this B into r term as Q G. We are interested so first of all define our steady state solution as x equal to x s some dimensionless quantity and theta equal to theta s. So then we will define the perturbation variable we are looking at only two-dimensional system over here that makes matter little easy so we will define our perturbation variable y 1 as x minus x s and y 2 as theta minus theta s.

So given this going back to going back to this what is our what is our dynamics x equal to f of x x equal to f of x for example, if I write x and this is not to be confused with dimensionless quantities the notation is little unfortunate but let us let us stick with that x with theta my x dot equal to f of x my f will be f 1 and f 2 where f 1 is this whole quantity and f 2 is this whole quantity or in other words my f 1 f 1 is minus x by k zero tau R plus e raise to theta by one plus theta by gamma into one minus x and my f 2 is B into r minus Q R.

So these are my f 1 and f 1 and f 2. Then this is my steady state solution x equal to x of s so that steady state steady state solution last time we saw that steady state solution are these three steady state solution or depending upon operating condition. So this is when Q R is equal to Q G and so on. So I am actually trying to look at look at those solutions x equal to x s, theta equal theta s and for which I define my perturbation variables y 1 and y 2 as x minus x s and theta minus theta s such that I can write I can write my solution of x dot equal to f of x and in terms of in terms of my dimensionless quantities my y dot my y dot which is A of y this is the equation I am looking for can now be written in the following manner.

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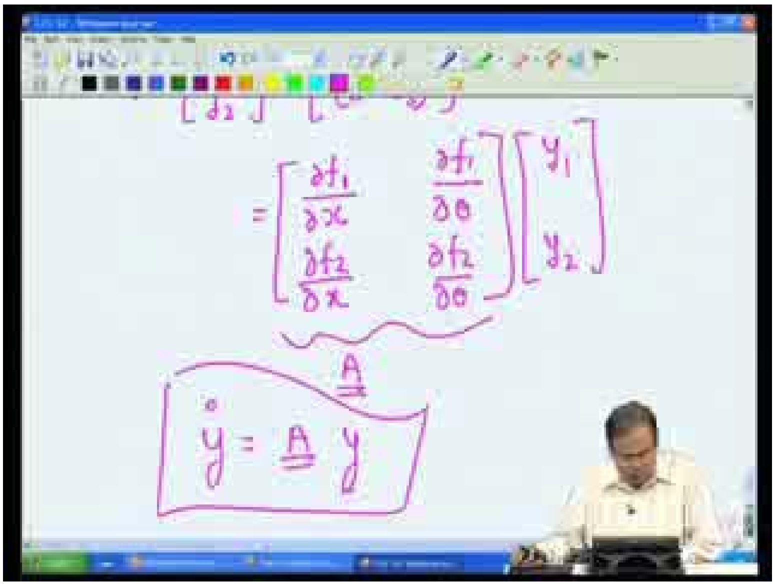

Remember our y dot which is y 1 dot and y 2 dot which is equal to in the real terms dimensionless concentration and dimensionless temperature that is deviation from the steady state value. This is the real quantity that we are looking at and these are my f 1 and f 2. So this quantity this quantity is now nothing but del f 1 del x del f 1 del theta del f 2 del x del f 2 del theta into y 1 and y 2 where what is del f del x derivative of this f 1 with respect to x del f 1 del f theta derivative of f 1 with respect to theta and so on and this vector matrix we called as Jacobean matrix A bar and wrote the dynamics y dot equal to and we are interested in looking at solution of this of this y. Now how do we get the solution of this y?

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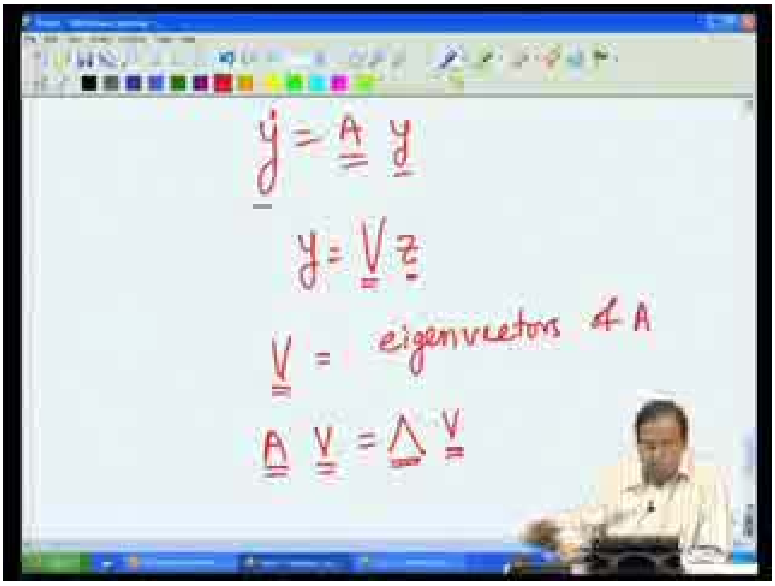

To do that let us let us look at our dynamics y dot equal to y is the vector and A is a matrix. Now I want to look at look at solution of this of this matrix. Let us define let us define y and we will we will come to that in a minute what is all this y as matrix V into z. what is matrix V? Matrix V is the matrix of Eigen vectors of Eigen vectors of A. What are what is Eigen vectors matrix of Eigen vectors, given this matrix A it satisfies this particular condition this you know from your mathematics.

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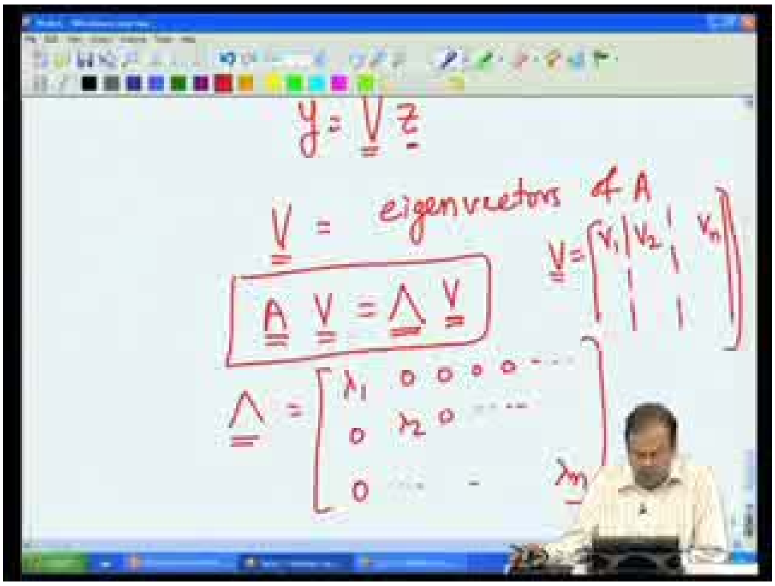

Where what is what is the matrix lambda matrix is a diagonal matrix of Eigen values lambda 1, lambda n. So and what is this what is this matrix we want so there are lambda 1 lambda 2 lambda n represent n Eigen values of this Jacobean matrix A, then V 1 is the Eigen vector corresponding to matrix Eigen value lambda 1. So matrix consisting of V 1 V 2 these is this makes my Eigen vector matrix V and we know this from the definition of Eigen value.

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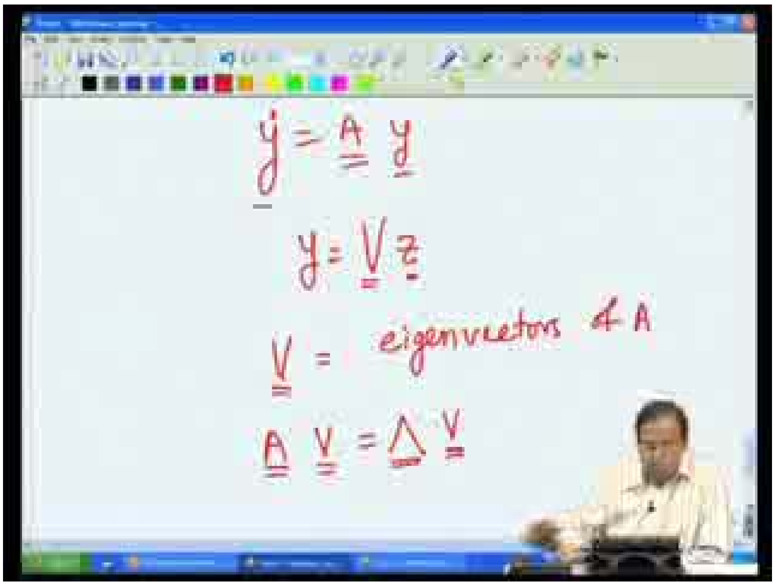

So if we now look at look at our y dot equal to A y and define y equal to V into z then what do we get? Then we get y dot which is V z dot is A V z by putting e for y equal to V z on both sides of these equation. So what do I get I get y dot or other let us not worry about y dot anymore I get I get V z dot equal to A V z. So if I post pre-multiply both sides by V inverse I get z dot equal to V inverse A V z.

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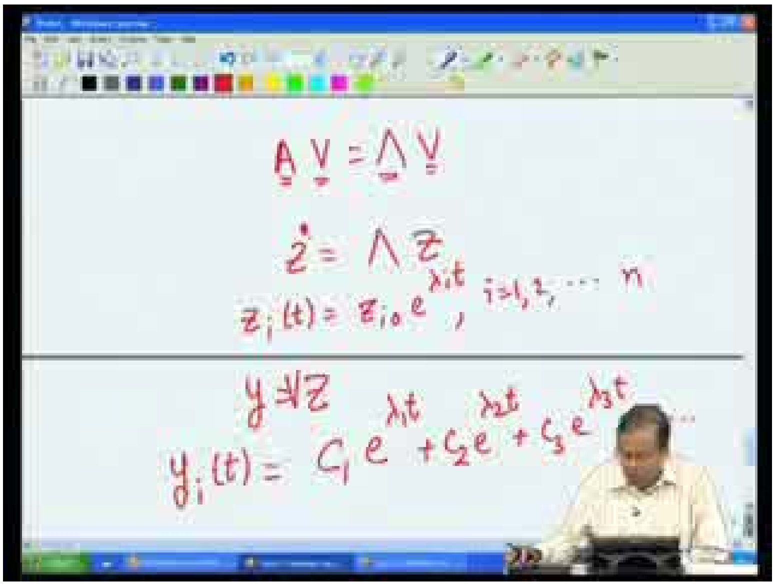

What is B inverse A V? Remember our A V equal to lambda V, where lambda is a diagonal matrix. So this gives us V inverse A V is nothing but z dot equal to equal to lambda z, lambda is a diagonal matrix so we can write the solution of this as z i of t is some z i zero e raise to lambda 1 t and similarly or lambda i t so for i equal to 1,2 up to n. Or in other words in other word what do we have here we have define our y as z V into V into z and individually z solution is given like this it implies that the solution y of t is given by some constant c 1 let us say e raise to lambda 1 t plus c 2 e raise to lambda 2 t plus c 3 e raise to lambda 3 t and so on. That constant will come from all this combinations of Eigen vector and z i zero values and so on.

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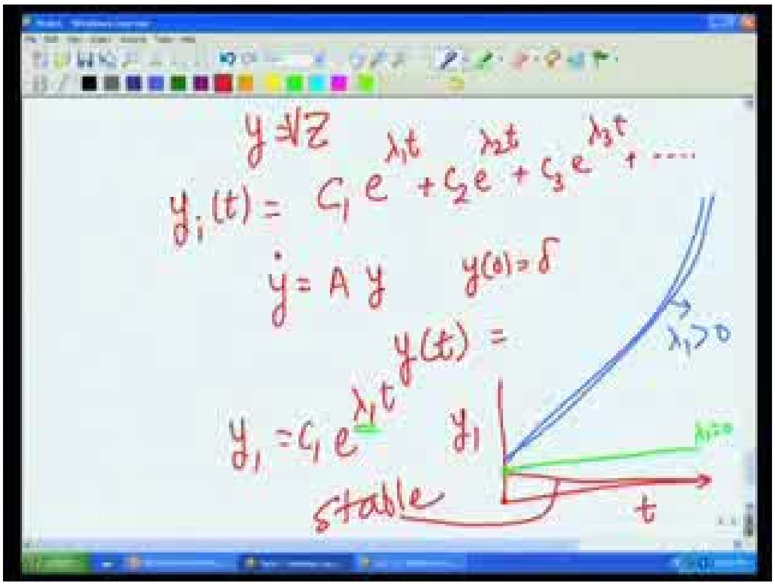

So what do we what do we what do we get here. Now what are what are we interested in we are interested in looking at the dynamics of this solution given y of zero adds up value delta let us say we know that this components individual components y i are given by summation of this summation of this e raise to lambda t terms. We are interested in knowing what happens to y of t what happens to y of t? Now even without even without knowing much of mathematics we can take a simple example let us say that we have just had one-dimensional problem lambda 1 t.

Now suppose y is equal to zero this is against time and this is I am plotting y 1 y equal to zero is the solution. So suppose I am start at t equal to zero I start somewhere over here. Now what will happen to as time progresses will depend upon what is the magnitude of this lambda 1 or what is the sign of lambda 1. Suppose lambda 1 is zero then what happens solution will remain something like this. This is for lambda 1 equal to zero. Suppose lambda 1 is a positive term then what will happen to this solution as time increases y 1 will keep on increasing I do not know in the in the exponential form in the exponential form.

So this is for lambda one greater than zero. But what will happen if lambda one is less than zero for lambda one less than zero as time increases y one will go to zero and is not this is what we looking for our stable system indeed that is given a small perturbation if I go back to my original steady state that is it is stable and attractive I call such system as asymptotically stable and this simple example would have convinced you that that depends upon the sign of the Eigen value.

So to summarize we say that system is asymptotically stable if real value because Eigen values can be complex. So we are interested only in the real part of the Eigen value. so if real part of the Eigen value is negative for all Eigen values not just one or two but for all Eigen values then we call such system as asymptotically stable. If among all n values if anyone is positive the real part is positive the system is unstable, if the real part is zero we cannot say anything about stability of this of this system. We will stop here for this session but in the next session we will look at the behavior of this system and try to analyze whether our stable our steady state will got for our CSTR problem whether they are stable or unstable. Thank you.