Continuum Mechanics And Transport Phenomena
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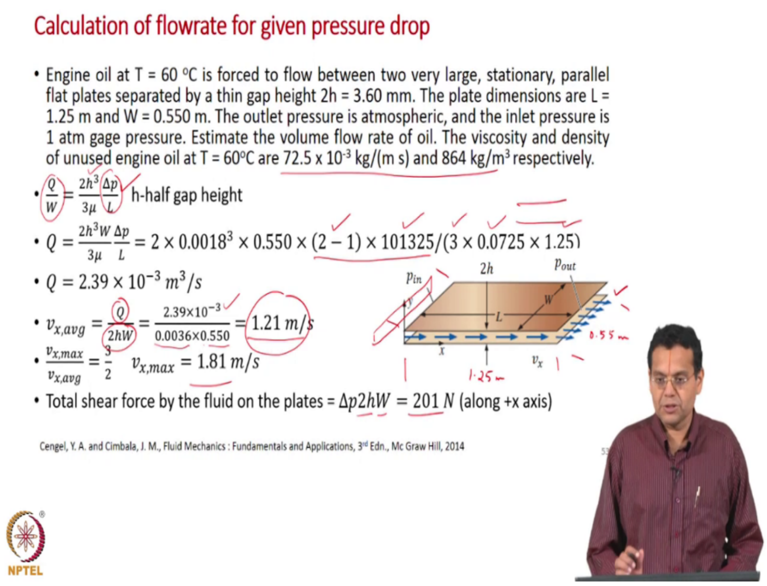
Lecture - 96
Planar Poiseuille Flow: Shear Stress Distribution

Example: (Refer Slide Time: 00:14)



Now we will discuss a numerical example where we apply the equations which we were derived for the case of plane Poiseuille flow. Another objective of this example is to discuss the shear stress distribution in detail, two main objectives I would say. Let us read the example engine oil at a temperature of 60 degree centigrade is forced to flow between two very large, stationary, parallel flat plates separated by a thin gap height 2 h = 3.6 millimeter and that is what is shown here to very large. why does it say very large?

Remember we said our width is very very large so, that we need not consider variation along the z direction and along the length if the length is very short, then there will be region where the profile will keep changing along the flow direction.

So, if you consider the very long plate somewhere let us we are in between and then there will not be any and effects and then the fully developed profile is a very good approximation. So, that is why the question says between two very large stationary, we have been using the word fixed parallel plates which means that there is no change in the distance between them the vertical distance between them. And of course, the gap is thin and the gap height is given to us.

The plate dimensions are the length of the plate is given as L = 1.25 meters and the width is given as W = 0.55 meters. The outlet procedure atmospheric pressure and the inlet pressure is one atmospheric gauge pressure which means it is two atmospheric absolute pressure. Estimate the volume flow rate of oil.

Remember we discussed the equation relating pressure drop and flow rate has two uses or two different ways of using it either you specify the pressure drop find out the flow rate that is this question or give the flow rate find out the required pressure drop. And we are giving the properties the viscosity and density of unused engine oil at the operating temperature of 60 degrees are given by, viscosity = 72.5 x 10^{-3} kg/(m.s) and density = 864 kg/m^3.

Solution:

So, it is a simple substitution as I told you the ideas to get some field for the numerical values.

$$\frac{Q}{W} = \frac{2h^3}{3\mu}\frac{\Delta p}{L}$$

This is an expression which we derived relating the flow rate and the pressure gradient. Let us look at the expression it is flow rate per unit width, h is a half gap height in our nomenclature and $\frac{\Delta p}{L}$ is a pressure gradient. So, let us rewrite for Q

$$Q = \frac{2h^3W}{3\mu}\frac{\Delta p}{L}$$

Now, 0substitution the given values,

$$Q = \frac{2h^3W}{3\mu}\frac{\Delta p}{L} = \frac{2\ x\ 0.0018^3\ x\ 0.550\ x\ (2-1)\ x\ 101325}{3\ x\ 0.0725\ x\ 1.25} = 2.39\ x10^{-3}\ \frac{m^3}{s}$$

And of course, we get a better idea if we expressed in terms of average velocity.

$$v_{x,avg} = \frac{Q}{2hW} = \frac{2.39\ x\ 10^{-3}}{0.0036\ x\ 0.550} = 1.21\frac{m}{s}$$

We also found out the relationship between the maximum velocity and the average velocity so,

$$\frac{v_{x,max}}{v_{x,avg}} = \frac{3}{2}$$

$$v_{x,} = \frac{3}{2} v_{x,avg} = \frac{3}{2} x1.21 = 1.81 \frac{m}{s}$$

We also found the expression for the total shear force by the fluid on the plates which was

$$= \Delta p 2hW = 201\ N$$

It is shear force and of course, direction you have discussed few times it is along the positive x axis.

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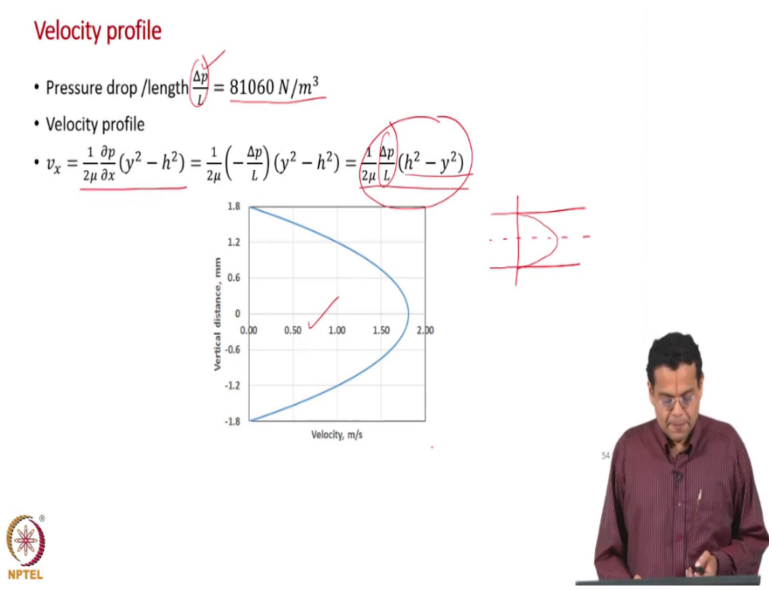

Now, we can draw the velocity profile now, for that we need the pressure drop per length which is a pressure gradient,

$$\frac{\Delta p}{L} = 81060 \frac{N}{m^3}$$

Now, let us draw the velocity profile, the expression for velocity profile was found out to be

$$v_x = \frac{1}{2\mu}\frac{\partial p}{\partial x}\left(y^2 - h^2\right) = \frac{1}{2\mu}\left(-\frac{\Delta p}{L}\right)\left(y^2 - h^2\right) = \frac{1}{2\mu}\frac{\Delta p}{L}\left(h^2 - y^2\right)$$

And using this expression for velocity profile, you can draw this velocity profile (as shown in the above slide image) and the velocity profile is drawn in such a way that the vertical distance is along the y axis and the velocity is along the x axis.

Of course, we know that the velocity is the dependent variable, y is the independent variable, but purposefully the profile has been drawn here taking velocity along the x axis and vertical distance along the y axis so, that you can easily visualize this profile with the geometry of the plate.

And if you look at the profile, the maximum velocity is about 1.8 and that is what you have found out analytically also in the previous slide, of course, the velocity 0 at both the plates.

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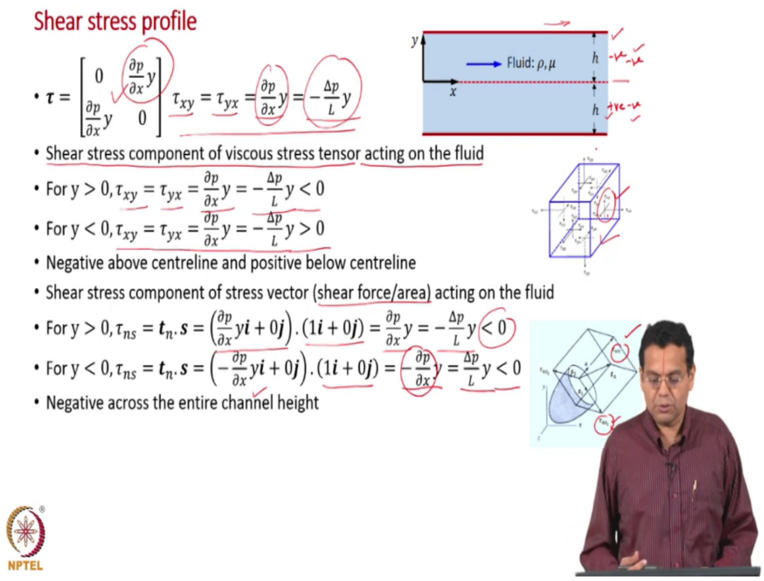

Now, as I told you one of the main objectives of this example is to discuss about the shear stress profile; few books discuss about shear stress profile. Let us discuss that in detail. I like to recall two figures (in the above slide image) as we have done in the previous few slides back. We will use this nomenclature again in this discussion also. In the bottom figure whatever we have they represent components of the viscous stress tensor, and what you see in the top figure they are components of the stress vector; of course, both the cases we will be interested only in the shear stress component. So, let us keep that in mind and proceed.

We have found out the expression for the viscous stress tensor few slides back given by

$$\tau = \left[0 \ \frac{\partial p}{\partial x}y \ \frac{\partial p}{\partial x}y \ 0\right]$$

So,

$$\tau_{xy} = \tau_{yx} = \frac{\partial p}{\partial x} y = -\frac{\Delta p}{L} y$$

So, now shear stress component of viscous stress tensor acting on the fluid, which means we are discussing this stress element, the component shown in that stress element, ok. So, let us read it again shear stress component of viscous stress tensor. This distance is important as you go along you will understand why we are really distinguishing the nomenclature; shear stress component of viscous stress tensor, of course acting on the fluid. Now, for y > 0 which means above the axis

$$\tau_{xy} = \tau_{yx} = \frac{\partial p}{\partial x} y = -\frac{\Delta p}{L} y < 0$$

So, this shear stress component is negative above the axis. Now let us evaluate this for below the centre line below the axis, for y < 0

$$\tau_{xy} = \tau_{yx} = \frac{\partial p}{\partial x} y = -\frac{\Delta p}{L} y > 0$$

To conclude these two lines the shear stress component of viscous stress tensor. So, we plot the components of viscous stress tensor that is negative above centre line and positive below centre line.

We will see graph of this next slide, but right now conclusion is that if you plot the stress tensor component alone, then that will be negative and then positive, that is the idea here. Now let us see what happens to the shear stress component of the stress vector.

Now once again acting on the fluid; how do we evaluate that? To evaluate that we need the stress vector and for y > 0,

$$\tau_{ns} = t_n . s = \frac{\partial p}{\partial x} y = -\frac{\Delta p}{L} y < 0$$

So, this shear stress component of stress vector is less than 0 above the centre line. Now, for y < 0 which is below the centre line below the axis

$$\tau_{ns} = t_n . s = -\frac{\partial p}{\partial x} y = \frac{\Delta p}{L} y < 0$$

So, the shear stress component of stress vector is negative along the entire channel head. In one we are plotting the component of the stress tensor; in the other we are plotting the compound of the stress vector. What really is required is the compound of the stress vector only, that is what is required; that will determines the shear force acting on the fluid or acting on the plate tells the direction of shear force acting on the fluid or acting on the plate, that is why here within bracket it is shown as shear force per area.

So, let us summarize all this pictorially the next few slides. Once again I want to repeat that if you are plotting the component of stress tensor it is negative and positive, but if we are plotting component of stress vector it is negative throughout. We can remember this so, that we understand the figures here.

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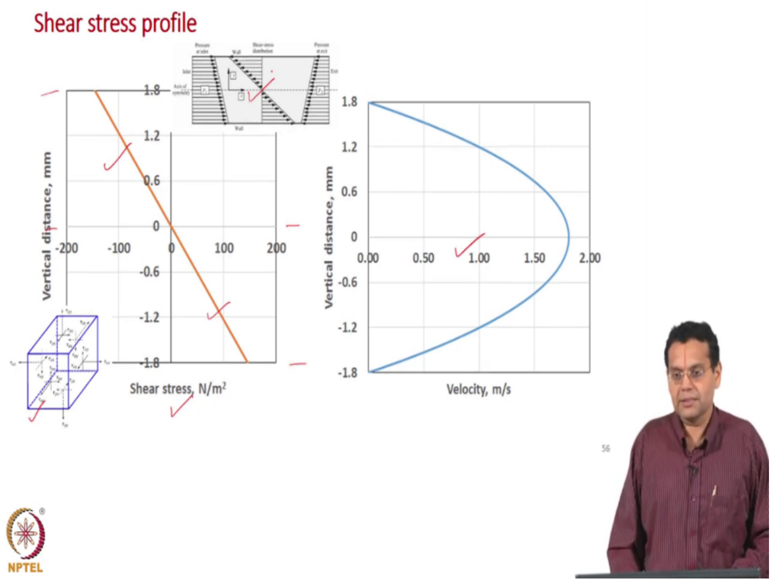

Now what this slide shows is, shear stress profile on the left hand side and the velocity profile on the right hand side. This we have already discussed. Now, what is plotted on the left hand side is the component of the stress tensor and what did we discussed? Above the centre line, it is negative and below the centre line it is positive that is what we are seeing here.

And remember once again, the vertical distance is along the y axis, the shear stress is along the x axis. So, above the axis it is negative and below the axis it is positive and that is the distribution shown in this book in this diagram. So, we should know what does the distribution shown in that figure. What is plotted there is the variation of the component of viscous stress tensor that diagram does not show the shear stress component of the stress vector. It shows the component of the stress tensor that should be kept in mind that if you plot it is negative in the above the axis and positive below the axis.

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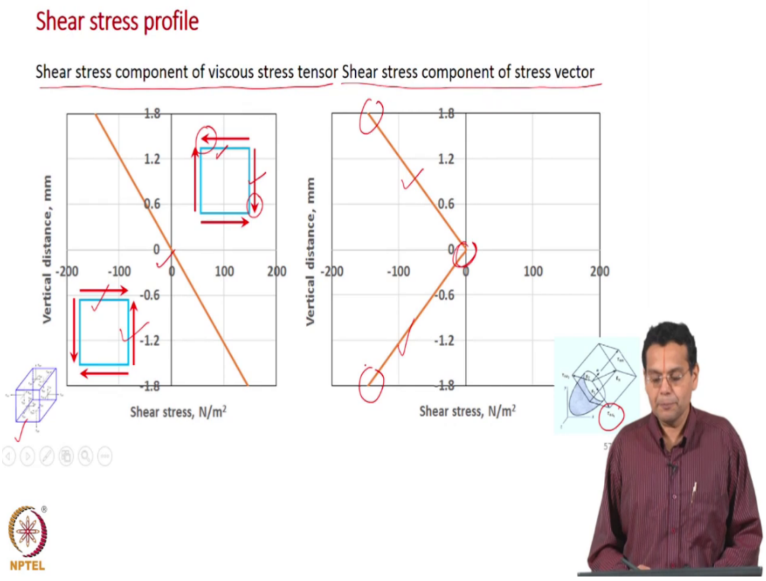

Now the title of the slide says once again shear stress profile the left hand side diagram is known to us from the previous slide. Same has been plotted again. Look at the title now shear stress component of viscous stress tensor that is why this stress element is shown here. Component of stress tensor is plotted as we have discussed in the previous slide.

Additionally, what is shown here is a stress element? The representation of stress element changes whether it is above the axis or below the axis. What happens above the axis, the component is negative. How do you represent a negative component? On a positive phase, the direction of the force is along the negative axis and let us say, here also positive phase and the direction of force is along the negative axis so, that is how our stress element looks.

Below the axis, how does stress element look because the component is positive for a positive plane the force is along the positive axis, for a positive plane force is along positive axis. So, the stress element representation is different for above the axis and below the axis, ok. This also kind of connects very well with our sign convention for the stress element.

And coming to the right hand side figure, what is shown? What we will discuss? The shear stress component of stress vector is always negative throughout the entire height. So, right hand side plots shear stress component of stress vector which means that we are plotting this one and that is negative throughout the region, throughout the entire height the stress shear stress component of stress vector is negative and that is what we have. Once again of course, that also varies linearly but it is 0 here and of course, negative here 0 here and once again negative here.

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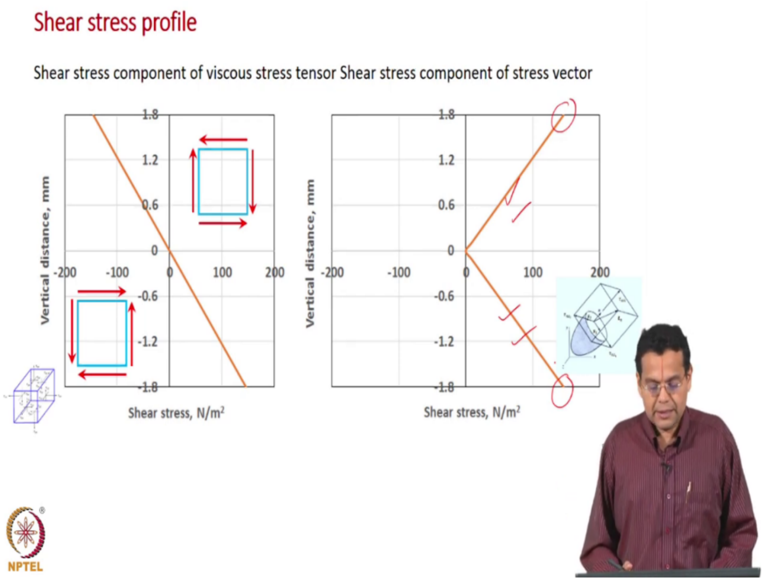

Now one more slide, left hand side is just same as last slide, right hand side now it is positive; why is it positive, in the previous slide whenever we say shear stress component of stress vector, it is acting on the fluid. Suppose we want to represent shear stress exerted by the fluid, then we get a positive graph like this which is positive throughout the entire height between the plates.

And if you multiply these values whatever shear stress we have got right and with the area of the plate which is the surface area of the plate of course, include both the; include both for the top plate and the bottom plate, then you will get the total shear force acting on the plates.

And just to summarize the last few slides, let me go back reason for discussing this in detail is that some books will show you shear stress profile like this, some books will show you profile like this. So, we should know to distinguish between the two profiles what is the profile that is plotted in this way, what is the shear stress that is being plotted in this way and what is the shear stress that is being plotted in this way; the title may still say shear stress profile.

So, just to summarize the previous slides, we are looking at the shear stress profile how shear stress varies along the height between the two plates, but we are looking at how the shear stress component of viscous stress tensor varies which is negative above axis and positive below axis. We are also looking at how shear stress component of stress vector varies, but and that is negative both above and below the axis.

And so, here right hand side is velocity profile, left hand side is the component of the stress tensor, and here again left hand side is component of the stress tensor along with the stress element right hand side component of the stress vector. And here the right hand side is the stress acting on the fluid, here is the shear force excreted by the fluid.

So, all we have seen I would say three ways of shear stress profile; one is the profile like this and one is the profile like this and one is the profile like this. So, we should know clearly the distinction between all these three representations of shear stress profile. And this also kind of nicely summarizes whatever discussed earlier with respect to sign convention, and then also physically understand what is the role of the component of viscous stress tensor and what is the role of component of the stress vector.

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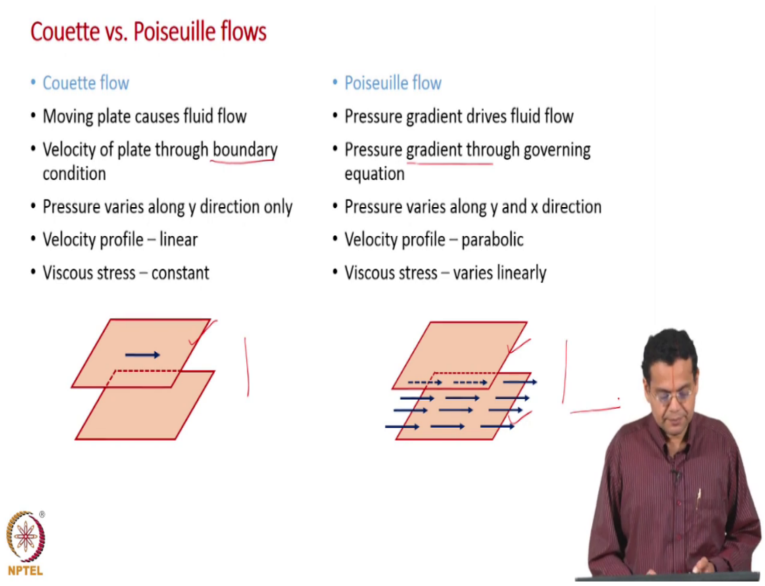

So, now that we have discussed in detail about Couette and Poiseuille flow. Let us distinguish between these two flows.

- In the case Couette flow, the moving plate causes fluid flow, in the case of Poiseuille flow, pressure gradient drives fluid flow.

Now how does this moving plate is something external, pressure gradient is something external; how this external condition enters the solution of the problem differs between the two cases. The first case of Couette flow, this velocity of plate entered the solution through the boundary condition; second case a pressure gradient entered the solution for the governing equation itself. So, first case remember we said at the top plate $v_x = v_p$, but in the second case velocity was 0 at both the plates, but we had pressure gradient incorporated through the governing equation itself.

- Couette flow pressure varies along the y direction only which is the hydro static pressure distribution. And in the case of Poiseuille flow, there is variation along y direction once again hydro static, but we do have variation along the x direction as well, there is decrease in pressure along the x direction.
- Velocity profile is linear in the case of Couette flow parabolic in the case Poiseuille flow.
- Viscous stress is a constant in the case of a Couette flow but it varies linearly in the case of Poiseuille flow.

Quick comparison of how these two flows vary in terms of what causes the flow, in terms of governing equation and boundary conditions and in terms of profiles.

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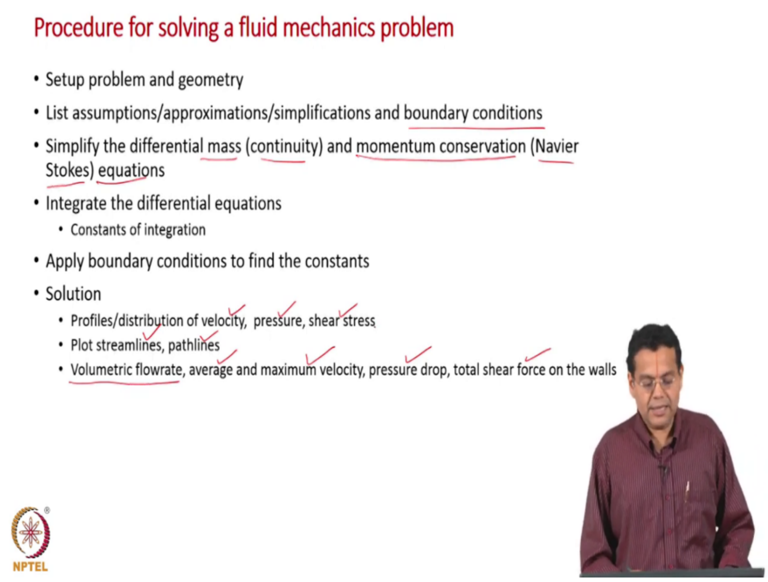

Now given the scope of this course, we have solved two very simple problems, but I would say in detail connecting the entire a part of linear momentum balance but this is a good starting point or we have learnt enough to solve other more complex problems. So, let us look at the general procedure for solving a fluid mechanics problem, that procedure is almost constant across any fluid mechanics problem.

- First is setting up setting up of the problem and the geometry.

Let us say Couette flow, Poiseuille flow and then geometry; in this case, we took planar Couette flow and then planar Poiseuille flow. We could have also taken the usual cylindrical coordinate Couette flow and then Poiseuille flow and then the dimensions etcetera. And then of course, selection of coordinate axis all comes under the first step, then

- List all the assumptions or approximation simplifications and the boundary conditions.

Remember these depend on the problem and these depend on the level of sophistication we want to achieve. The more the number of assumptions we make, we get of course, simpler and simpler solutions. Of course, if we keep relaxing assumptions you will go more closer to reality. So, when I say you will solve more involved problems meaning one implication is that you will solve the same problem with less number of assumptions. Of course, the boundary condition on also has to be listed based on the physics.

- Based on the assumptions, we can simplify the governing equations.

What are the governing equations? Remember always the differential mass balance goes without saying though we have been emphasizing on the momentum balance, always we solve both the differential mass balance and the momentum balance together.

So, simplify the differential mass and momentum conservation equations in terms of nomenclature continuity and the Navier Stokes equations. Then what do we do, we integrate the differential equations which will result in the constants of integration. In the present case, we resulted in two constants of integration.

- Now, we have the boundary conditions; so, apply the boundary conditions to find the constants which means the solution is ready; you have solved the given fluid mechanics problem.

Now, what do we mean by solution? What are the different levels at which the solution can be found out or reported?

- First we can find out the profiles or distribution of velocity, pressure, shear stress; we have seen shear stress could be either component of the stress tensor or component of the stress vector. More practical relevance is the component of stress vector because that only determines the force acting on the plates or any other surrounding geometry.
- Now, we can also plot stream lines, and path lines. Remember for stream lines and path lines what we require is the velocity distribution, earlier when we started off we were given a velocity distribution. Now we have found out the velocity distribution so, we can plot for stream lines and path lines.

Remember when we discussed stream lines, we said the objective is to represent a result of a simulation or result of a prediction. We have predicted the velocity profile; we can represent that as a stream lines of course, in this case they are just straight lines; they are just straight lines. Because we do not have any y velocity component only x velocity component so, path lines stream lines they are all the straight lines, it horizontal lines.

What else can be found out in terms of average values?

- We can find out the volumetric flow rate, we can find out the average and maximum velocity. If flow rate is given, we can find out the pressure drop; we also found out a total shear force acting on the walls.

So, the way in which I separated the last three bullets are first bullet for profiles, second is for let us say flow visualization, third are average values; flow rates, average velocity, total share force, pressure drop etcetera.

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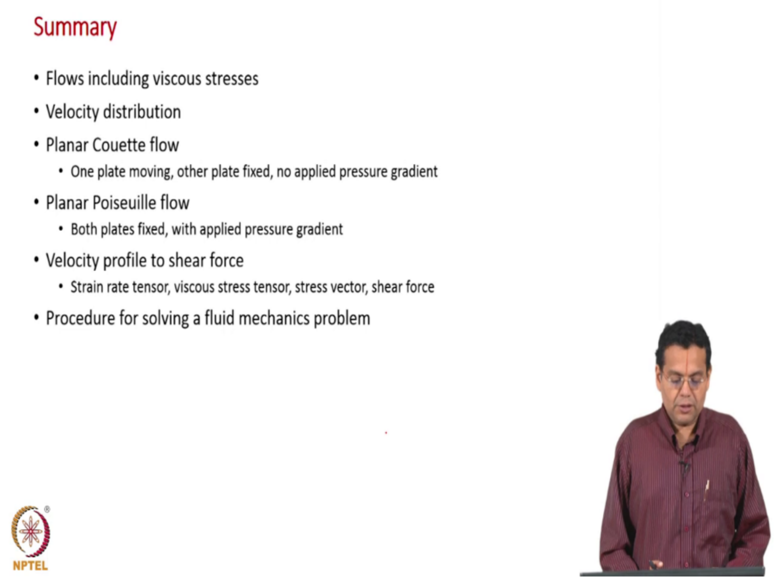

So, now we are ready to summarize the third develop applications for Navier Stokes equation namely flows including viscous stresses. Only one keyword if you want to summarize all of them velocity distribution. Of course, now we discuss in detail, but one main objective was to find out the velocity distribution.

We consider two geometries; first one is a planar Couette flow in which one plate was moving other plate fixed with no applied pressure gradient, the plate cause the moment of the fluid. Second planar Poiseuille flow both plate fixed, the applied pressure gradient cause the movement of the fluid.

And in terms of analysis we have done a detailed analysis right from velocity profile to the shear force calculation. How did you do that? We evaluated a strain rate tensor, viscous stress tensor, stress vector, shear force. And we also discussed what are the general procedure for solving a fluid mechanics problem that brings us to the close of the fluid mechanics part of this course and we will start the transfer phenomena part of the course from now onwards.