Continuum Mechanics And Transport Phenomena
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Lecture – 95
Planar Poiseuille Flow - Shear force

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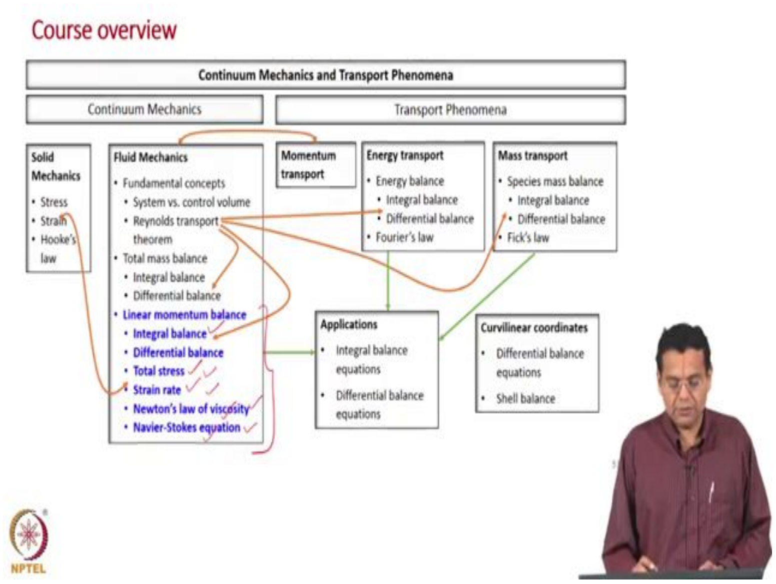

Having derived the velocity distribution and the pressure distribution, now, we will do what we did for plane Couette flow. That is link all the concepts which we have learnt with this Poiseuille flow which means that almost all the concepts which we have discussed under linear momentum balance, we started with integral balance. Then started deriving the differential balance, discussed about total stress-strain rate, Newton's law of viscosity and derived the Navier-Stokes equation. Now, we are going to connect and link all these with these example. Let us see how do we do that.

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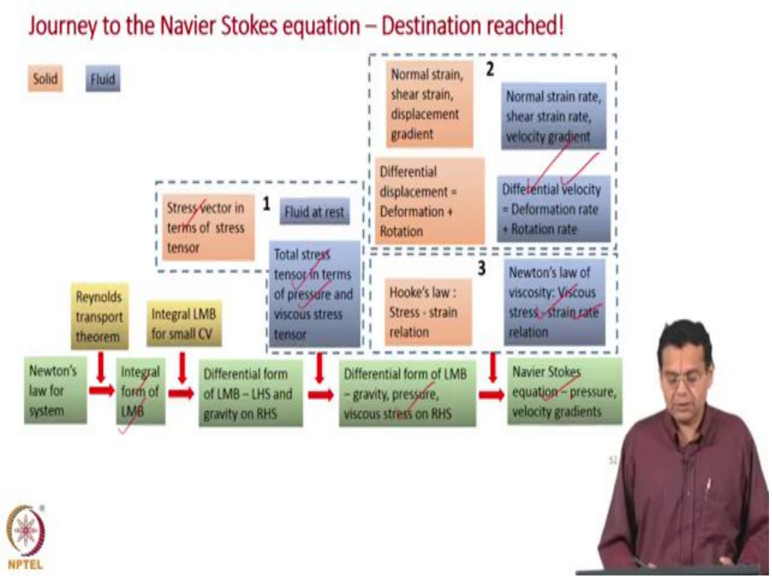

So now, let us start. Now, to begin with, we introduced total stress and then viscous stress tensor.

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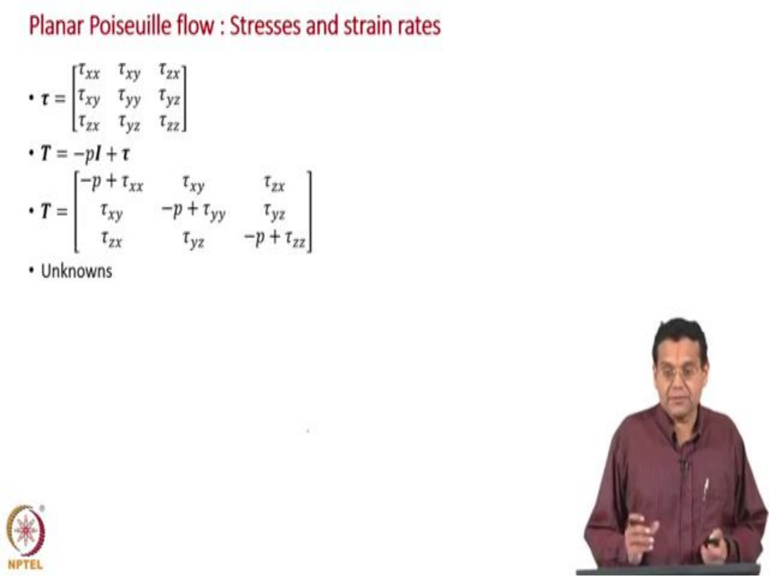

So, we introduced the viscous stress tensor and this is the viscous stress tensor components (expression given in the slide image). And then of course, we could express the total stress in terms of hydrostatic stress plus viscous stress which means, it is just adding minus p to the diagonal elements. At this stage we just introduce them as physically meaningful variable, but they were unknowns. So, that is where we introduced these, but in terms of they have just variables.

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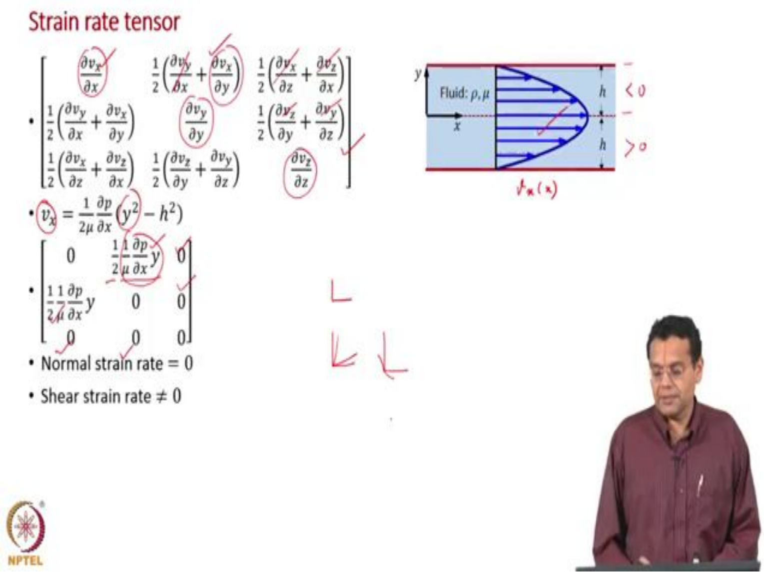

$$Strain\ rate\ tensor = \Big[\frac{\partial v_x}{\partial x}\ \frac{1}{2}\left(\frac{\partial v_y}{\partial x}+\frac{\partial v_x}{\partial y}\right)\ \frac{1}{2}\left(\frac{\partial v_x}{\partial z}+\frac{\partial v_z}{\partial x}\right)\ \frac{1}{2}\left(\frac{\partial v_y}{\partial x}+\frac{\partial v_x}{\partial y}\right)\ \frac{\partial v_y}{\partial y}\ \frac{1}{2}\left(\frac{\partial v_z}{\partial y}+\frac{\partial v_y}{\partial z}\right)\ \frac{1}{2}\left(\frac{\partial v_x}{\partial z}+\frac{\partial v_z}{\partial x}\right)\ \frac{1}{2}\left(\frac{\partial v_z}{\partial y}\right.$$

Now, later we discussed about strain rate. And, so now, are in a position to evaluate the strain rate there again in fact, we first introduced that as a strain rate tensor. But now having derived the parabolic velocity profile, we can evaluate the components of the strain rate tensor.

$$v_x = \frac{1}{2\mu}\frac{\partial p}{\partial x}\left(y^2 - h^2\right)$$

So, this is the velocity profile which we derived which is a quadratic in y. And, now we will have to evaluate the components of the strain rate sensor using this velocity distribution.

$$\Big[\frac{\partial v_x}{\partial x}\ \frac{1}{2}\left(\frac{\partial v_y}{\partial x}+\frac{\partial v_x}{\partial y}\right)\ \frac{1}{2}\left(\frac{\partial v_x}{\partial z}+\frac{\partial v_z}{\partial x}\right)\ \frac{1}{2}\left(\frac{\partial v_y}{\partial x}+\frac{\partial v_x}{\partial y}\right)\ \frac{\partial v_y}{\partial y}\ \frac{1}{2}\left(\frac{\partial v_z}{\partial y}+\frac{\partial v_y}{\partial z}\right)\ \frac{1}{2}\left(\frac{\partial v_x}{\partial z}+\frac{\partial v_z}{\partial x}\right)\ \frac{1}{2}\left(\frac{\partial v_z}{\partial y}+\frac{\partial v_y}{\partial z}\right)\ \frac{\partial v_z}{\partial xz}\Big] = \Big[0\ \frac{1}{2\mu}$$

So, we can write the off diagonal elements and in terms of physical significance. So, if you look at the components of the strain rate tensor the diagonal components are 0. And which means that the normal strain rate is 0 and you do have non-zero off diagonal elements which means that the shear strain rate is not equal to 0. What does it mean? When you say normal strain rate is 0, remember this along the x axis and y axis. So, if you take a line element along the x axis, there is no rate of change of length and if you take an element along the y axis, there is no rate of change of length.

Now, coming to shear strain rate if you look at the expression for shear strain rate, it depends on $\frac{\partial p}{\partial x}$ and y. In the top portion of the region, where y is greater than 0 which means above the central line of course, $\frac{\partial p}{\partial x}$ is less than 0. So, in this region this component, the strain rate tensor component, which is the shear strain rate divide by 2 and that is less than 0. It is negative ok. And, below the central line of course, $\frac{\partial p}{\partial x}$ is negative y is also negative.

And, so, the component is greater than 0 or positive. And which mean we know that positive shear strain rate means that if you consider two line elements perpendicular to each other at some time t, next some let us say time t + Δt they come towards each other. They approach each which means there is a decrease in angle between them. That is what happens below the central line. Above the central line if you once again consider two line elements at some time t sometime later t + Δt there is an increase in angle between them.

So, it becomes let us say obtuse and they go away from each other. So, that is how we interpret the components of the strain rate tensor. So, the whether they come approach each or go away from each other depends on whether we consider the two perpendicular line elements above the axis or below the axis. Next we discuss the Newton's law of viscosity to relate viscous stresses and strain rates.

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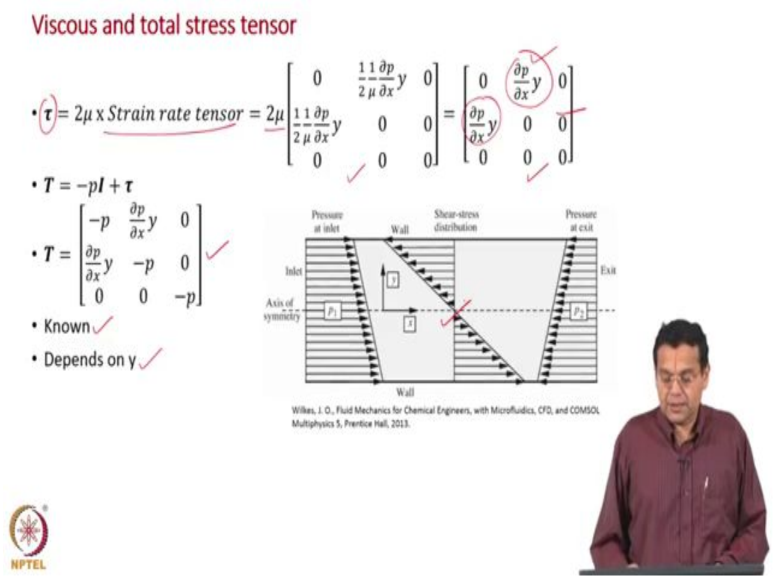

$\tau = 2\mu$ *x strain rate tensor*

That is the relationship between the viscous stress tensor and the strain rate tensor. And in the previous slide we got the expression for strain rate tensor.

$$\tau = 2\mu \left[0 \ \frac{1}{2\mu}\frac{\partial p}{\partial x}y \ 0 \ \frac{1}{2\mu}\frac{\partial p}{\partial x}y \ 0\ 0\ 0\ 0\ 0 \right] = \left[0 \ \frac{\partial p}{\partial x}y \ 0 \ \frac{\partial p}{\partial x}y \ 0\ 0\ 0\ 0\ 0 \right]$$

And you can also find out the total stress tensor adding $-$p to the diagonal elements. So,

$$T = -pI + \tau = \left[-p \ \frac{\partial p}{\partial x}y \ 0 \ \frac{\partial p}{\partial x}y \ -p \ 0\ 0\ 0 \ -p \right]$$

Now, earlier we introduced the viscous stress tensor, total stress tensor as variables. Now, they become known. They were unknown earlier. Now the velocity profile is known. So, we can evaluate the viscous stress tensor and the total stress tensor and they become known. And, if you notice there is one difference between the viscous stress tensor for the Couette flow and this viscous, viscous stress tensor for the Poiseuille flow. The difference is that in the earlier case the viscous shear stresses these were just constants. It was just $\frac{v_p}{h}$. And v_p was the velocity of the plate, h was the distance between the plates. It was just a constant.

But in this case of Poiseuille flow the viscous stress tensor and the shear stress components, they depend on y. So, they vary linearly with y. In the earlier case it was just a constant. That is what this bullet says that there is a dependence on y and linear dependence on y. And, we have seen this nice diagram from James O Wilkes when we discussed pressure distribution.

And the shear stress distribution is also shown in this diagram nicely, we will discuss that more about that in detail little later, but just to show that the there is a viscous shear stress distribution which varies linearly. We have evaluated the components of the viscous stress tensor.

Now, we also discussed about stress vector and the relationship between stress vector and stress tensor. So, if we know the stress tensor, we can evaluate the stress vector and also can evaluate the shear force acting on the plates and that is what we will do now.

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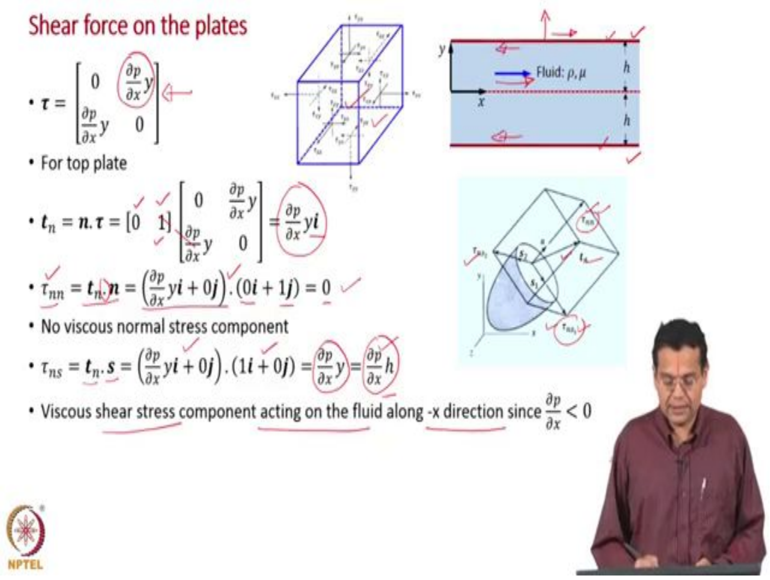

So, when I say stress tensor in the present context it means viscous stress tensor.

$$\tau = \left[0\ \frac{\partial p}{\partial x}y\ \frac{\partial p}{\partial x}y\ 0\right]$$

So, this is the stress tensor, only the two dimensional form is shown here other terms are 0. Now, when we began solid mechanics we learnt how to evaluate stress vector if we know the stress tensor and we will use that now. What is idea? To evaluate the shear force acting on the plates.

Now, we have a top plate and a bottom plate and we will evaluate the shear force acting on the top plate and the bottom plate. Now, for the top plate to evaluate the stress vector we know stress vector is given by the dot product between the normal vector and the stress tensor or matrix multiplication of the normal vector and the stress tensor. What is the normal vector?

$$t_n = n.\tau$$

Normal vector for the top plate is along the positive y direction and in terms of a vector in terms of its elements we represent as [0, 1]

$$t_n = [0\ 1]\left[0\ \frac{\partial p}{\partial x}y\ \frac{\partial p}{\partial x}y\ 0\right] = \frac{\partial p}{\partial x}yi$$

So, you get a component along the x direction and there is no component along the y direction. So now, this is the stress vector which we obtained using the stress tensor and the normal to the plane.

In this case the plane is a top plate. Now then, once we found out the stress vector we found out the normal component and the shear stress component. I like to distinguish the word component here. We are using in two different ways, see when I use the component here I mean it is the component of the viscous stress tensor. So, what we should recall is this stress element. When I say component here it is a component of the stress vector. So, you should recall this figure. This stress vector was resolved into the normal stress and then to shear stresses. In this case it is because it is two dimensional we will have one normal stress and then one shear stress.

$$\tau_{nn} = t_n.n = \left(\frac{\partial p}{\partial x} y i + 0j\right).(0i + 1j) = 0$$

Of course, a normal stress in this case we will turn out to be 0. So, we will have only shear stress, but because it is 2 dimensional we have only the n vector and the s_1 vector and instead of s_1 vector I will call it as s vector.

It has component along the x direction only. So, without even doing this we can know that there is no normal stress component of the stress vector. Now so, there is no viscous normal stress component. Now, let us find out the shear stress component of the stress vector. So, it is a projection of stress vector along the tangential direction.

$$\tau_{ns} = t_n.s = \left(\frac{\partial p}{\partial x} y i + 0j\right).(1i + 0j) = \frac{\partial p}{\partial x} y = \frac{\partial p}{\partial x} h$$

So, there is a viscous shear stress component at y = h.

What is the direction? We have discussed already that $\frac{\partial p}{\partial x}$ is negative ok which means that this will be negative. And, so, the shear stress is acting along the negative x direction. Why did we say that? h of course, a positive number $\frac{\partial p}{\partial x}$ is less than 0 and hence τ_{ns} is negative. And hence the shear stress acts along the negative direction. Remember all our discussion on the stresses are all acting on the fluid.

So, shear stress component acting on the fluid is along the negative x direction that also we can easily understand. The plates are fixed, both the plates are fixed. And fluid is moving in this direction. So, which means that the shear stress on the fluid is along the negative x direction; so, the fluid tries to let us say pull the plate to the right and the plate tries to plate tries to pull the fluid to the left. So, intuitively you can understand that their shear stress component acting on the fluid is along the negative x direction.

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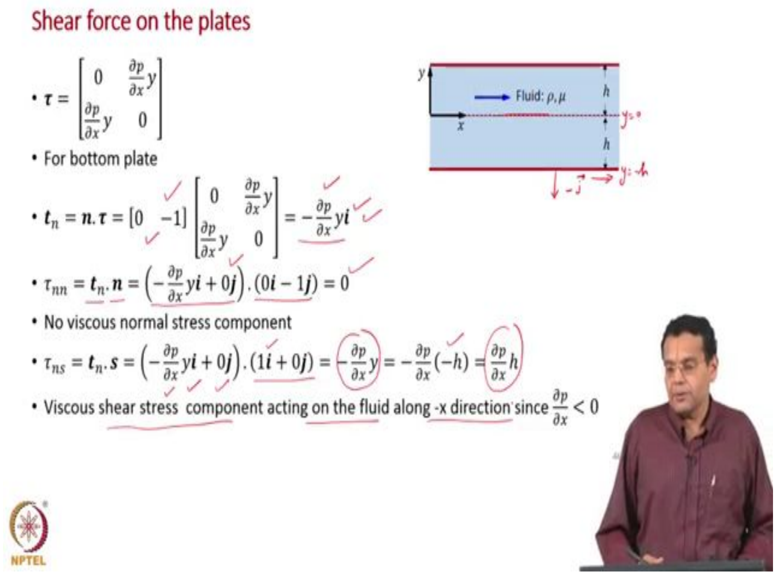

So, now let us repeat this for the bottom plate.

$$\tau = \begin{bmatrix} 0 & \frac{\partial p}{\partial x}y & \frac{\partial p}{\partial x}y & 0 \end{bmatrix}$$

So, for bottom plate now the n vector is along this direction. We always draw as an outward normal. And so,

$$t_n = n.\tau = [0 \ -1 \ ]\begin{bmatrix} 0 & \frac{\partial p}{\partial x}y & \frac{\partial p}{\partial x}y & 0 \end{bmatrix} =- \frac{\partial p}{\partial x}yi$$

Once again it has component along the x direction there is no component along the y direction. And, that sign has changed because our normal vector is now $[0 \ -1 \ ]$. Why is that $[0 \ -1 \ ]$? It is a normal vector pointing along the negative y axis and so, which we represented as $-$j vector that is why it is $[0 \ -1 \ ]$. Of course, do a matrix multiplication with the stress tensor you get the stress vector. Now, like we did it in the previous case let us find out the normal component and the shear stress component of the stress vector.

$$\tau_{nn} = t_n.n = \left(-\frac{\partial p}{\partial x}yi + 0j\right).(0i - 1j) = 0$$

$$\tau_{ns} = t_n.s = \left(-\frac{\partial p}{\partial x}yi + 0j\right).(1i + 0j) =-\frac{\partial p}{\partial x}y =-\frac{\partial p}{\partial x}h$$

So, no viscous normal stress component. And here again the shear stress component acting on the fluid is along the negative x direction. Of course, that cannot change because through the entire domain fluid flows along the positive x axis.

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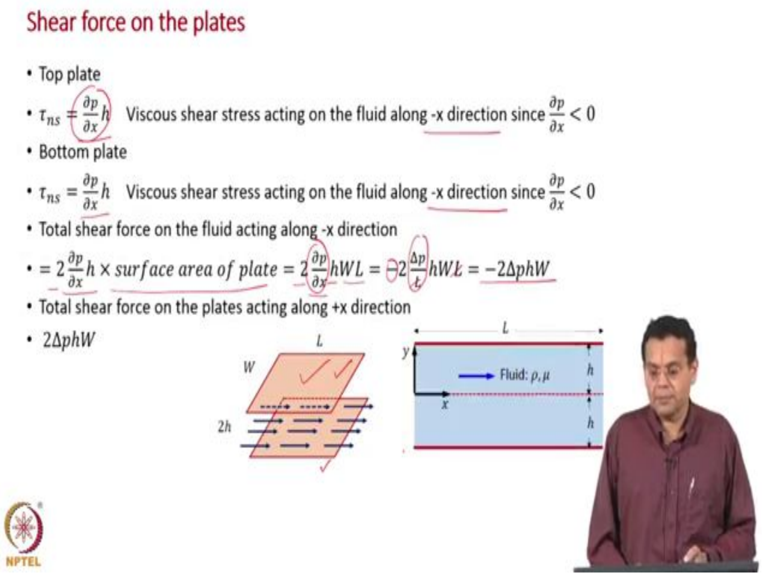

So, certainly the shear stress acting on the fluid is along the negative x axis. Now, what we will do is find out the shear force on the plates. So, let us summarize what we have done in the last two slides. With the top plate the shear stress component of stress vector or simply put the shear stress was given by we found out that it is

$$\tau_{ns} = \frac{\partial p}{\partial x}h$$

Similarly, for the bottom plate the same expression was obtained and the shear stress acting on the fluid is along the negative x direction.

$$\tau_{ns} =-\frac{\partial p}{\partial x}h$$

Now, we will find out the shear force acting on the fluid. How do you find out? This is the expression for shear stress. We will have to multiply by area. What is area we should multiply? We should multiply by the surface area of the plate and then twice of that because we have to account for the top plate and the bottom plate. And that is what we have done here.

$$Total\ shear\ stress = 2\frac{\partial p}{\partial x}h\ x\ surface\ area\ of\ plate = 2\frac{\partial p}{\partial x}hWL = -2\Delta phW$$

We have seen that $\frac{\partial p}{\partial x} = -\frac{\Delta p}{L}$ and that replacement has been done here.

So, what we have done is from the viscous stress tensor found out the stress vector, found out the normal component it turned out to be 0, found out the shear stress component. And found what is the shear stress acting on the fluid at the top plate, at the bottom plate; found out the total shear force acting on the fluid.

Hence, found out what is the total shear force acting on the plates. Throughout we work with shear stress shear force acting on the fluid. Because all our discussion right from beginning is with respect to force acting on the fluid, that is why whenever we discuss about tau etcetera it is on the fluid. And the last step we just changed the direction and say that the total shear force acting on the plates is negative, what we have found out in the previous step. Of course that this $2\Delta phW$.

Towards a beginning of a linear momentum balance we derived the integral form of linear momentum balance from Newton's second law of motion by applying Reynolds transport theorem. We also connect the even the integral linear momentum balance to this example. Let us say how do we do that.

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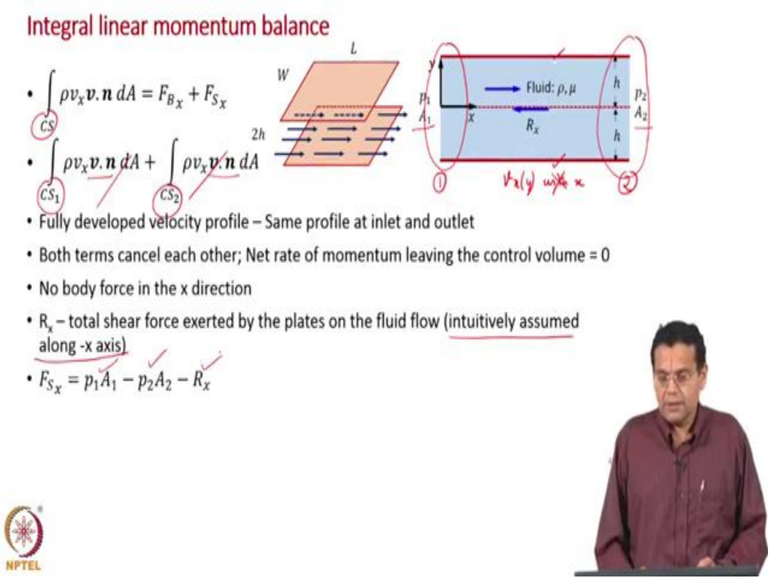

What is the idea? To once again to evaluate the shear force acting on the plates; so, let us do that. So, let us write down the integral linear momentum balance.

$$\int_{CS} \rho v_x v.n\, dA = F_{B_x} + F_{S_x}$$

Of course, we are writing for the steady state condition. Left hand side is a net rate of x momentum leaving through the control surface. Right hand side your body forces and surface forces acting along the x direction. We have inflow at let us say section 1 and outflow at section 2. And, so, CS is split into CS_1 and CS_2.

$$\int_{CS_1} \rho v_x v.n\, dA + \int_{CS_2} \rho v_x v.n\, dA = F_{B_x} + F_{S_x}$$

Now, remember we assumed the condition such that we get a fully developed flow. We said, $v_y = 0$, $v_z = 0$ and, from the continuity equation we arrived at the condition that the flow is fully developed which means that v_y as a function of y does not change with x. which means that whatever velocity profile we have at section 1 is same as the velocity profile what we have at section 2. And for inflow $v.n$ is negative, for outflow $v.n$ is positive. And v_x is along the positive x axis which means that these two terms will cancel each other. The left hand side terms cancel each other.

$$\int_{CS_1} \rho v_x v.n\, dA + \int_{CS_2} \rho v_x v.n\, dA = 0$$

What does it mean? Net rate of momentum leaving the control volume is equal to 0. Whatever rate of momentum enters same leaves the control volume ok. We are writing the linear momentum balance along the x direction and there is no body force along x direction.

Now, we will consider R_x as the total shear force exerted by the plates on the fluid flow. So, whatever these plates exert shear force on the fluid flow is taken as R_x. And we know by physics that it should be along the negative x axis; so, intuitively assumed along the negative x axis. So, now, what happens to the surface force? We have the inlet pressure as p_1 and the area as A_1 at the exit we have the pressure as p_2 and the area as A_2. So, we have

$$F_{S_x} = p_1 A_1 = p_2 A_2 - R_x$$

Like to mention, that we have come across a similar example in one of the applications of integral linear momentum balance. So, I would recommend that you can refer those slides as well; a nomenclature also been almost kept the same.

There it was a compressible flow and there again we are finding out the frictional force. In this case there we termed as frictional force. Now, we have more formal terminology namely the total shear force exerted by the plates on the flow.

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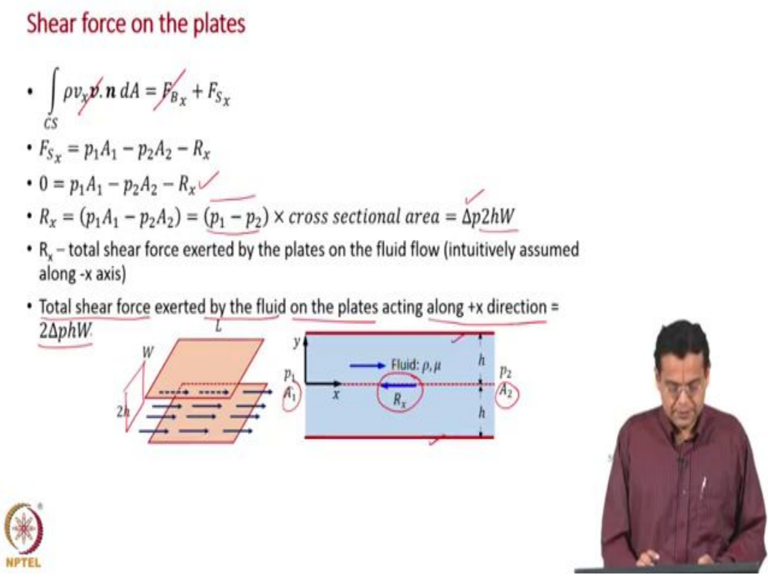

So, let us put these all together in the integral linear momentum balance.

$$\int_{CS} \rho v_x v.n\, dA = F_{B_x} + F_{S_x}$$

We found out the surface first just now in terms of pressure area and the shear force.

$$F_{S_x} = p_1A_1 = p_2A_2 - R_x$$

So, left hand side is 0, right side right hand side body force is 0. So, left only with surface forces and so,

$$0 = 0 + p_1A_1 = p_2A_2 - R_x$$

Rearrange for R_x so,

$$R_x = p_1A_1 - p_2A_2$$

Of course, $A_1 = A_2 = A$ So,

$$R_x = (p_1 - p_2)A$$

So, the cross sectional area is 2hW; So,

$$R_x = \Delta p\,(2hW) = 2\Delta phW$$

And what does it represent? Total shear force exerted by the plates on the fluid. And what is the direction? Remember we have taken R_x along on the negative x axis. So, this is the total shear force exerted by the plates on the fluid flow along the negative x axis.

And just like we did earlier what we are interested is the shear force on the plates. So, total shear force exerted by the fluid on the plates is along the positive x direction. Of course, magnitude is $2\Delta phW$. Of course, it cannot change we have found the expression using the integral linear momentum balance. So, now to summarize what we have done in the last few slides. We have gone through one complete cycle of all these topics ok. The derivation of Navier-Stokes started with the integral balance.

Then of course, under differential balance we discussed about total stress, strain rate, Newton's law of viscosity, Navier-Stokes equation. Now, we have gone completely almost in the reverse direction. We use the Navier-Stokes equation, solve for the velocity profile, found out strain rate, used Newton's law of viscosity, to found to find out total stress, stress vector, from that the shear force. And, we have also use the integral linear momentum balance to find out the total shear force acting on the plates. Same thing in terms of our journey slide, we had the integral form of linear momentum balance.

And then for the differential balance we discussed about total stress, viscous stress and then the strain rates, Newton's law of viscosity. Substituted linear momentum balance, the differential form, derived the Navier-Stokes. Now completely reverse direction now, solve the Navier-Stokes, got the velocity profile, evaluated at this strain rate, used Newton's law of viscosity, found out the stress tensor components. Found out the stress vector, also found out the shear force acting on the plates, also use the integral form to find out the total shear force acting on the plates.

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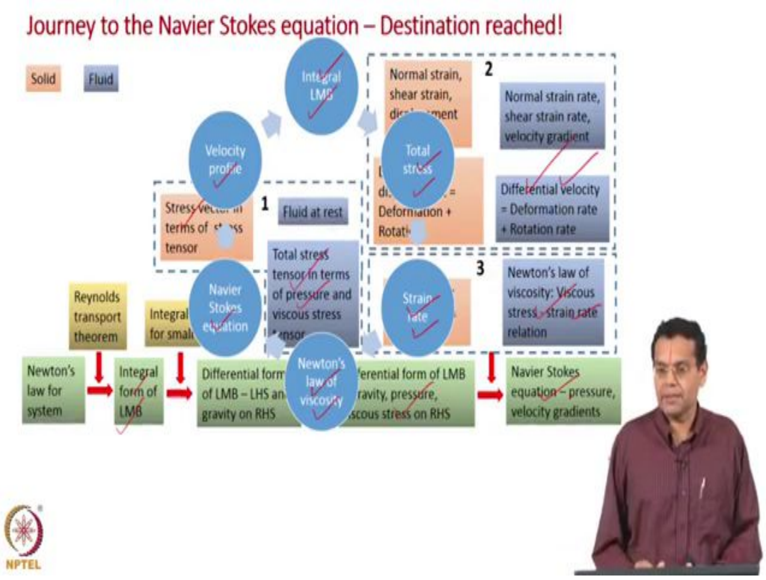

Of course once again shown in the form of a cycle; the integral balance, total stress, then strain rate, Newton's law of viscosity, Navier-Stokes, velocity profile. Once the velocity profile is known from the Navier-Stokes you can find out strain rate. Use Newton's law of viscosity find total stress, stress vector and then the shear force which can also be found out from the integral linear momentum balance.

So, this example is a very comprehensive example almost covering right from our beginning of the derivation, from the you can say almost let us say Newton's second law of motion till the Navier-Stokes ok. When we came to the forward path they were all led us to the derivation of Navier-Stokes. Now, you can understand why we really came through that particular path. And, then such a combination was strain rate tensor, the components of strain rate tensor. But, now we can all evaluate them in terms of values as well, in terms of the velocity profile, in terms of a measurable values you can evaluate them.