Continuum Mechanics And Transport Phenomena
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Lecture - 94
Planar Poiseuille Flow: Velocity and Pressure Distribution

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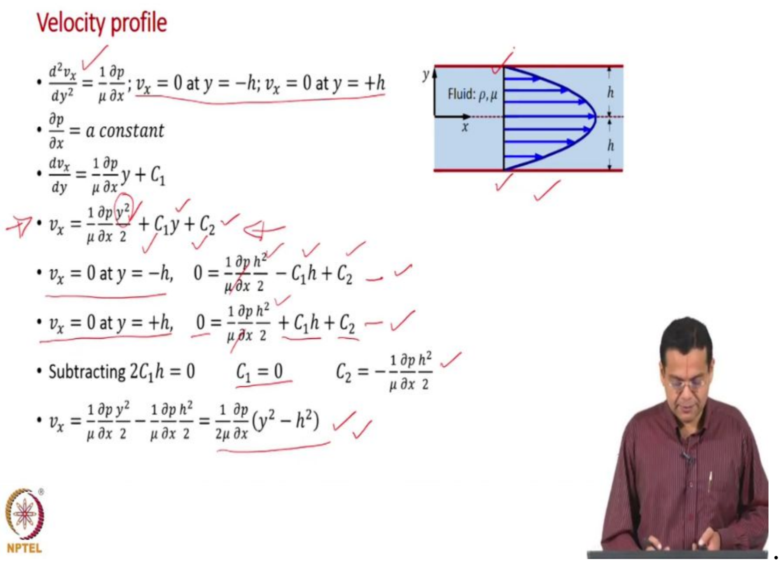

We are ready to solve as I told in the previous case, it is just pure maths at this stage. All the physics are discussed,

$$\frac{d^2v_x}{dy^2} = \frac{1}{\mu}\frac{\partial p}{\partial x}; \quad v_x = 0\ at\ y = -h; \qquad v_x = 0\ at\ y = +h$$

We know the physics behind this ordinary differential equation, it is a Navier-Stokes equation, simplified form of that. You also know the physics behind the boundary condition as well, where the no slip boundary conditions. So, just like your maths class, where you would have been given such an ordinary differential equation, given the boundary condition; now we have got a chance, opportunity to derive the governing equation and state the boundary condition as well, you know the physics behind them as well.

Now to proceed further to integrate, you should know that

$$\frac{\partial p}{\partial x} = a$$

So, let us integrate once we get

$$\frac{dv_x}{dy} = \frac{1}{\mu}\frac{\partial p}{\partial x} y + C_1$$

Let us integrate once again, you get

$$v_x = \frac{1}{\mu}\frac{\partial p}{\partial x}\frac{y^2}{2} + C_1 y + C_2$$

Now we will have to evaluate the constants, we will use the boundary conditions, $v_x = 0\ at\ y = -h$. So, let us substitute in this equation. So,

$$0 = \frac{1}{\mu}\frac{\partial p}{\partial x}\frac{h^2}{2} - C_1 h + C_2$$

Let us use the second boundary condition, $v_x = 0\ at\ y = +h$. So,

$$0 = \frac{1}{\mu}\frac{\partial p}{\partial x}\frac{h^2}{2} + C_1 h + C_2$$

So, we have two simultaneous equations in the two constants. So, let us subtract, we will have to eliminate one of the constants, if we subtract will eliminate C_2;

$$C_1 = 0$$

And if you substitute C_1 either in one equation you will find out C_2 as

$$C_2 = -\frac{1}{\mu}\frac{\partial p}{\partial x}\frac{h^2}{2}$$

So, let us substitute back in this equation,

$$v_x = \frac{1}{\mu}\frac{\partial p}{\partial x}\frac{y^2}{2} + C_1 y + C_2$$

$$v_x = \frac{1}{\mu}\frac{\partial p}{\partial x}\frac{y^2}{2} + -\frac{1}{\mu}\frac{\partial p}{\partial x}\frac{h^2}{2}$$

$$v_x = \frac{1}{2\mu}\frac{\partial p}{\partial x}\left(y^2 - h^2\right)$$

So, this velocity profile is parabolic, because it depends on y^2. And such a parabolic velocity profile is shown here, that profile is not at all new to us; we almost from the beginning of the classes we have come across that parabolic velocity profile several times.

So, it is very nice now to really derive the velocity profile, the equation which represents this velocity profile. The equation which determined this parabolic velocity profile is this equation; of course, at y equal to if you substitute, $v_x = 0\ at\ y = +h\ or - h$. Regarding maximum velocity we will shortly discuss.

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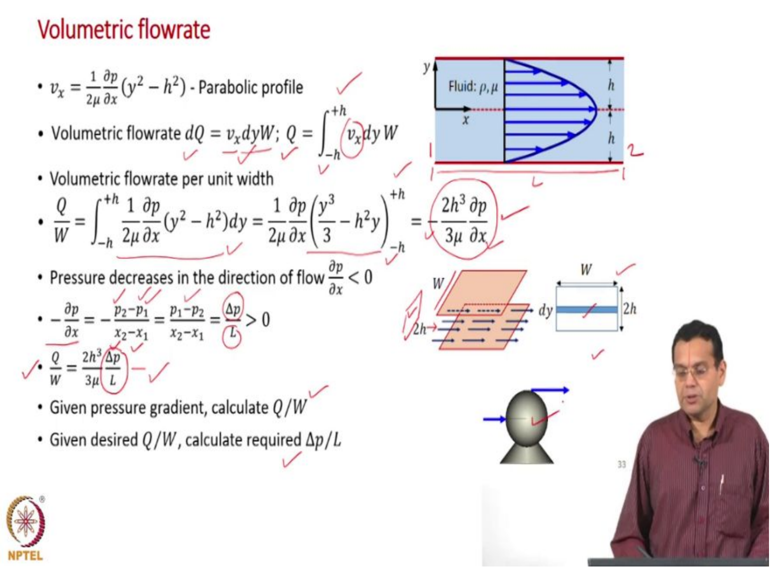

What else can be found out from the velocity profile? We can find out what is the volumetric flow rate flowing between the plates. How do you evaluate that, let us write the velocity profile which I have derived.

$$v_x = \frac{1}{2\mu}\frac{\partial p}{\partial x}\left(y^2 - h^2\right)$$

Of course, as I discussed it is the parabolic velocity profile. Now how do you evaluate the volumetric flow rate, you know volumetric flow rate is velocity into the area,

$$dQ = v_x dy W$$

$$Q = \int_{-h}^{h} v_x dy W$$

Now the velocity varies along the y direction. So, I cannot take the entire area, what I do is take a small strip of height dy and then find out what is the volumetric flow rate through the small strip and then integrate and that is what we shown here.

Also like to mention that, we have come across similar steps when we discussed an application for integral mass balance, including velocity profile. In fact, we took the same case of flow between two parallel plates which are fixed and we took a similar velocity profile and found out the inlet flow rate, outlet flow rate etcetera. And we discuss, we have to similar discussion. So, you can refer that discussion as well.

Now let us evaluate this volumetric flow rate, just to normalize what we usually evaluate is $\frac{Q}{W}$ volumetric flow rate per unit width of the plate, and this is the width of the plate. So, let us do that

$$\frac{Q}{W} = \int_{-h}^{h} v_x dy = \int_{-h}^{h} \frac{1}{2\mu}\frac{\partial p}{\partial x}\left(y^2 - h^2\right) dy$$

So, let us integrate, if you integrate we get

$$\frac{Q}{W} =- \frac{2h^3}{3\mu}\frac{\partial p}{\partial x}$$

So, $\frac{\partial p}{\partial x}$ is negative. So, this term along with the negative sign is a positive quantity. That is what I just now told you, pressure decreases in the direction of flow. So, $\frac{\partial p}{\partial x} < 0$. Now what we usually do is this $\frac{\partial p}{\partial x}$ is a negative quantity. So, we will do a simple variable replacement which is usually followed in fluid mechanics, you take two locations x_2 and x_1

$$-= \frac{p_2 - p_1}{x_2 - x_1} = \frac{p_1 - p_2}{(x_2 - x_1)} = \frac{\Delta p}{L} > 0$$

This is the pressure gradient, a constant pressure gradient. What is L? L is the length of the plates and now Δp is positive, we say it as pressure drop that is $p_1 - p_2$, p_1 is higher p_2 is lower.

So, Δp is pressure drop is the positive value; L is of course, the length of the plate. So, $\frac{\Delta p}{L}$ is the pressure gradient. So, what is given to us is the pressure gradient; of course, L is based on the length of the plate. So, either you say pressure drop is given or pressure gradient is given. And this I think we should keep this in mind for further discussion, $\frac{\partial p}{\partial x}$ is negative $-\frac{\partial p}{\partial x}$ is represented as, $\frac{\Delta p}{L}$. So, that moment you look at Δp it is positive, easy to understand as well.

So, we are writing the expression for flow rate in terms of, $\frac{\Delta p}{L}$. Now what is use of this equation; two ways of looking at it, have been always telling that we are given the pressure gradient. So, if the pressure gradient is given, or putting it to the other way if you tell the allowable pressure drop then I can calculate, what is the flow rate that can be pumped ok, flow rate per unit width. Other way if you tell me that, I need to pump so much of liquid between the plates Q by W, then you can calculate what is the required pressure drop. That is more practical as well, that will determine the rating of the pump. So, it will determine the capacity of the rating of the pump. Those are the practical applications.

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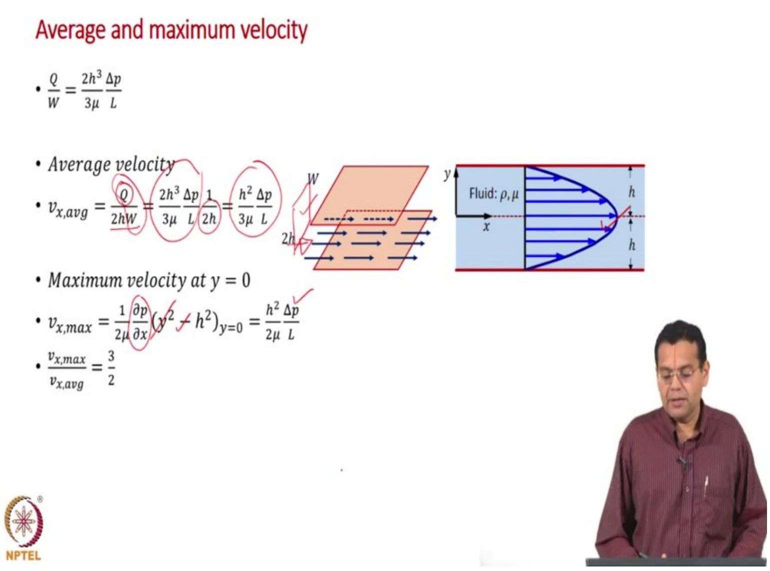

We can also find out the average velocity and maximum velocity. Let us do that .

$$\frac{Q}{W} = -\frac{2h^3}{3\mu}\frac{\partial p}{\partial x} = \frac{2h^3}{3\mu}\frac{\Delta p}{L}$$

$$v_{x,avg} = \frac{Q}{2hW} = -\frac{2h^3}{3\mu}\frac{\partial p}{\partial x}\frac{1}{2h} = \frac{h^2}{3\mu}\frac{\Delta p}{L}$$

This is the expression for the average velocity. We have been looking at the profile, we know v_x varies along y; but suppose you want to represent as an average velocity, then this is the expression.

Now we can also find the expression for maximum velocity which happens at y = 0. And how do you find out? In the expression which I derived for the velocity distribution, for the velocity profile substitute y = 0 and then you get

$$v_{x,max} = \frac{1}{2\mu}\frac{\partial p}{\partial x}\left(y^2 - h^2\right)_{y=0} = -\frac{h^2}{2\mu}\frac{\partial p}{\partial x} = \frac{h^2}{2\mu}\frac{\Delta p}{L}$$

We can also find out the ratio of this maximum velocity to the average velocity which is

$$\frac{v_{x,max}}{v_{x,avg}} = \frac{3}{2}$$

So, the maximum is 1.5 times the average velocity, as I told you all this calculations, velocity profile, and then volumetric flow rate, average velocity, maximum velocity all can be analogously done for flow through circular pipe. To make it easy for us and then restrict to Cartesian coordinates we are deriving it for flow between parallel plates.

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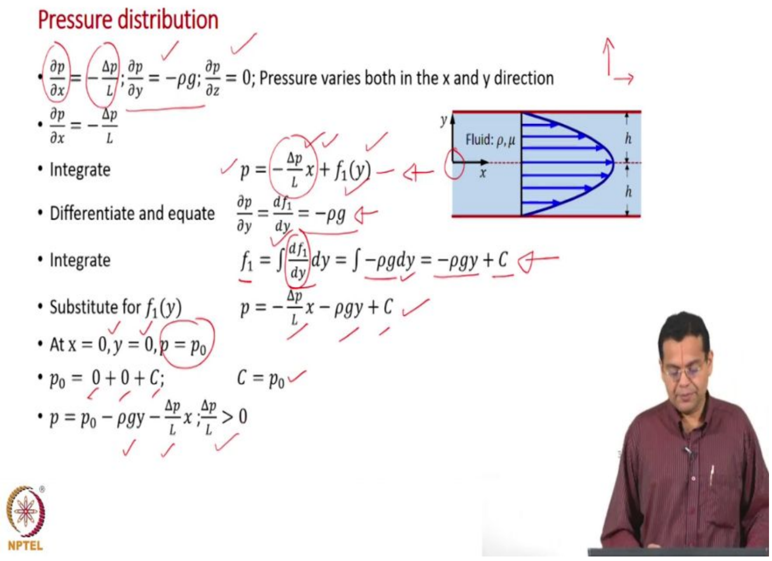

We can also derive the pressure distribution; let us do that

$$\frac{\partial p}{\partial x} =- \frac{\Delta p}{L}; \quad \frac{\partial p}{\partial y} =- \rho g; \quad \frac{\partial p}{\partial z} = 0$$

Now I have written as an expression for $\frac{\partial p}{\partial x}$ because we are going to evaluate the pressure distribution, we need expression or values for the pressure gradients along the x, y, z direction. So, that is why written this as $\frac{\partial p}{\partial x} =- \frac{\Delta p}{L}$, and $\frac{\Delta p}{L}$ is a constant. For example, $\frac{\Delta p}{L}$ could be a let us say 5 unit of pressure is Newton per meter squared, because a pressure gradient for example, could be a value like 5 kilo Newton per meter cube.

Now, the pressure gradient to the y direction is hydrostatic and of course, there is no pressure gradient in the z direction. Of course, pressure varies both in the x and y direction; remember in the last case, for the case of planar couette flow, pressure variation was only in the y direction, there was no pressure variation along the x direction. In this case pressure varies both along the x direction and the y direction.

So, now, how do we proceed, what we are going to do now, the steps we have already seen when we discussed an example after Bernoulli's equation. One of the examples which we discuss about the Bernoulli's equation, we derive an expression for the pressure distribution. What we are going to do now, is exactly similar what we have done earlier.

So, I would recommend that you can refer those slides, one of the examples after we example which we discussed after Bernoulli's equation. After the irrotational form of Bernoulli's equation, we derive Bernoulli's equation both for rotational, irrotational; the example which we discussed after discussing irrotational Bernoulli's equation, the steps there and the steps here are same. So, let us start the pressure gradient in the x direction, which is

$$\frac{\partial p}{\partial x} = -\frac{\Delta p}{L}$$

Let us integrate. In fact, the wordings, the variables, the constant are also have use this same way. So, if you integrate, we get

$$p = -\frac{\Delta p}{L}x + f_1(y)$$

Remember $\frac{\Delta p}{L}$ it is a constant and then it is partial integration. So, our constant can be a function of y. So, we denote it as $f_1(y)$, the same nomenclature has been used earlier also.

So, now what should we do, we have to evaluate $f_1(y)$. So, we will differentiate this equation

$$\frac{\partial p}{\partial y} = \frac{df_1}{dy}$$

And we know that

$$\frac{\partial p}{\partial y} = -\rho g$$

So, we are equating to two equations,

$$\frac{df_1}{dy} = -\rho g$$

So, now, we will have to find out f_1. So, let us integrate. So,

$$f_1 = \int \frac{df_1}{dy} dy = \int -\rho g dy = -\rho g y + C$$

Here C is not a function; because this integrations usual integration, it is not partial integration. Now let us substitute the expression for f_1,

$$p = -\frac{\Delta p}{L}x - \rho g y + C$$

Now will have to evaluate C, the constant; now we will have to be given value of pressure, at some point let us say at x = 0, y = 0, we have p = p_0

$$C = p_0$$

So, you have

$$p = p_0 - \rho g y - \frac{\Delta p}{L}x; \qquad \frac{\Delta p}{L} > 0$$

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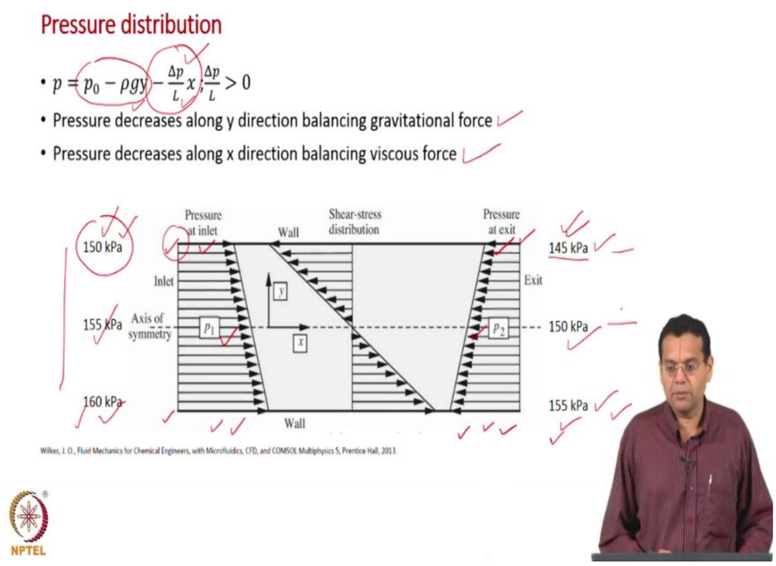

Let us continue with this in the next slide;

$$p = p_0 - \rho g y - \frac{\Delta p}{L}x; \qquad \frac{\Delta p}{L} > 0$$

That is the equation for the pressure distribution. What we conclude from this? First observation is that, pressure varies both along the x direction and the y direction. In the earlier case, there is a pressure variation only in the y direction, in the case of planar couette flow. Now we have variation both in the x direction, y direction.

Now, let us discuss the y direction, because that arises because of hydrostatic condition; and the term is same as what we have seen earlier. Earlier we are only these two terms, $p_0 - \rho g y$ and that is why I am writing that term first. So, pressure decreases along y direction balancing gravitational force, this is same as what you have scene for planar couette flow. Now for Poiseuille flow, pressure decreases along x direction also, balancing viscous force and that is clear from here, you have, $-\frac{\Delta p}{L}x$, $\frac{\Delta p}{L}$ is positive remember.

So, pressure decreases along y direction balancing gravitational force there is nothing new, it is same as planar the couette flow; but this is something different from the earlier case, pressure decreases along x direction also balancing viscous force.

Now, this representation is a very good representation of what we are discussing now from the book by Wilkes Fluid Mechanics for Chemical Engineers; and of course, it as microfluidic CFD COMSOL Multiphysics etcetera. Now let us understand the pressure distribution shows; shear stress distribution that we will see later. For the moment we will discuss the pressure distribution.

Now first observation is that, we know that person is compressive. So, it always acts into the control volume; that is why right side also it is into the control volume, left side also into the control volume, first observation. Second observation, pressure increases with a depth. So, it increase as we go down the y axis that is why pressure is low here and then high here, because of it is a hydrostatic pressure distribution.

Same is the case on the exit also low here and then high here. So, both the cases pressure increases as we go down the y axis. Now if you look at this length and this length; the left hand side pressure inlet has a larger length, at the exit does smaller length. What does it mean? It tells about decrease in pressure and that is of course, constant throughout the height, whichever if you take at any position then there is a decrease in length; the decreases also constant continuous till the bottom.

So, let us quickly repeat, the arrows are towards each other into the control volume because it is compressive; and pressure increases as we go down both at the inlet and the exit. And then in terms of let us say magnitude, the inlet pressure is more, the outlet pressure is less; and that is why that is shown in the length of arrows as well, and that difference is same at any vertical position.

Just to get little more understanding, so I have taken some numerical values. Now as I told you, moment you talk about pressure for incompressible flow it is relative; so somewhere you should fix a pressure, some we always talk in terms of difference in pressure. So, let us take this pressure as 150 kilo Pascal; what does it mean, at $x = 0$ $y = h$, at that point let us say 150 kilo Pascal. Now let us say the distance between the plates is such that, the pressure here is 160 kilo Pascal, it should be increasing. So, let us take there is some 10 kilo Pascal increase; and because pressure varies linearly with height at the central line it will be 155 kilo Pascal.

Now, let us come to the exit. Let us say the conditions are such that now flow, viscosity, etcetera such that, the pressure drop is 5 kilo Pascal. So, inlet pressure is 150, exit pressure is 145 kilo Pascal at y = h. Now at the exit also we have seen, at are at any position, any x position pressure increases if you go down. So, here again at the top it is 145 kilo Pascal, at the bottom the same 10 kilo Pascal increase should be there and that is why this pressure is 155 kilo Pascal.

Once again the pressure variation is linear and so this pressure is 150 kilo Pascal. So, at any position if you take, let us say at y = h the difference between inlet and outlet is 5 kilo Pascal. If you take y = − h, the difference between inlet and outlet is 5 kilo Pascal. Even if you take central line, once again it is 5 kilo Pascal. So, very good representation numerical values we are discussing but the pictorial representation is from the book, a very good representation of pressure distribution.