Continuum Mechanics And Transport Phenomena
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Lecture – 92
Planar Couette Flow – Shear Force

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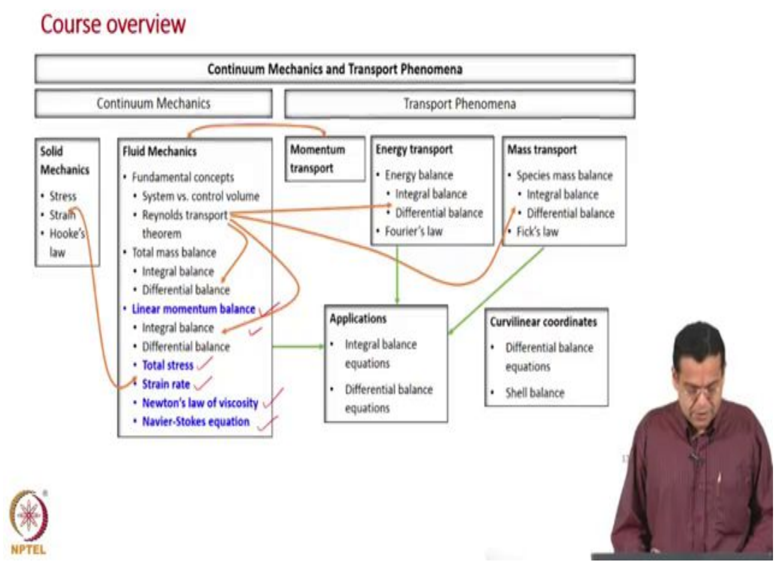

What we will do is using this example of flow between two parallel plates we will link all the concepts which have discussed as we progressed from the linear momentum balance or the integral balance to the Navier-Stokes. So, we discussed about total stress strain rate Newton's law of viscosity and Navier-Stokes equation we are going to link all this concepts we will also understand the better the path we have followed. Now, let us do that.

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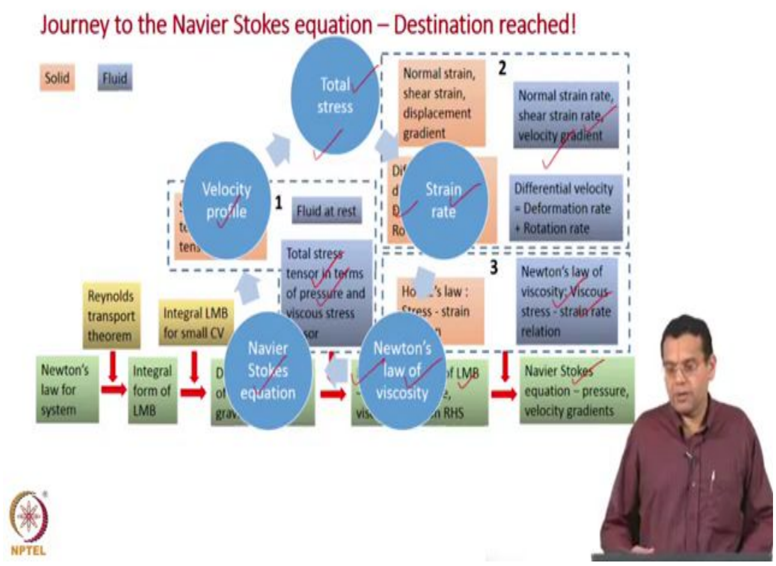

So, we will use this as usual this journey to Navier-Stokes equation of course, the destination reach slide. So, now we began with the total stress that is where we began. Of course, I will discuss only the fluid mechanics part.

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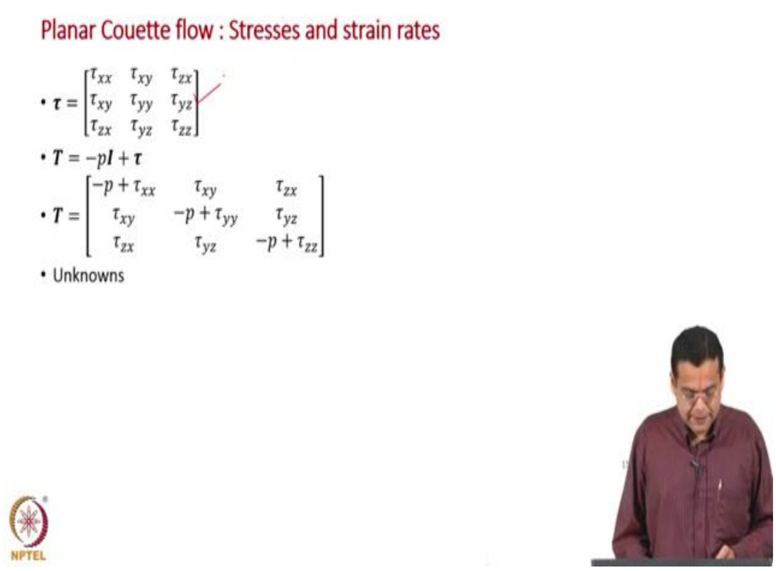

So, now we introduced viscous stresses represented by the viscous stress tensor; we introduced all the components of viscous stress tensor, normal and then shear. We also introduced the total stress tensor, hydrostatic stress plus viscous stress. Just addition of $-p$ to all the diagonal elements. So, this is where we begin with. Point is that they are unknowns at that point they are unknowns and they are just variables.

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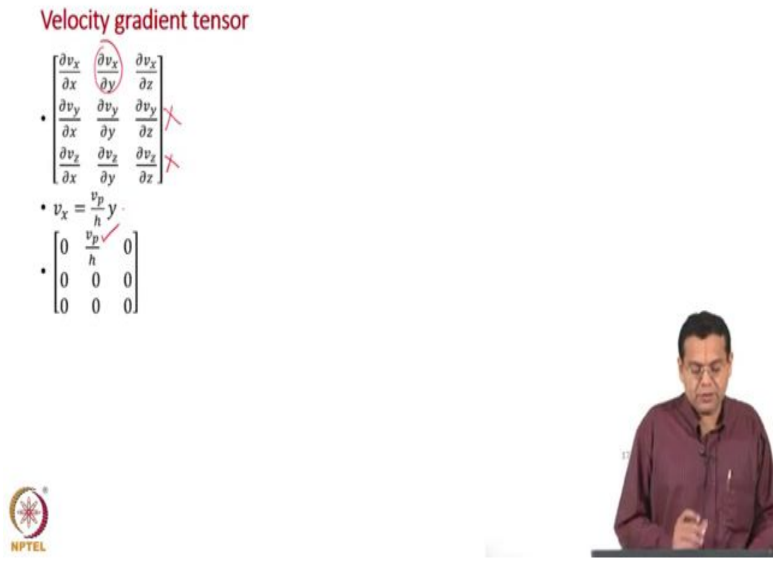

Now, what we have to do next? We discussed about strain rate here, introduced velocity gradient tensor. So, now, we can evaluate the velocity gradient tensor why is that because we know we now know the velocity profile which means I can evaluate the velocity gradient tensor the component of the velocity gradient tensor.

$$v_x = \frac{v_p}{h} y$$

So, the velocity gradient tensor is

$$\left[\frac{\partial v_x}{\partial x} \; \frac{\partial v_x}{\partial y} \; \frac{\partial v_x}{\partial z} \; \frac{\partial v_y}{\partial x} \; \frac{\partial v_y}{\partial y} \; \frac{\partial v_y}{\partial z} \; \frac{\partial v_z}{\partial x} \; \frac{\partial v_z}{\partial y} \; \frac{\partial v_z}{\partial z} \right] = \left[0 \; \frac{v_p}{h} \; 0\; 0\; 0\; 0\; 0\; 0\; 0 \right]$$

Of course because only the v_x is the non-zero velocity component, there is no element in the second row and third row. So, earlier once again we introduced as a physically meaningful matrix at tensor namely velocity gradient tensor, but now we are able to evaluated because we have found out the velocity profile.

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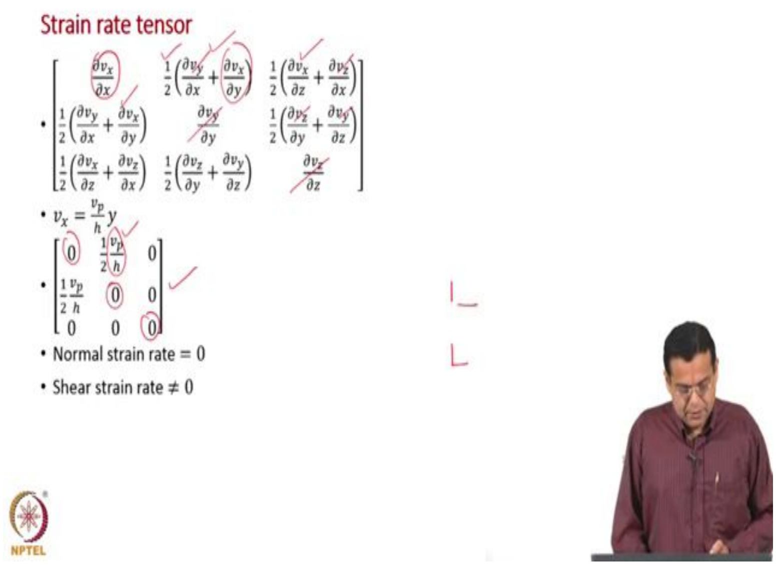

Ok. Now, then we introduced about strain rate tensor. This is the expression these are the components of the strain rate tensor.

$$\left[\frac{\partial v_x}{\partial x}\ \frac{1}{2}\left(\frac{\partial v_x}{\partial y}+\frac{\partial v_y}{\partial x}\right)\ \frac{1}{2}\left(\frac{\partial v_x}{\partial z}+\frac{\partial v_z}{\partial x}\right)\ \frac{1}{2}\left(\frac{\partial v_y}{\partial x}+\frac{\partial v_x}{\partial y}\right)\ \frac{\partial v_y}{\partial y}\ \frac{1}{2}\left(\frac{\partial v_y}{\partial z}+\frac{\partial v_z}{\partial y}\right)\ \frac{1}{2}\left(\frac{\partial v_x}{\partial z}+\frac{\partial v_z}{\partial x}\right)\ \frac{1}{2}\left(\frac{\partial v_y}{\partial z}+\frac{\partial v_z}{\partial y}\right)\ \frac{\partial v_z}{\partial z}\ \right]$$

Now, once again we can evaluate the components of the strain rate tensor using the velocity field which we have derived.

$$v_x = \frac{v_p}{h}y$$

So, what are the components?

$$\left[\frac{\partial v_x}{\partial x}\ \frac{1}{2}\left(\frac{\partial v_x}{\partial y}+\frac{\partial v_y}{\partial x}\right)\ \frac{1}{2}\left(\frac{\partial v_x}{\partial z}+\frac{\partial v_z}{\partial x}\right)\ \frac{1}{2}\left(\frac{\partial v_y}{\partial x}+\frac{\partial v_x}{\partial y}\right)\ \frac{\partial v_y}{\partial y}\ \frac{1}{2}\left(\frac{\partial v_y}{\partial z}+\frac{\partial v_z}{\partial y}\right)\ \frac{1}{2}\left(\frac{\partial v_x}{\partial z}+\frac{\partial v_z}{\partial x}\right)\ \frac{1}{2}\left(\frac{\partial v_y}{\partial z}+\frac{\partial v_z}{\partial y}\right)\ \frac{\partial v_z}{\partial z}\ \right] = [0\ \frac{1}{2}\frac{v}{h}$$

So, this is the strain rate tensor where we have non-zero values for two of the strain rate tensor components. Now, what do we conclude the normal strain rate they are the diagonal elements they are 0, shear strain rate is not equal to 0. What is shear strain rate? The off diagonal elements are shear strain rate. So, $\frac{v_p}{h}$ is shear strain rate that is non-zero. What does it mean? If you take a fluid element either horizontally or vertically there is no rate of change of the length.

And, if you take two perpendicular elements let us say at some time t; at time t + Δ t they will come together. Why is it? The shear strain rate is positive which means the angle between them should decrease. So, that is the meaning of these two value which have obtained one is of course 0, other is non-zero.

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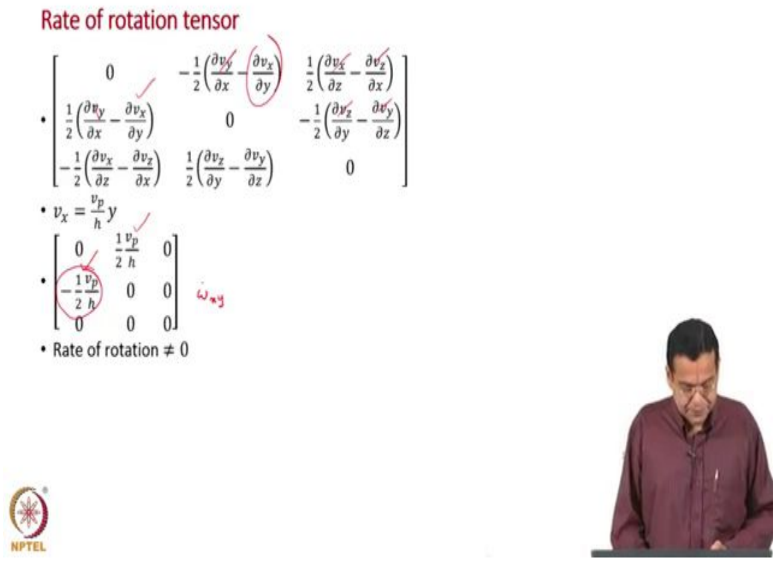

Now, we also discussed about rate of rotation tensor

$$\left[0 \; -\frac{1}{2}\left(\frac{\partial v_y}{\partial x}-\frac{\partial v_x}{\partial y}\right) \; \frac{1}{2}\left(\frac{\partial v_x}{\partial z}-\frac{\partial v_z}{\partial x}\right) \; \frac{1}{2}\left(\frac{\partial v_y}{\partial x}-\frac{\partial v_x}{\partial y}\right) \; 0 \; -\frac{1}{2}\left(\frac{\partial v_z}{\partial y}-\frac{\partial v_y}{\partial z}\right) \; -\frac{1}{2}\left(\frac{\partial v_x}{\partial z}-\frac{\partial v_z}{\partial x}\right) \; \frac{1}{2}\left(\frac{\partial v_z}{\partial y}-\frac{\partial v_y}{\partial z}\right) \; 0\right]$$

And these are the components of the rate of rotation tensor. Now, of course, a diagonal elements are anyway 0 because it is antisymmetric tensor and we can evaluate the other components using the velocity profile

$$v_x = \frac{v_p}{h} y$$

Now let us evaluate the other terms,

$$\left[0 \; -\frac{1}{2}\left(\frac{\partial v_y}{\partial x}-\frac{\partial v_x}{\partial y}\right) \; \frac{1}{2}\left(\frac{\partial v_x}{\partial z}-\frac{\partial v_z}{\partial x}\right) \; \frac{1}{2}\left(\frac{\partial v_y}{\partial x}-\frac{\partial v_x}{\partial y}\right) \; 0 \; -\frac{1}{2}\left(\frac{\partial v_z}{\partial y}-\frac{\partial v_y}{\partial z}\right) \; -\frac{1}{2}\left(\frac{\partial v_x}{\partial z}-\frac{\partial v_z}{\partial x}\right) \; \frac{1}{2}\left(\frac{\partial v_z}{\partial y}-\frac{\partial v_y}{\partial z}\right) \; 0\right] = [0$$

Now, what do we conclude remember, this term represents rate of rotation. So, in our nomenclature $\dot{\omega}_{xy}$ and which is not 0 which means it is a rotational flow it is a rotational flow.

And, it is negative which means the fluid element will undergo a clockwise rotation. Remember, we said clockwise is negative anticlockwise is positive. So, the fluid element will undergo clockwise rotation. That is the conclusion from here.

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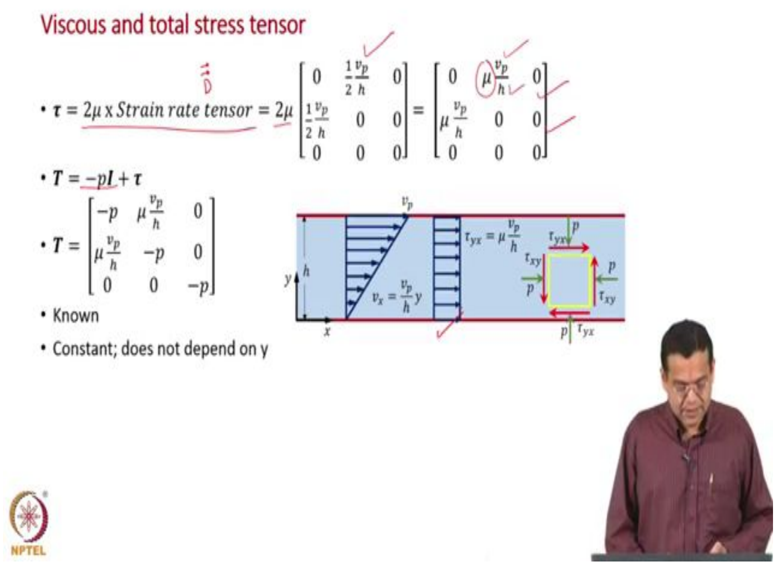

Then we discussed about the Newton’s law of viscosity where we related the viscous stress to the strain rate. And, we can do that as well here viscous stress tensor is

$$\tau = 2\mu\, x\ \textit{Strain rate tensor} = 2\mu \begin{bmatrix} 0 & \frac{1}{2}\frac{v_p}{h} & 0 \\ \frac{1}{2}\frac{v_p}{h} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \mu\frac{v_p}{h} & 0 \\ \mu\frac{v_p}{h} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

And, we can also evaluate the total stress tensor. As we have seen

$$T = -pI + \tau$$

So,

$$T = \begin{bmatrix} -p & \mu\frac{v_p}{h} & 0 \\ \mu\frac{v_p}{h} & -p & 0 \\ 0 & 0 & -p \end{bmatrix}$$

What is the big difference now? Earlier we introduced them as variables, now we are able to evaluate that not only variables they were unknown.

When you first discussed viscous stress total stress there were unknown, now they become known; not alone known they are able to evaluate in terms of the velocity profile because you know the velocity profile you are able to evaluate the viscous stress tensor in terms of the variables which describe this particular flow. What are the variables which describe the flow? One is of course, property of the fluid μ other is the velocity of the top plate, v_p other is the distance between the two plates, h. So, we are able to evaluate the viscous stress tensor and the total stress tensor.

What is other observation? If you look at the viscous stress tensor the it is just the constant μ is the constant, v_p is a constant, h is a constant. So, does not depend on y and that is what is shown here. The shear stress component of the viscous stress tensor or viscous shear stress does not depend on y just a constant. This stress element will discuss in the next slide. We also discussed about stress element in the beginning and we also discuss about stress vector. We will discuss about stress element in this example.

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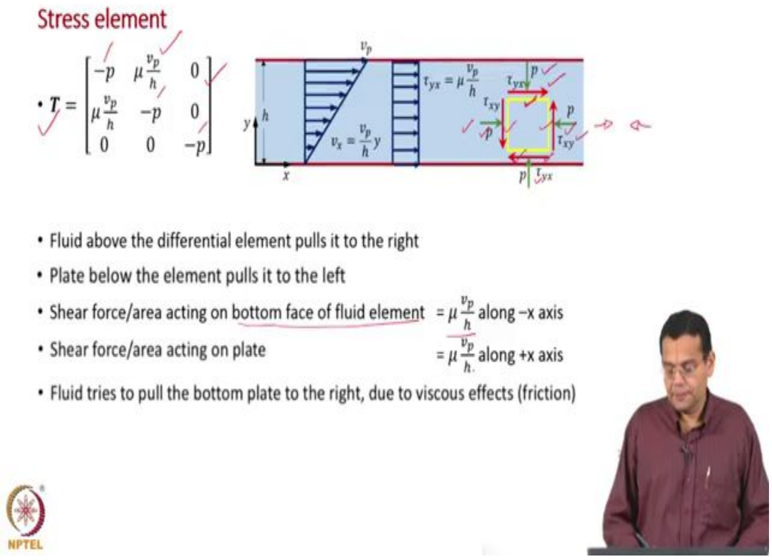

$$T = \left[-p\ \mu\frac{v_p}{h}\ 0\ \mu\frac{v_p}{h}\ -p\ 0\ 0\ 0\ -p \right]$$

How do you represent the stress element? How do you represent this total stress tensor as a stress element? We have done this in the very beginning of solid mechanics and of course, later on in fluid mechanics also.

Now, if you look at the normal stresses we have only pressure as a normal stress and it is compressive. So, we are representing it as compressive forces and that is why the arrows are towards each other. So, on a positive phase the force is towards the negative x direction; on a negative phase the force is towards positive x direction, similarly in the y direction as well. So, that represents the compressive pressure force.

Now, coming to the shear stress components is positive μ, v_p, h were all constants and it is a positive as well. So, on a positive phase the force should be along the positive direction. So, if you take this phase it is the normal to the phase is along the positive y-axis. So, the force is along positive x-axis and for this phase the normal is towards the positive x-axis and force is along the positive y-axis, similarly the other forces as well.

So, now, the earlier we discussed stress element we should have some artificially given values 30, -10, -15. Now, after evaluating the velocity profile and then going through the strain rate tensor, viscous stress tensor, total stress tensor we are able to draw a stress element which represents the state of stress at a point in this configuration. So, now, it is much more meaningful; earlier they were just numbers the we did that all to have practice understand what a stress element is. Now, we are really using it to tell about the state of stress in a fluid which is flowing between 2 phase, 1 fix, other moving. And, we can evaluate that as well because we can we know μ, v_p, h all are known to us.

How, do you interpret this? If you look at this shear stress the fluid above the different element pulls it the right because that is at a higher velocity. So, that will try to pull it to the right and plate below that is at rest that is fixed. So, that it will pull the fluid element to the left and that is why you have the force in this direction. So, you have a fluid element and the top plate is set in motion. So, it is in moving. So, the fluid above that if you considered fluid element here fluid here it is higher velocity so, it will try to pull the fluid element to the right; this plate is stationary and that will try to pull the fluid to the left and that is what is seen there.

Now, in terms of evaluating you can evaluate the shear force per area that is what our shear stress tells you and shear force acting on bottom phase of fluid element. Remember, always our viscous stress, total stress where with respect to acting on the fluid. So, shear force per area which is our shear stress acting on bottom phase of fluid element is of course, this $\mu\frac{v_p}{h}$ and what is the direction? It acts along the negative x axis. Fluid at the bottom plate because of plate is stationary the plate tries to pull the fluid to the left. So, shear force acting on the fluid element is towards the negative x-axis.

And, of course, shear force acting on the plate is same magnitude $\mu\frac{v_p}{h}$, but along positive x axis fluid is trying to pull the bottom plate towards the positive x axis that is why shear force per area acting on the plate is towards the positive x axis bottom plate to the right of course, due to viscous effects.

In one way what we have done is gone through a full cycle I would say we started with the stress went to strain rate went to Newton's law of viscosity derive the Navier-Stokes equation. Now, having derived the Navier-Stokes equation we have applied that found the velocity profile and then we found the strain rate applied the Newton's law of viscosity and found out the viscous stress and total stress.

So, we have gone through one full cycle I would say and that is what is shown here also in terms of our journey slide; we discussed about the viscous stress, total stress, then we discussed about the strain rate, related viscous stress and strain rate through Newton's law of viscosity, substituted in the linear momentum balance to derive the Navier-Stokes equation.

Now, use a Navier-Stokes equation found out the velocity profile once I know the velocity profile I can know the components of strain rate tensor. I can find out the viscous stress tensor components using Newton's law of viscosity of course, which means this becomes known of course, we can draw the stress element as well. So, in one way we have gone through a complete cycle and that is what I have represented here.

Total stress, then strain rate, then Newton's law of viscosity then Naveir-Stokes equation in the velocity profile. Once you know the velocity profile then find strain rate used Newton's law of viscosity to find the total stress. Of course, I would not say perfectly cyclic, but these arrows does not mean one goes to the other, but we have gone through the cycle in the forward direction and once again in the reverse direction.

When we went to the forward direction they were all variables which aided at which aided us in the development of the Navier-Stokes equation we kept on introducing physically meaningful variables. For example, viscous stress tensor, total stress tensor, strain rate and then the Newton's law of viscosity they were all operations which for which made helped us in deriving the Navier-Stokes equation. Following that path we derived the Navier-Stokes equation.

Now, once we have derived the velocity profile we can evaluate all the variables which we have introduced. For example, $\frac{\partial v_x}{\partial y}$ was the derivative now we can evaluate $\frac{\partial v_x}{\partial y}$ and then physically meaningful variables like the strain rate and then the viscous stress tensor, and then you can also show the stress element etcetera. So, that; so this one way this helps us to understand the path we have followed and why we followed that.