Continuum Mechanics And Transport Phenomena
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Lecture – 90
Planar Couette Flow Governing Equations

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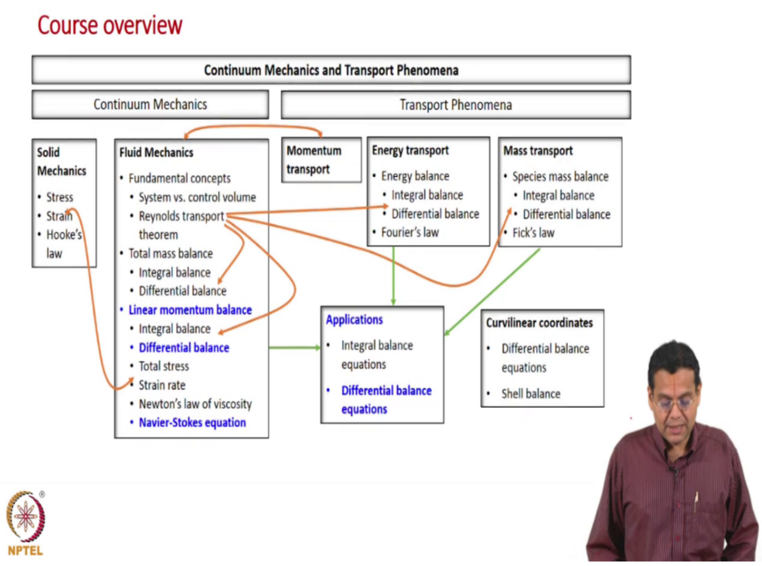

We have been discussing applications of differential form of a the linear momentum balance namely the Navier-Stokes equation.

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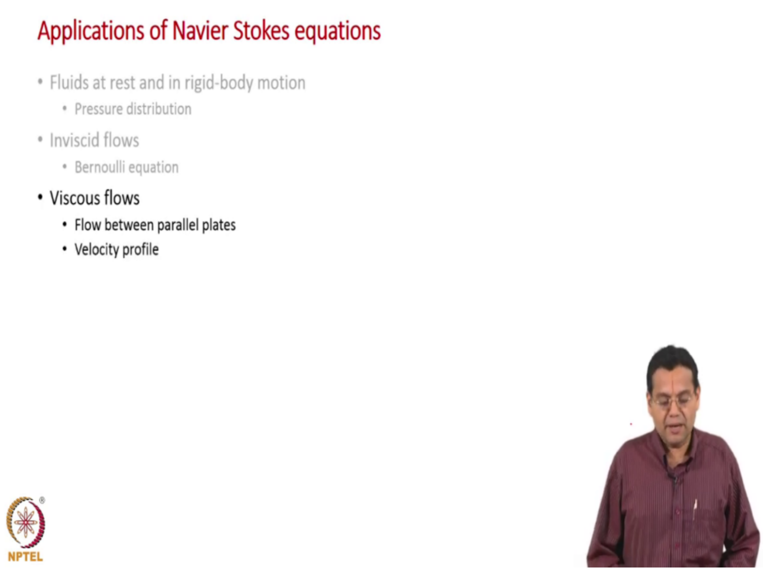

And we are discussing the applications hierarchically going from the simplest to the complex applications. So, first we started with the fluids at rest and rigid body motion, looked at the pressure distribution and then we have discussed inviscid flows and which we discuss the mainly the Bernoulli's equation. Now, we are going to discuss viscous flows. So, initially it was fluid at rest and rigid body motion, and there are no viscous stresses at all.

Then we considered inviscid flows where the viscous stresses were negligible. Now we are going to consider viscous flows which means that there are viscous stresses. And We are going to discuss about flow between parallel plates which are come across a several times and what is the main objective is to derive the velocity profile.

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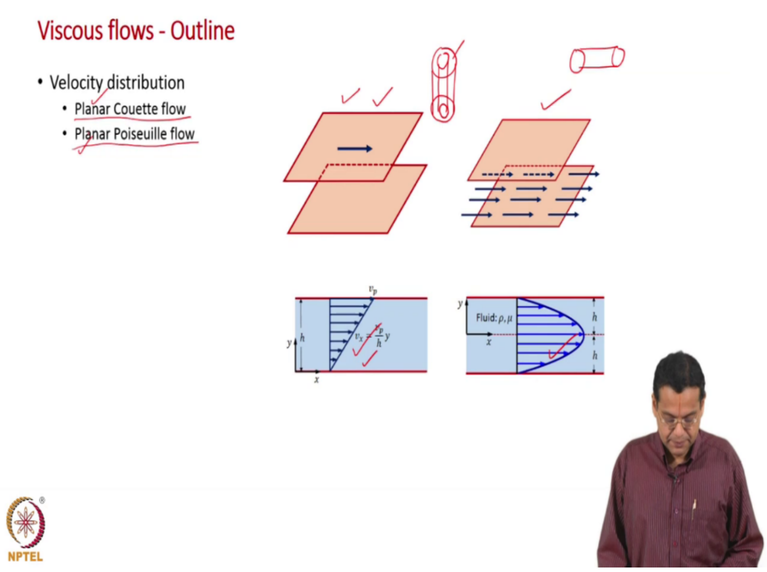

So, let us look at the outline for this set of applications only of course, two words as I told you velocity distribution. The velocity distribution we are going to discuss for two geometries; the two geometries which are come across several times earlier what are they? Flow between two parallel plates, one in which the top plate is set in motion, other in which both plates are fixed, but pressure driven flow.

For example, something pushes water through the between the two plates let say a pump which pushes the water between the two plates. And we have seen this velocity profile even use them we are going to finally, I would say derive those expressions. We have been telling this towards the end of fluid mechanics part, we will be deriving the expressions and that is what we are up to now derived the velocity profile,

In terms of terminology, so, far I have been using words like configuration, geometry example etcetera. Now, time has come to use more formal words they are called planar Couette flow the first one and the second one is called the planar Poiseuille flow. Now, why is it planar Couette flow, planar Poiseuille flow? Moment you say Couette flow, it represents the flow between two concentric cylinders we are come across this earlier, when we discussed about viscosity example of Newton's law of viscosity when discuss about viscometer as an example of application of Newton's law of viscosity.

So, the top figure represent the Couette flow. So, it is in cylindrical coordinates because we are considering a similar flow in terms of Cartesian coordinates, it is called planar Couette flow. Now, similarly when you say Poiseuille flow what represents Poiseuille flow is, flow through a circular pipe. And because we are considering analogous flow condition under Cartesian coordinate, it is called planar Poiseuille flow.

So, Couette flow and Poiseuille flow in general represents flow in cylindrical coordinates and because we are considering analogous as flows in Cartesian coordinates and we are calling them as planar Couette flows, planar Poiseuille flow.

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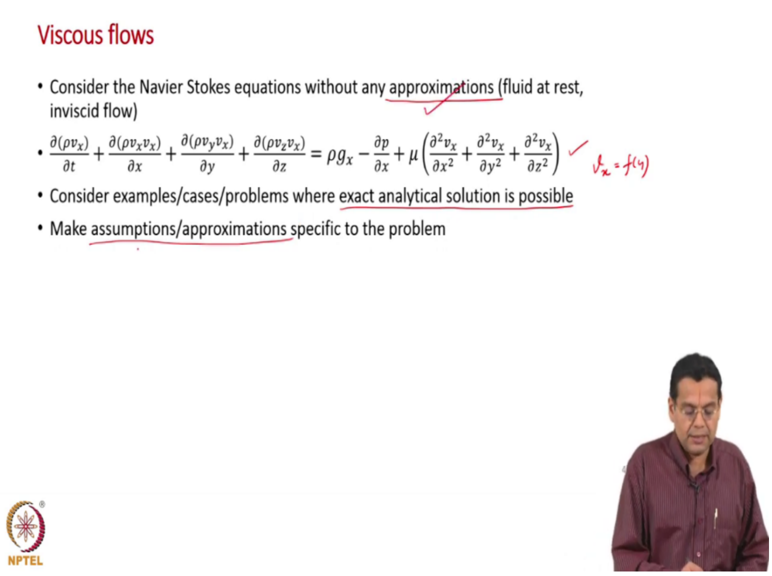

So, let us start the Navier-Stokes equation, now we are going to consider the Navier-Stokes equation without any approximations. So, far we have been making approximations fluid at rest, rigid body motion, inviscid flow. So, we have been not considering the Navier-Stokes in entirety, we have neglected certain terms.

Now, you are going to consider the full form of the Navier-Stokes equation and

$$\frac{\partial(\rho v_x)}{\partial t} + \frac{\partial(\rho v_x v_x)}{\partial x} + \frac{\partial(\rho v_y v_x)}{\partial y} + \frac{\partial(\rho v_z v_x)}{\partial z} = \rho g_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right)$$

That is the Navier-Stokes equation. First we said the velocity is 0 flow fluid that rest, then we said the entire fluid body as one velocity, all the points has a same velocity. So, you can represent as one acceleration term all the point of the same velocity. So, we can represent as an acceleration term on the left hand side then we neglect all the viscous terms on the right hand side. So, now, we are considering all the terms.

Now, the Navier-Stokes equation we have written only one direction, solving Navier-Stokes equation including all the terms is so difficult really a formidable work because you will have three direction equations you will have continuity equation also. So now, what do you going to do is, consider examples or cases, geometry, problems etcetera where we can get a exact analytical solution.

So, we will limit our being introductory course, we consider only those cases where we can get a analytical solution. What do you mean by that? For example, let us say you can get v_x as some function of y, x velocity as a function of y. Other approach is numerical approach where you get v_x as different points of y, not as a expression you get only in terms of numerical data numerical answer.

So, you are going to consider only cases where exact analytical solution is possible. So, what should you do for that?. Make assumptions or approximations not for a entire class of flow, but specific to the problem, this should be kept in mind. When we see approximation here, we made a approximation for a class of flow. What do you mean by class of flow? Class of flow where fluid is under rest condition where the fluid moves as a rigid body they are a class of flows.

Similarly, a class of flows called inviscid flow. Now all are viscous flows only, but these assumption approximations are problem specific. Given a particular problem, given a particular geometry etcetera for that particular example we will make approximation assumptions that will understand as we go along.

So, though we take the entire Navier-Stokes equation, now the assumptions are not for a whole class, but specific to the problem that is the difference between this approximation and earlier assumption.

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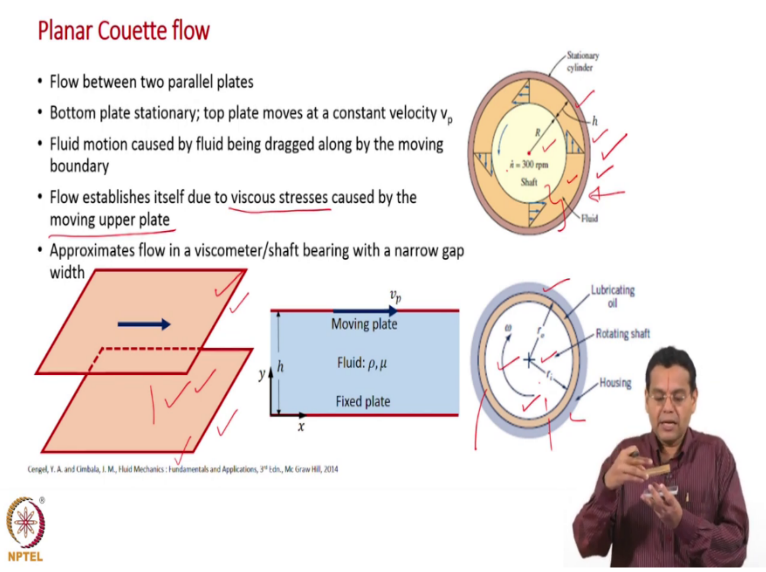

So, now let us take the planar Couette flow now that we have understood the terminology. Now, what follow should be very easy to us. We have discuss the several times flow between two parallel plates we have come across this several times, flow between two parallel plates and the bottom plate is stationary and the top plate moves at a constant velocity v_p.

And, fluid motion is caused by fluid being dragged along with the moving boundary. Because the top plate moves that plate drags the fluid along with it and hence causes motion in the region between the two plates. What is the more formal way of saying this? That is little more informal; we say fluid is being dragged by the plate, more formal way of saying that is the flow establishes itself that is what happens due to the viscous stresses caused by the moving upper plate, that now becomes very clear.

The previous statement is little more informal saying that that moving plate drags the fluid along with it, hence cause sets the motion between the two plates really what happens? Because of the upper plate moving there are viscous stresses and because of that the flow get established. Now what are the flows which are similar to this, or what is the flow that this geometry approximates?

One is familiar to us the other is almost similar to that, what is that that is familiar to us? We have come across the viscometer arrangement, when we discuss few classes back the application of Newton's law of viscosity. What is that we saw? We have a inner cylinder which is rotating, the outer cylinder is stationary. We have a fluid let us say water or any viscous fluid between the two cylinders and then we measure the torque and use that to calculate viscosity. We also said that this is actual Couette flow, actual Couette flow is this which is a flow between two coaxial cylinders.

Now, our geometry is flow between two parallel plates; one is moving other is stationery. Here again the flows between two cylinders and one is rotating other is stationery. So, that way they are analogous. When is it is valid? It is valid when the gap between the two cylinders is very small. When the gap is very small then the flow with between the two cylinders can be approximated to flow between two plates that is the condition.

Now so, that is the field of instrumentation or measurement what is the other example? More of engineering application is example shown here which is a shaft bearing. What is shaft bearing mean? You have a rotating shaft and then you have a housing which is similar to a stationary cylinder and then you have a lubricating oil of course, this is always or almost a high viscous oil so, that the friction is reduced between the rotating shaft and the housing.

So, once again this configuration is also similar to our case or our configuration approximates this configuration. Why is it? Because you have a rotating cylinder something like a moving plate and then you have a housing or stationery cylinder, which is something similar to our stationery plate. And, the gap is usually not very large in a shaft bearing it is a very small gap. And, there is a high viscous oil there in our case viscous fluid array a fluid back our viscous stresses are going to be included and that is between the two plates.

So, what we are discussing is a Cartesian co-ordinate system so, that our complex is reduced and though this arrangement cannot be realized in lab, we cannot have two plates like this. In a lab we cannot have to plate like this make one move and then get a velocity field between them. But how do you realize is like these two cases have a fluid between two cylinders and rotate one cylinder you can get a similar velocity field. And, if the gap is very small, and both are Cartesian configuration approximates the cylindrical geometry that is the idea.

So, in terms of nomenclature now x co-ordinate, y co-ordinate and this is the our front view and gravity acts along the y axis and there is moving plate, the velocity of the plate is v_p. And, there is a fixed plate and fluid has properties ρ and then viscosity μ the distance between the two places is h.

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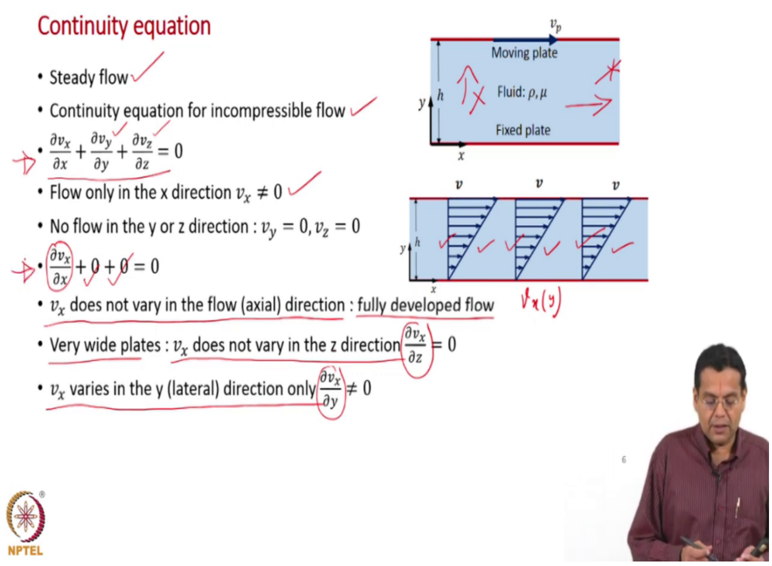

So, now first we will have to start with the continuity equation. Let us start the continuity equation, we will consider steady flow. What does it mean? Suppose you have a these two plates, and let us say before you start the experiment just imagine suppose if you are able to realize this in the lab, and this two plates a stationary. Let us say time t = 0 you start moving the plate, there is some initial transience we are not going to consider that.

You set the plate in motion and now the velocity field reaches a steady state condition and that is where we are going to focus. So, we are going to consider a steady flow and we are going to consider incompressible flow, in this case it is incompressible fluid also if it is water.

$$\frac{\partial(v_x)}{\partial x} + \frac{\partial(v_x)}{\partial y} + \frac{\partial(v_x)}{\partial z} = 0$$

So, that is the continuality equation for incompressible flow. Now, we will consider flow only in the x direction, as I told you sometime back we are not going to solve the complete Navier-Stokes and continuity equation, we are going to we are going to make assumptions specific to the problem. When I say specific to the problem, we are going to consider steady flow of course, incompressible flow comes along with the Navier-Stokes equation.

$$v_x \neq 0; \quad v_y = 0; \quad v_z = 0$$

Now, this is an assumption which you are going to make for this particular problem flow only in the x direction. What does it mean? I got a flow only in the x direction and in the y direction there is no flow and in the z direction there is no flow. No flow in the y or z direction that is also intuitively understandable because you are setting the plate to motion moment in this direction.

So, we expect the flow only in this direction, we are not there is no driving force which can cause the flow in the y direction or the z direction. So, no flow in the y or z direction. Now, let us see: what is the implication of that is what we conclude from the continuity equation. Let us substitute in the continuity equation we have

$$\frac{\partial(v_x)}{\partial x} + 0 + 0 = 0$$

So,

$$\frac{\partial(v_x)}{\partial x} = 0$$

What does it physically mean? $\frac{\partial(v_x)}{\partial x}$ tells you change of v_x with respect to x. Now, because that is 0 we conclude that v_x does not vary in the flow direction, the flow direction in this case the axial direction that is what you shown here, v_x can be a function of y that is what is shown here we are going to see that shortly, v_x will linearly vary with y. We are not telling anything about that, what we are saying is whatever the function v_x of y that does not change as you go along the flow directions.

That is what we say yes fully developed flow. So, now, what is the conclusion? We said $v_x \neq 0$ then we assumed that, $v_y = 0$, $v_z = 0$ if you do so, and use the continuity equation you conclude that the flow should be fully developed. You cannot have a condition where $v_y = 0$, $v_z = 0$ and v_x varies with the x that cannot happen that will violate the continuity equation that is the conclusion.

Now, what we do next is as part of our assumptions specific to the problem, I say that the plates are very wide in the z direction which means that we are somewhere in between and so, there is no variation of x velocity in the z direction very wide plates when you say y this is this width a very wide plate and so, there is no variation in the z direction v_x does not vary in the z direction. So, we assume once again in the assumption that

$$\frac{\partial v_x}{\partial z} = 0$$

Now what do we infer from all this? We said v_x of course, it cannot varying the z direction based on the continuity equation. Now, based on this assumption we have we are saying that v_x is assume not to vary in the z direction. So, only direction v_x can be varying is the y reaction ok. This v_x is not varying with x direction is a conclusion from the continuity equation, v_x not varying z direction is an assumption. So, v_x we will vary only in the y direction which we call as the lateral direction.

$$\frac{\partial v_x}{\partial y} \neq 0$$

So, only this derivative will remind. What is that? Variation of v_x in the y direction that derivative will not be equal to 0. So, about this slide tells you set of combination of assumptions and conclusion on continuity equation. We have made lot of assumptions specific to the problem and one conclusion or derived result I would say from the continuity equation. What are the assumptions?

$v_y = 0, v_z = 0, \frac{\partial v_x}{\partial z} = 0, \frac{\partial v_x}{\partial y} \neq 0$ and conclusion from the continuity equation, $\frac{\partial v_x}{\partial x} = 0$.

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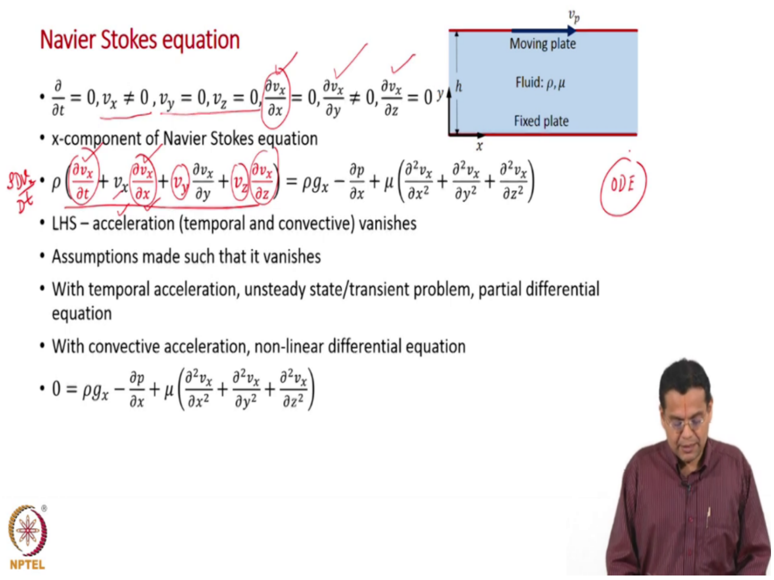

So, let us list all of them in the in the slide

$$\frac{\partial}{\partial t} = 0; \quad v_x \neq 0; \quad v_y = 0, \ v_z = 0, \frac{\partial v_x}{\partial z} = 0, \frac{\partial v_x}{\partial y} \neq 0, \ \frac{\partial v_x}{\partial x} = 0$$

So, now, let us put all these I would say assumptions and conclusion from continuity equation, in simplifying the Navier-Stokes equations. So, first we start with continuity equation state flow assumptions, arrive conclusions from continuity equation and then simplify the Navier-Stokes equation. Let us take the x component of Navier-Stokes equations let us see what happens.

$$\frac{\partial(\rho v_x)}{\partial t} + \frac{\partial(\rho v_x v_x)}{\partial x} + \frac{\partial(\rho v_y v_x)}{\partial y} + \frac{\partial(\rho v_z v_x)}{\partial z} = \rho g_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial(v_x)}{\partial t} + v_x\frac{\partial(v_x)}{\partial x} + v_y\frac{\partial(v_x)}{\partial y} + v_z\frac{\partial(v_x)}{\partial z}\right) = \rho g_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right)$$

Now, we assumed the conditions that

$$\frac{\partial}{\partial t} = 0; \quad v_x \neq 0; \quad v_y = 0, \ v_z = 0, \frac{\partial v_x}{\partial z} = 0, \frac{\partial v_x}{\partial y} \neq 0, \ \frac{\partial v_x}{\partial x} = 0$$

Now, we made assumptions in such a way that a left hand side vanishes that is a other way of looking at it. We have the assumptions would have look disconnected. Now the reason for making those assumptions is that we made those assumptions so, that the left hand side vanishes; both the temporal and convective terms vanishes.

Now, why should we do that? What would happen if the left hand side were to be present? Let us discuss that suppose let us say we do not assume steady state. What will happen? You will have this $\frac{\partial v_x}{\partial t}$ term on the left hand side. You will see shortly that we will result in a ordinary differential equation, that is also obvious because the time variable is gone now for the present case only y is independent variable so, we are going to get a on a differential equation.

Suppose if we have not assumed steady state, then we would have the time derivative also and we would have resulted in a unsteady state transient problem and we would have to solve a partial differential equation. We want to avoid that being interact being keeping in mind of this scope of the course; you want to study cases where the result is a the equation simplifies the on an ordinary differential equations that is why we assumed steady state.

Now, second what would happen if we are the convective acceleration terms on the left hand side? Now if you look at those terms we have remember the velocity components there all unknowns. So, we have product of two unknowns either the variable as such or variable derivative of the variable which means that they are non-linear terms. Non-linear terms are usually much more difficult to solve, that is why we made a set of assumption in such a way that the convective acceleration also drops.

So, the set of assumptions are driven by one objective for this particular problem, to make it simple that to drive both the temporal acceleration and the convective accelerations term to 0 so, that we get a linear or ordinary differential equation that is a whole idea. Otherwise assumptions may look little adhoc.

So, of course, if you take other courses in higher semester you may not make this assumption you may solve the entire Navier-Stokes equation under transient condition with convective terms etcetera.

$$0 = \rho g_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right)$$

So, now, the left hand side is 0 and right hand side you have all the forces. We will take up the right hand side now, we are not discuss the right hand side how it get simplified will take up that right now we will take it now.

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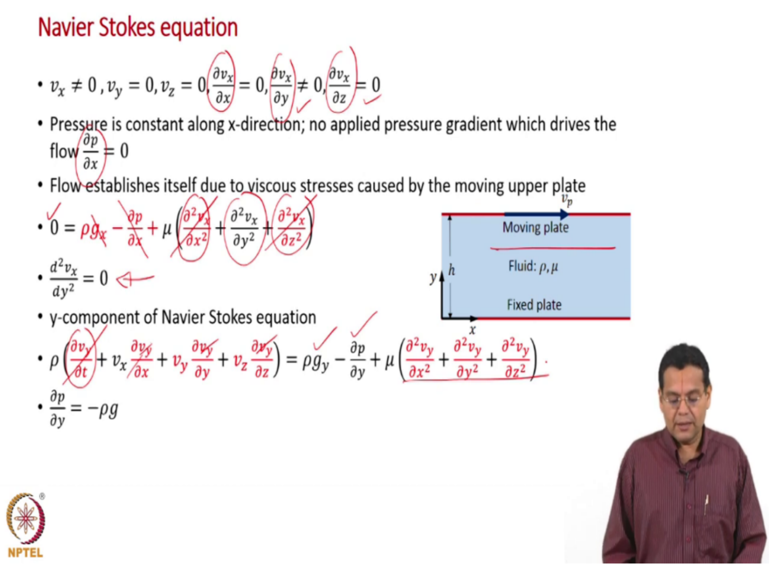

Now, let us write down the assumptions required. So,

$$\frac{\partial}{\partial t} = 0; \quad v_x \neq 0; \quad v_y = 0, \ v_z = 0, \frac{\partial v_x}{\partial z} = 0, \frac{\partial v_x}{\partial y} \neq 0, \ \frac{\partial v_x}{\partial x} = 0$$

Now, in this case as I told you what drives the fluid motion is because of the plate. You do not have any pump or anything which is pushing the fluid between the two plates. So, the motion of the fluid is caused by the movement of the plate. So, pressure is constant along the x direction. Why is that? Because there is no applied pressure gradient which drives the flow; so,

$$\frac{\partial p}{\partial x} = 0$$

You will see the next example which is a Poiseuille flow because both plates are stationary in the Poiseuille flow case, right now we on the Couette flow case, the Poiseuille flow case both plates are stationary. So, you need some external device a pumping mechanism to pump the fluid between the two plates.

But now no such pump is required we are not imposing this plates itself causes motion. So, pressure is constant long x direction because no applied pressure gradient which drives a flow. The flow establishes itself due to viscous stresses caused by the moving upper plate. So, now, let us see what happens to the right hand side. Left hand side we have already seen that it vanishes become 0 right hand side we have ρg_x and gravity acts along y direction. So,

$$g_x = 0$$

Now let us come to the viscous terms. Now, $\frac{\partial v_x}{\partial y} \neq 0$ So, we do have a second derivative and other terms are 0.

$$0 = 0 - 0 + \mu\left(0 + \frac{\partial^2 v_x}{\partial y^2} + 0\right)$$

$$\frac{\partial^2 v_x}{\partial y^2} = 0$$

So, we are left out the very simple equation and because y is only variable it becomes a ordinary differential equation. We just reduced, I would say of formidable Navier-Stokes equation, very complex Navier-Stokes equation to a simple linear second order ordinary differential equation, thanks to the assumptions whatever you are made. But, for that it would have been very difficult, but we simplify the problem making several assumptions so, that we result in a ordinary differential equation. In this case it does not even look like a Navier-Stokes equation, but do remember we started with a Navier-Stokes equation and simplified to this particular equation.

Now, let us look at the y component of Navier-Stokes equation. Let us write that the y component of Navier-Stokes equation and let us simplify that.

$$\rho\left(\frac{\partial(v_y)}{\partial t} + v_x\frac{\partial(v_y)}{\partial x} + v_y\frac{\partial(v_y)}{\partial y} + v_z\frac{\partial(v_y)}{\partial z}\right) = \rho g_y - \frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2}\right)$$

Now, left hand side very simple $v_y = 0$, So, all terms just vanish other way of saying is of course, there is no v_y itself. So, there is no question of derivative here moment you do not have v_y none of the derivative exist. So, just because $v_y = 0$ left hand side vanishes you not discuss anything further at all ok. Now, right hand side we do have ρg_y because gravity acts in a y direction and then because of that there is a pressure gradient and what happens to the viscous term $v_y = 0$. So, all of them vanished. So,

$$0 = \rho g_y - \frac{\partial p}{\partial y} + \mu(0 + 0 + 0)$$

$$\frac{\partial p}{\partial y} =- \rho g$$

So, the y component of Navier-Stokes equation simplifies to the equation which we discussed for fluids at rest. It just hydrostatic balance, the hydrostatic stress and the gravity forces balance each other same equation which we discussed for fluids and at rest along the y direction. Why is that? Because there is no flow y direction; so, it becomes same as what you have discussed for fluids at rest.

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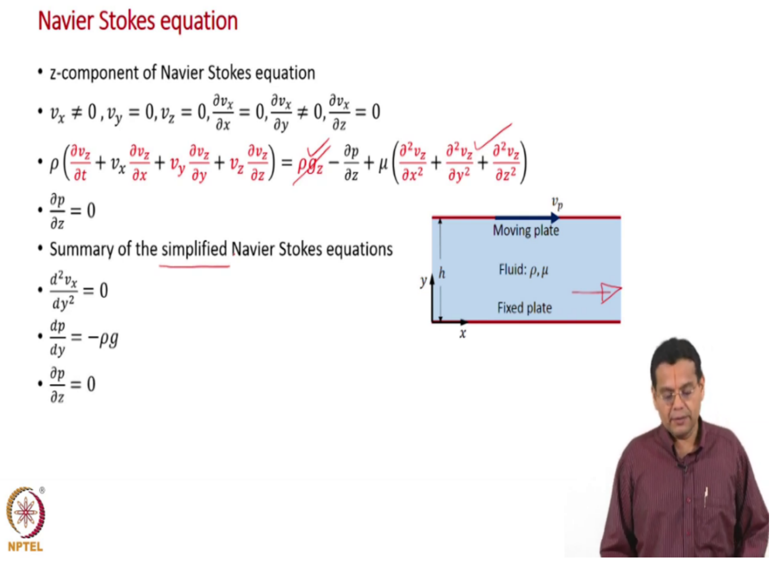

Now, let us take the z component of Navier-Stokes equation; let us write down that assumptions are known, we have seen this few times now and let us see what happens to the z component.

$$v_x \neq 0; \quad v_y = 0, \; v_z = 0, \frac{\partial v_x}{\partial z} = 0, \frac{\partial v_x}{\partial y} \neq 0, \; \frac{\partial v_x}{\partial x} = 0$$

$$\rho\left(\frac{\partial(v_z)}{\partial t} + v_x\frac{\partial(v_z)}{\partial x} + v_y\frac{\partial(v_z)}{\partial y} + v_z\frac{\partial(v_z)}{\partial z}\right) = \rho g_z - \frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2}\right)$$

Left hand side once again because $v_z = 0$ the left hand side vanishes, right hand side there is no gravity along z directions. So, that goes to 0 and all the viscous terms become 0 because $v_z = 0$. So,

$$0 = 0 - \frac{\partial p}{\partial z} + \mu(0 + 0 + 0)$$

$$\frac{\partial p}{\partial z} = 0$$

This gives us the pressure gradient is 0 along the z direction. So, there is no variation of pressure in the z direction. So, there is no variation of pressure along the x direction also because, there is no impossible pressure gradient. So, pressure does not vary along x direction, z direction and vary in the y direction because of gravity only.

Now, let us summarise the simplified Navier-Stokes equation. In the x direction we had

$$\frac{d^2 v_x}{dy^2} = 0$$

Then in the y direction same as equation is same as what we are discuss for fluid at rest is a hydrostatic pressure distribution

$$\frac{\partial p}{\partial y} = -\rho g$$

And in z direction there is no pressure gradient

$$\frac{\partial p}{\partial z} = 0$$