Continuum Mechanics And Transport Phenomena
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Lecture - 09
Substantial Derivative Part 2

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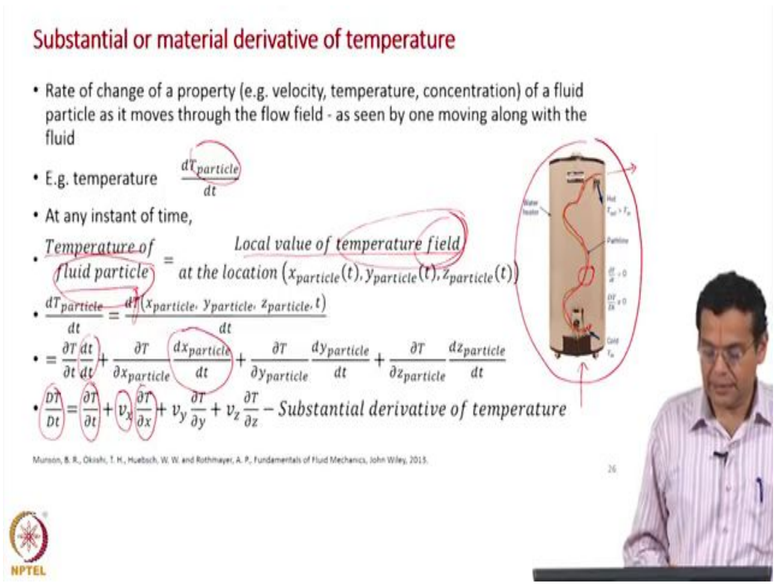

Let us do the same thing. Follow the same steps for another variable namely temperature. Two objectives of this, specifically emphasize that the material derivative can be applied for any property. We have done first for velocity (in the last lecture). It can be done for temperature, can be done for concentration. Now that is why all whatever we are discussing are all discussed under the heading called Fundamental concepts. They are not necessarily related to fluid flow, though a fluid mechanics book discusses all these concepts. These concepts are applicable for fluid flow, energy, heat flow and then later on species flow also.

To emphasize that also here I am illustrating the substantial derivative of temperature, let us take an example what do we mean by substantial derivative of temperature. You have here is a household geyser (see above refer slide). Water flows in, gets heated and then comes out. Now you have a particle. What is the meaning of substance derivative? Rate of change of property of a fluid particle, in this case rate of change of temperature. So, as you follow the fluid particle what is a rate of change of its temperature is a substance derivative $\frac{DT}{Dt}$.

Now, take example of temperature and what we are interested in is $\frac{dT_{particle}}{dt}$ just like we had velocity earlier we have T of temperature particle here and just like we wrote earlier at any instant at any particular location along the path of the particle, we said velocity of the particle is equal to the velocity field. Similarly here the temperature of the fluid particle are more specifically temperature sensed by the fluid particle should be equal to the local value of temperature field.

At any instant of time,

Temperature of fluid particle = Local value of temperature field at the location $(x_{particle}(t),\ y_{particle}(t),\ z_{particle}(t))$.

So, once again relating Lagrangian quantity to a Eulerian quantity. Now so, the T particle in the time derivative replaced in terms of the Eulerian temperature T.

$$\frac{dT_{particle}}{dt} = \frac{dT\left(x_{particle}, y_{particle}, z_{particle}, t\right)}{dt}$$

And so, total derivative once again we differentiate with partially with respect to the four independent variables; x particle, y particles, z particle and time.

$$= \frac{\partial T}{\partial t}\frac{dt}{dt} + \frac{\partial T}{\partial x_{particle}}\frac{dx_{particle}}{dt} + \frac{\partial T}{\partial y_{particle}}\frac{dy_{particle}}{dt} + \frac{\partial T}{\partial z_{particle}}\frac{dz_{particle}}{dt}$$

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + v_x\frac{\partial T}{\partial x} + v_y\frac{\partial T}{\partial y} + v_z\frac{\partial T}{\partial z}$$

Now this special derivative is called substance derivative $\frac{DT}{Dt}$. I cannot use it over acceleration that is applicable certainly for velocity only. We call this only as substantial derivative of temperature.

So, now to little bit imagine about this we said we follow the fluid particle something like sitting on the fluid particle. You would have seen in the movies a person becomes like a fly and gets into house and so on. So, likewise you imagine yourself as a something like a temperature resistant and get into it and then you start this stopwatch and then what is the rate of change of temperature, you would measure and that is $\frac{DT}{Dt}$, that is how you can imagine that.

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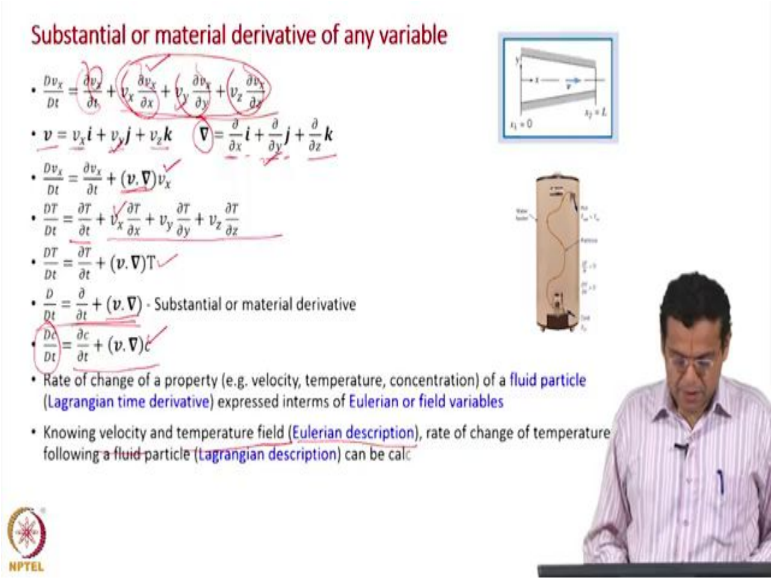

So, now we will extent this to any variable. Throughout the course we will come across substantial derivative of course velocity, we came across density, we came across temperature, concentration. I think we will come across pressure as well and substantial derivative all have the same physical significance rate of change of property as you follow the fluid particle.

$$\frac{Dv_x}{Dt} = \frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z}$$

Now, of course this x component of the velocity $\frac{Dv_x}{Dt}$ and substance derivative of x component of velocity; now, the combination of terms (in right hand side) can be represented in terms of vector notation just to slightly introduce you to vector notation, so that the expression become very simple and elegant. So,

$$v = v_x i + v_y j + v_z k$$

Now, we know the velocity field. It is a vector field and it has three components v_x, v_y and v_z and we are familiar from calculus, the gradient operator is

$$\nabla = \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k$$

So, now if you take a dot product of the velocity field with gradient operator (del),

$$v.\nabla = \frac{\partial v_x}{\partial x} + \frac{\partial v_x}{\partial y} + \frac{\partial v_x}{\partial z}$$

So, this set of terms can be represented very nicely in terms of v dot the gradient operator.

$$\frac{Dv_x}{Dt} = \frac{\partial v_x}{\partial t} + (v.\nabla)v_x$$

Now in this particular case we have $\frac{\partial}{\partial t}$ and then, v_x here and then all these terms without the v_x is represented by $v.\nabla$ and then to be more formally we say operating on v_x.

Similarly, we can write for the substantial derivative of temperature,

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + v_x\frac{\partial T}{\partial x} + v_y\frac{\partial T}{\partial y} + v_z\frac{\partial T}{\partial z}$$

And,

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + (v.\nabla)T$$

once again this can be put in terms of the v vector and the gradient vector. The $\frac{\partial}{\partial t}$ plus $v.\nabla$ is called the substantial or material derivative.

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + (v.\nabla)$$

So, what does this expresses in terms of the vector notation. So, either we can use $\frac{D}{Dt}$ in terms of vector notation or in the expanded form, it is better to know both the equivalent forms of the same expression.

$$\frac{DC}{Dt} = \frac{\partial C}{\partial t} + (v.\nabla)C$$

As I told you we can represent substance derivative for any variable. For example, I have shown above for concentration. So, just to repeat, it says rate of change of a property it could be velocity, temperature and then concentration of a fluid particle and that is a Lagrangian time derivative. Left hand side is Lagrangian time derivative. Most importantly we are expressing that in terms of Eulerian or field variables. That is the most significant importance of substantial derivative. Right hand side is the Eulerian field, concentration field, temperature field, velocity field etcetera. And if you use the derivatives in this particular combination $\frac{\partial}{\partial t} + (v.\nabla)$ operating on any variable, the physical significance what you get becomes Lagrangian.

That is a significant advantage. If you measure a property in terms of Eulerian variables, you can get the Lagrangian derivative meaning. Now, that is what I just explained to you knowing the velocity and temperature field accordingly concentration field or any other field which is

Eulerian description, the rate of change of temperature following a fluid particle which is the Lagrangian description can be obtained.

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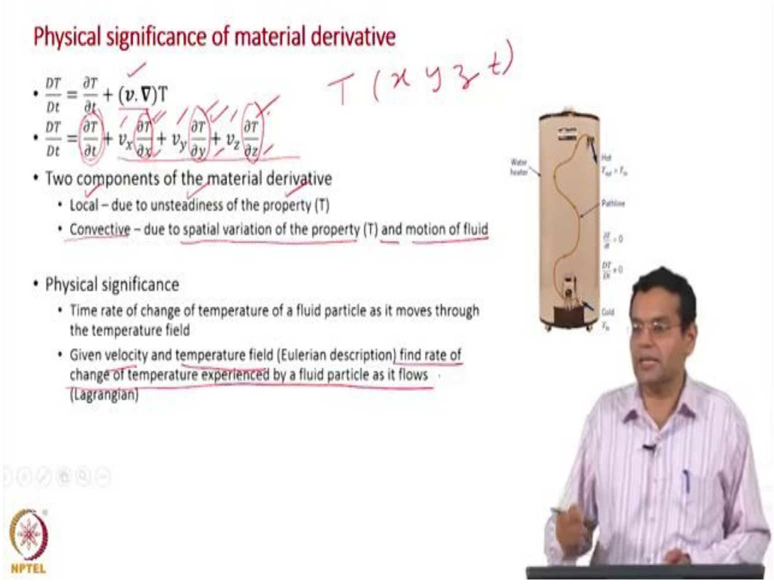

What are the physical significance of this material derivative? What do you mean by that? We have seen the significance of course it has two components in it. Let us see what are the meaning of those two components. Now, once again we will start with temperature in the present case, so that it is easy to understand and then go to velocity.

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + (v.\nabla)T$$

Now, this is written in the in terms of the velocity vector and the gradient vector. I have expanded that and written here in terms of the full expression, So that you can slowly get familiar to both the vectorial representation and the full representation.

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} + v_z \frac{\partial T}{\partial z}$$

Now there are two components to it. First is called the local component, $\frac{\partial T}{\partial t}$. Why is it local component? As I told you remember when you derived it when we took the derivative we kept the x y z constant. So, it is a particular location. So, this tells what is the rate of change of temperature at a particular location; how can it happen? Because of fluctuation in temperature, change in temperature in this particular room. Let us say if you are following a fluid particle, the particle is at a particular location, the temperature field has some fluctuation for whatever region goes from one value to other value it fluctuates. There is some transient in that, some unsteady state, steady state is not there. So, particle is at a particular location experiences the change in temperature because of the unsteadiness of the temperature field; so, the first local component due to the unsteadiness of the property in this particular case temperature.

What is the next? What is the significance of the next component? It is called the convective component, $v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} + v_z \frac{\partial T}{\partial z}$ or $(v.\nabla)T$. Why is it happening? Two contributions are there. First one there should be two conditions for that to be non-zero. What is that? First of all we have $\frac{\partial T}{\partial x}$, $\frac{\partial T}{\partial y}$ and $\frac{\partial T}{\partial z}$. What do they represent? Change of temperature with respect to spatial coordinates x direction, y direction and z direction etcetera. So, now which means that there should be this, there should be a spatial variation of the property. If there is no spatial variation, these terms will become 0 and second term will not contribute.

So, first requirement is that there should be a spatial variation. Secondly, there should be fluid flow, only then v_x, v_y, v_z will be non-zero, only then you will have non-zero value. So, for the convective components represents the contribution to the substantial derivative due to spatial variation of the property and motion of the fluid if either of them is absent, that term will be a 0 and it will not contribute to the derivative experienced by the fluid particle. So, in this case as the fluid particle flows through this heater (see refer slide), it can have two components. One at a position particle experiences change in temperature as it moves from one location to the other, it experience a change in temperature because of this spatial variation and we can talk about movement only when there is flow.

So, time rate of change of temperature of a fluid particle as it moves through the temperature field and given velocity and temperature field. what is required? You require the velocity, you require the temperature field. What is the temperature field? T as a function of x, y, z and time of course; so, given velocity and temperature of field Eulerian description, you can find out the rate of change of temperature experienced by a fluid particle as it flows the Lagrangian description.

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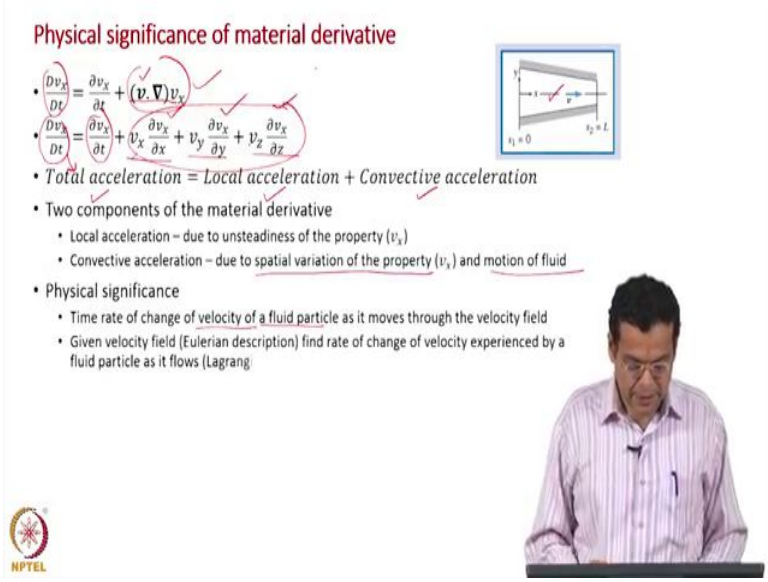

It just extent this to a the velocity we have written the expression for the acceleration of fluid particle in the x direction which is

$$\frac{Dv_x}{Dt} = \frac{\partial v_x}{\partial t} + (v.\nabla)v_x$$

This in terms of the vector notation and then we can expand this same in terms of the components.

$$\frac{Dv_x}{Dt} = \frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z}$$

This is of course more easily understandable compared to vector notation, but this more compact notion compared to this expression.

Now because it is derivative velocity, this has a name. Other than our substantial derivative velocity, this is called the total acceleration,

Total acceleration = Local acceleration $\left(\frac{\partial v_x}{\partial t}\right)$ + Convective acceleration $((v.\nabla)\,v_x)$

If temperature, concentration, pressure any other variable, we just call it as substantial derivative local component and then convective component because it is velocity. We call this a total acceleration, local acceleration and then convective acceleration.

Physical significance all are analogous. So, two components as we have seen earlier two components of the material derivative; one is the local component because once again because it is velocity, it is local acceleration. So, at a particular position, the particle experiences change of velocity with respect to time from fluctuation velocity due to unsteadiness of the property in this case, the x component of a velocity. Second one is the convective acceleration. As it moves from one place to other place, there is change in velocity ok. This example from one place to other we have seen. It will experience increasing velocity.

So, due to spatial variation of the property, this case velocity and motion of the fluid and physical significance we have seen earlier time rate of change of velocity of fluid particle and as it moves through the velocity field. of course in the earlier case we require the velocity field and the temperature field. In this case just of course only velocity field. If you know the velocity field, you can you know this v_x, v_y, v_z if you know the velocity field, you know all the derivatives. So, only velocity field is required here. So, given velocity field find rate of change of velocity experienced by a fluid particle as it flows through the domain and that is a Lagrangian description.

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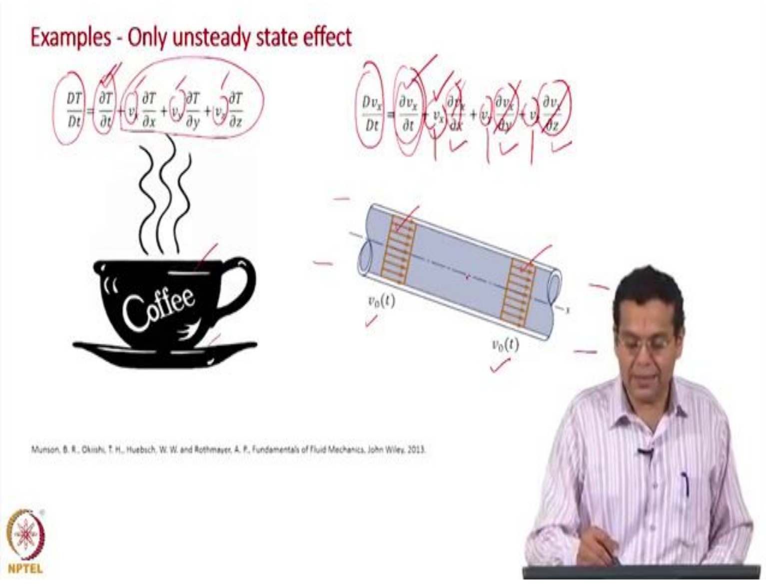

So, let us take some examples where the two different terms only contribute. The first two examples are the case where only the local component contributes $\left(\frac{\partial}{\partial t}\right)$. So, what I have shown here is a cup of coffee (see above refer slide). You can also imagine cup of tea. Tea is supposed to be more healthier than coffee that is what I came to know. So, the substantial derivative of temperature what is that mean as usual. So, imagine a particle there in the fluid particle in the cup of coffee and even if you blow air to cool it, there is not much motion of that, there is no fluid motion. So, because there is no fluid motion this v_x, v_y, v_z are all 0. There will be temperature variations may be it is slightly colder here, warmer there. There will be temperature variation, but there is no velocity components.

So, convective term will not contribute, but as the coffee is getting cooled if the particle is at a particular location, it experiences the rate of decrease in temperature. That is why the rate of change of temperature experiences the particle as only one component because of the unsteadiness. Sometime back we discussed unsteadiness is example for unsteadiness of the temperature field which is the temperature of the coffee reduces. So, the temperature as a unsteady component in it, there could be spatial variation also because there is hardly any flow. Convective term will not contribute as I told you it requires two condition. There should be a flow, there should be a gradient. One of them is not here. So, only the local component plays a role.

Let us take another example from fluid flow (see above refer slide). What you have here is pipe of constant cross section, so that there are no velocity gradients. We want to consider a case where only the first component contributes (only local component). That is why we take in a case where the pipe of constant cross section throughout. So, that is why the velocity distribution shown here, it is all same now, but the velocity can be a function of time the v_0 whatever can be a function of time let us say vary sinusoidally or some fluctuations are there.

So, what happens in this particular case, this will contribute at a if you are looking at a play if you are at particular location or the particle is at particular location, it experiences such change in velocity with respect to time, but now as you follow the fluid particle from one place to the other, there is a flow. Of course this case only one component is present, but otherwise these velocity components are present, but there is no gradient in the velocity.

This is the case where second term (convective term) does not contribute because the gradient in the velocity is not present. That is why is specifically took a case where the pipe has a constant cross sectional area though there is flow, but there is convective terms are not present. In this case they just becomes 0 and once again the acceleration experienced by a fluid particle is has only one contribution from the local component, local acceleration. The total acceleration becomes equal to the local acceleration.

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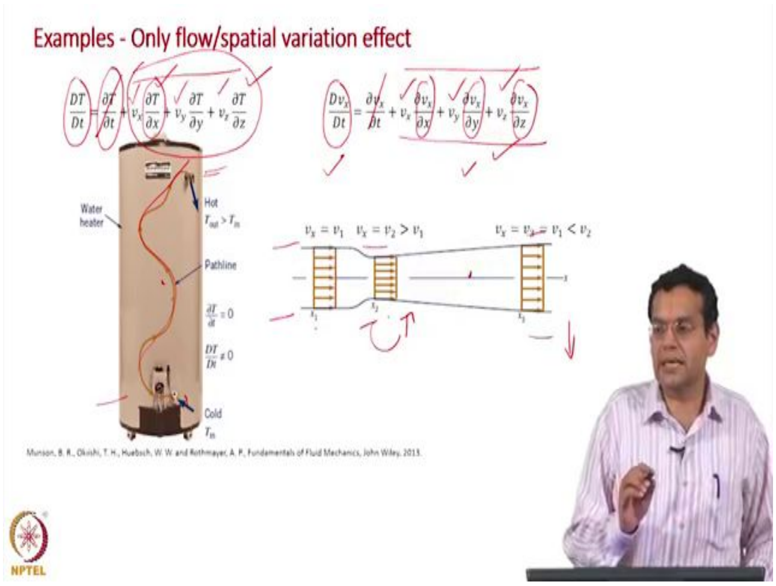

Let us take another set of examples where we have contribution from only the convective component, $((v.\nabla)\, v_x)$. What is that example, the household example, the geyser example which I have discussed sometime back? Usually this operates under steady state condition. You have a constant in flow, constant out flow and let us say water enters at some 30 degrees leaves at some 70 degree centigrade. So, these conditions are all same.

So, if the particle is at particular location, the temperature field has no unsteadiness in it. It is just a steady state and steady state field. So, the local term will not be present and that is what we actually mean by steady state. When we mean steady state at a particular location, there should not be any change of a property in this particular case temperature.

Now, what happens as there is suddenly flow through this medium? So, v_x components are present, it could be one dimensional or multi dimensional and then, there is gradient also along the path of the fluid particle, there is increasing temperature as the fluid flows through this. So, from one location to the other the fluid experiences such change in temperature.

Why does it experience? Because the temperature varies along the path; so, the convective component contributes and the local component does not contribute. So, the variation in temperature experienced by a fluid particle has only the convective component. In this particular case there is a gradient and flow of velocity and the velocity flow is also present.

Now, let us take another case where we have taken flow through a channel, but in this case we are specifically allowed for variation in the cross sectional area and you have a larger cross sectional area, a smaller cross section (see above refer slide). Once again it expands and the cross section area increases.

So, we have considered a case where there is spatial variation of velocity and now unlike the earlier case, we are just considering a steady velocity field. The velocity does not change respect to time. Of course, there is flow. So, v_x, v_y, v_z could be there this particular case only one component is present and then there is spatial variation also. So, these terms contribute.

So, the acceleration experienced by a fluid particle is only because of the convective component. The total acceleration becomes equal to the convective acceleration. In these particular examples if you are at the particular location, there is no change in the property, no rate of change of the property with respect to time. If you put an instrument in the pipe, it will not show any change in velocity. If you put an instrument in the geyser, it will not show any change in temperature, if you travel along then you will experience a change in temperature and velocity.