Continuum Mechanics And Transport Phenomena
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Lecture – 89
Bernoulli Equation: Example 2

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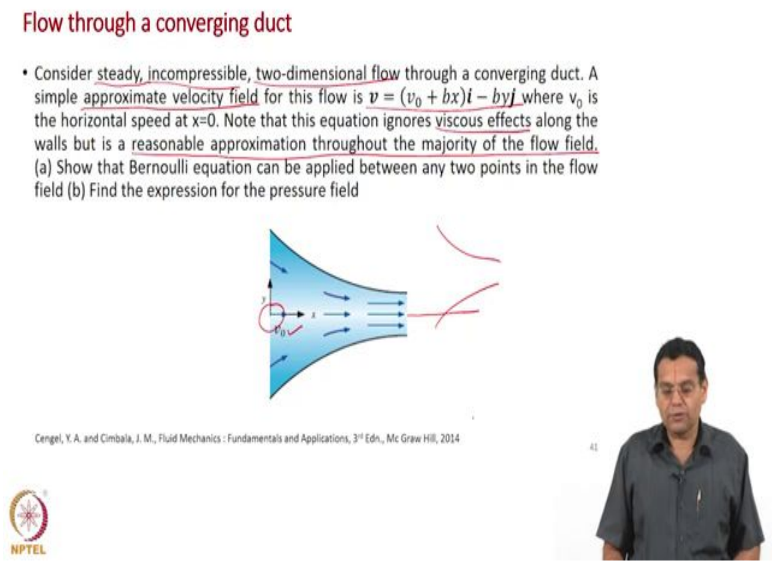

Let us take an example to illustrate the Bernoulli's Equation under irrotational flow condition ok. So, this is a flow through a converging duct. So, let us read the example; consider steady, incompressible, two dimensional flow think by this time we can easily understand these terms steady, incompressible and two dimensional flow through a converging duct and that is what is shown here converging duct. A simple approximate velocity field for this flow is given by this velocity field.

$$v = (v_0 + bx)\, i - byj$$

The velocity component in the x direction increases with the x because, it is converging and that is given by $v_0 + bx$, the x component of velocity is v_0 at x = 0. And: what about the y component of velocity? That is given by $-by$ and that increase in magnitude away from the center line the center line is y = 0 and in terms of magnitude, it increases as you go away from the center line that is what this tells you.

And, in terms of direction above the central line b is towards the center line because y is positive and if you come below the center line y is negative. So, once again the direction of y component is towards the center line and that is what this shows. And, where v_0 is the horizontal speed at x = 0 and that is what is denoted here. And, remember the y component of velocity is 0 at y = 0 along the central line; there is no y component of velocity. As we have seen earlier that is a symmetry line, why is it approximate field, why does the question say it is approximate field? Note that this equation ignores viscous effects, that is what we are discussing now.

This ignore viscous effects along the walls, but as a reasonable approximation throughout the majority of the flow field that is what we have been telling always. Whenever we say inviscid flow what is the region of validity? First of all high Reynolds number; secondly, away from the wall where the net viscous stresses are negligible that is what this question also tells you. Note that this equation ignores viscous effects along the walls, but as a reasonable approximation throughout majority of the flow field that is why it is approximate velocity field.

What is it we are asked to find out? Show that Bernoulli equation can be applied between any two points in the flow fields which means we should show that it is a irrotational flow field and then find the expression for the pressure field.

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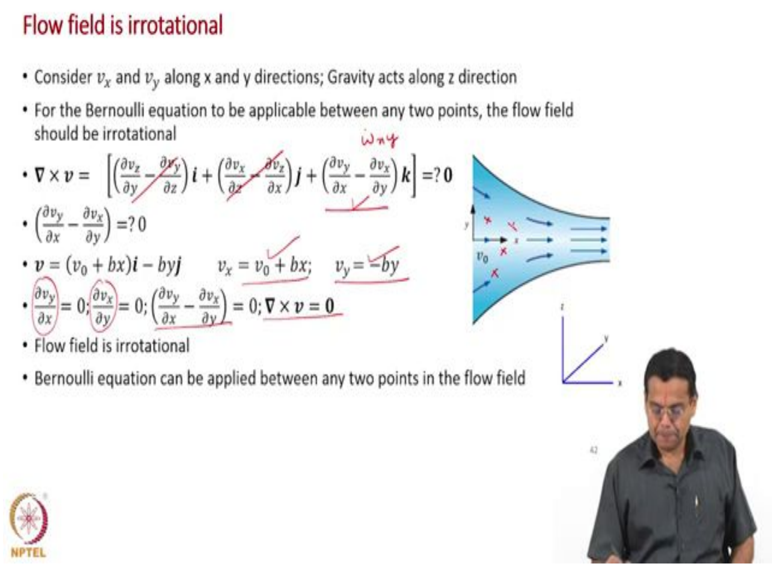

So, let us proceed. So, first objective is to show that the flow field is irrotational which means then you can apply the Bernoulli's equation between any two points. Now, we will consider this as something like a top view something like a top view and flows in the x y plane and gravity acts along the z direction and. Now, for the Bernoulli equation to be applicable between any two points the flow fields should be irrotational we have seen that which means that

$$\nabla \ x \ v = \left[\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) i + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) j + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right) k\right] = 0$$

Now, because we are considering the x y plane the component which is relevant to us is

$$\nabla \ x \ v = \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right) = 0$$

So, let us do that we are given the velocity field. So, which means we can identify the x component of velocity y component of velocity, let us evaluate the derivative

$$v_x = v_0 + bx; \qquad v_y = -by$$

$$\frac{\partial v_y}{\partial x} = 0; \qquad \frac{\partial v_x}{\partial y} = 0;$$

So,

$$\nabla \ x \ v = 0$$

Which means the flow is which means that the flow is irrotational and hence, Bernoulli equation can be applied between any two points in the flow field. What does it mean? You can take any point here, Bernoulli's equation should be valid. So, if you take any point here then Bernoulli's equation should be valid.

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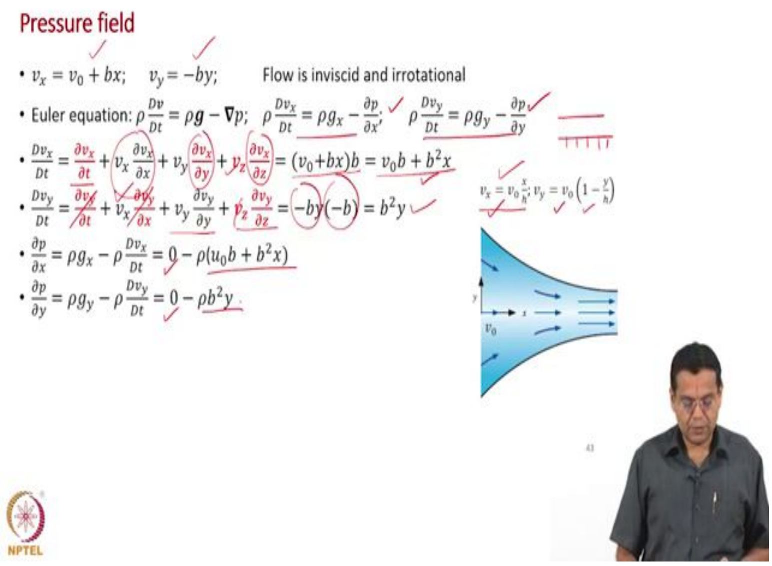

Now, let us evaluate the pressure field. What do we mean by pressure field? How pressure varies as a function of x comma y that is the idea.

$$v_x = v_0 + bx; \qquad v_y = -by$$

So, these are the velocity components. So, what we are going to find out is the pressure gradient in the x and y direction almost similar to what we had done earlier for an example under application of Euler's equation. You will see why is that so.

Now flow is inviscid and irrotational. So, we will take the Euler equation

$$\rho \frac{Dv}{Dt} = \rho g - \nabla p$$

Now, write the components of the Euler equation; x component and the y component,

$$\rho \frac{Dv_x}{Dt} = \rho g_x - \frac{\partial p}{\partial x}$$

$$\rho \frac{Dv_y}{Dt} = \rho g_y - \frac{\partial p}{\partial y}$$

Now, let us simplify the substantial derivative of v_x,

$$\frac{Dv_x}{Dt} = \frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = (v_0 + bx)\, b = v_0 b + b^2 x$$

Now let us simplify the term for v_y

$$\frac{Dv_y}{Dt} = \frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} = -by(-b) = b^2 y$$

Now let us substitute in the Euler equation and evaluate the gradients in their two directions or pressure. So,

$$\frac{\partial p}{\partial x} = \rho g_x - \rho \frac{Dv_x}{Dt} = 0 - \rho\left(v_0 b + b^2 x\right) = -\rho\left(v_0 b + b^2 x\right)$$

$$\frac{\partial p}{\partial y} = \rho g_y - \rho \frac{Dv_y}{Dt} = 0 - \rho b^2 y = -\rho b^2 y$$

So now, we have evaluated the gradient of pressure in the x direction and the y direction.

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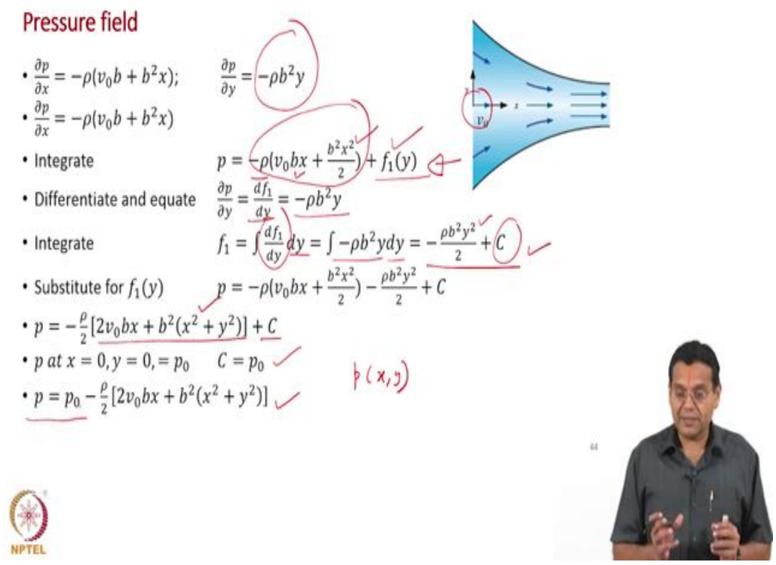

Now, we will proceed towards finding out the pressure field we will carry over the pressure gradient from the previous slide

$$\frac{\partial p}{\partial x} = -\rho\left(v_0 b + b^2 x\right); \qquad \frac{\partial p}{\partial y} = -\rho b^2 y$$

Now, how do we proceed what we do is start with the pressure gradient in the x direction

$$\frac{\partial p}{\partial x} = -\rho\left(v_0 b + b^2 x\right)$$

And then we will partially integrated. So,

$$p = -\rho\left(v_0 bx + \frac{b^2 x^2}{2}\right) + f_1(y)$$

Now you have a constant since it is partial integration that can be a function of y which we denote as $f_1(y)$ now we will differentiate it idea is to find out that function.

So, let us differentiate the expression for pressure which we found in the previous step. So,

$$\frac{\partial p}{\partial y} = \frac{df_1}{dy}$$

Now, this should be equal to the $\frac{\partial p}{\partial y}$ which I found out earlier,

$$\frac{\partial p}{\partial y} = \frac{df_1}{dy} = -\rho b^2 y$$

Now we have got a differential of the function. So, now, what should we do integrate idea is to find out f 1. So, let us integrate

$$f_1 = \int \frac{df_1}{dy} dy = \int -\rho b^2 y dy = -\frac{\rho b^2 y^2}{2} + C$$

This is just a constant or usual constant because it is usually integration it is not partial integration. Now we have found out the expression for f one that was our objective of course, the constant still remains. So, let us substitute this expression for f_1 in the pressure equation.

$$p = -\rho\left(v_0 bx + \frac{b^2 x^2}{2}\right) - \frac{\rho b^2 y^2}{2} + C$$

Now, let us do some simple rearrangements so that it looks nice,

$$p = -\frac{\rho}{2}\left[2v_0 bx + b^2\left(x^2 + y^2\right)\right] + C$$

We will have to evaluate the constant C now which means we need a condition. We need a value of pressure at some point let us say we are given the pressure at x = 0, y = 0 and the pressure is p_0. So, if we substitute here,

$$C = p_0$$

So, let us substitute,

$$p = p_0 - \frac{\rho}{2}\left[2v_0 bx + b^2\left(x^2 + y^2\right)\right]$$

That is the pressure field p. So, this gives you p as a function of (x, y), always we should know that there is no meaning for or we cannot find pressure as an absolute value for incompressible flow. It is always the difference in pressure that matters, you see we are when we are specific. When we are calculating here we calculate $p - p_0$.

So, we always find difference in pressure in incompressible flow ok. If it is compressible flow then of course, the absolute pressure has a meaning in the case of incompressible flow. If you want to say more formally you always find pressure up to an arbitrary constant or arbitrary given value of pressure and that is what is cleared from here also even earlier example, we found out pressure gradient we never found out pressure. So, pressure is always given or we always find in terms of pressure difference.

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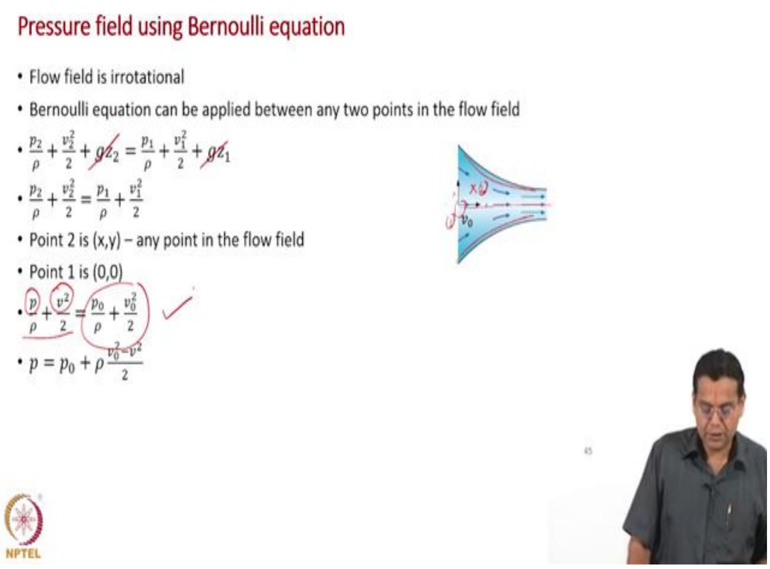

Now, what we will do is alternate method of finding pressure field using the Bernoulli equation. What did we do now? We took the Euler equation found out the two pressure gradients in two directions and then integrated differentiated once again integrated find f_1 found a function f_1 , constant C etcetera found the pressure distribution. Now, what we will do? We will straight away use the Bernoulli equation and then find the pressure field which in one way will be much simpler Bernoulli's equation was derived from the Euler equation.

So, both should be equivalent. Now because flow field is irrotational, I can apply Bernoulli equation between any two points that is advantage here. Because, if it is not irrotational, then by applying Bernoulli's equation, you cannot get the pressure field because then you will have to apply only along streamlines because it is irrotational and because we can apply Bernoulli's equation between any two points you can get the enter pressure field.

So, let us write down the Bernoulli's equation which we have derived between two points 1 and 2

$$\frac{p_2}{\rho} + \frac{v_2^2}{2} + gz_2 = \frac{p_1}{\rho} + \frac{v_1^2}{2} + gz_1$$

And this particular example, they are the same level. So, the gravity term or the potential energy term cancels out.

$$\frac{p_2}{\rho} + \frac{v_2^2}{2} = \frac{p_1}{\rho} + \frac{v_1^2}{2}$$

Now, what we will do is, we will consider point 2 as some position (x, y) we have the flow field like this; any point is considered as (x, y) and point 1 is (0, 0).

So, we will change the nomenclature in the previous equation

$$\frac{p}{\rho} + \frac{v^2}{2} = \frac{p_0}{\rho} + \frac{v_0^2}{2}$$

So, let us rearrange this. So,

$$p = p_0 + \rho\left(\frac{v_0^2 - v^2}{2}\right)$$

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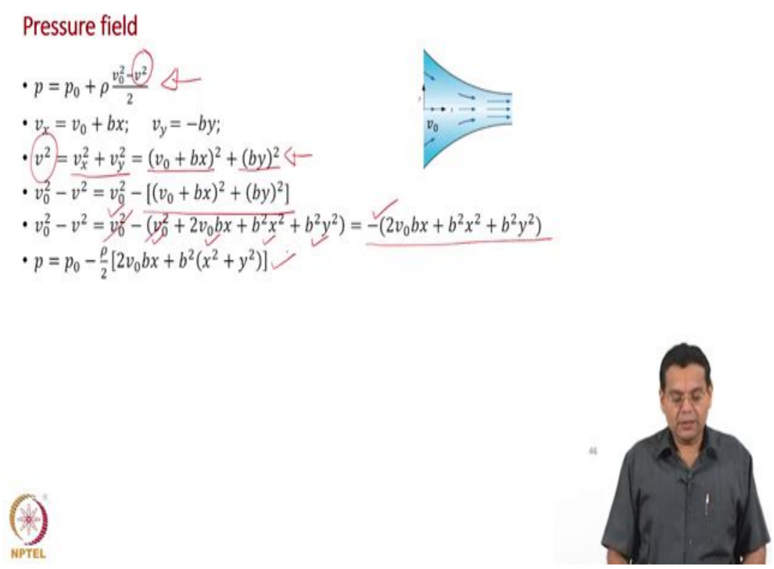

So, let us rewrite that expression.

$$p = p_0 + \rho\left(\frac{v_0^2 - v^2}{2}\right)$$

Now we are given the velocity field which means we know

$$v_x = v_0 + bx; \qquad v_y = -by$$

Now, I can find out the magnitude of velocity at any point x comma y represented by v

$$v^2 = v_x^2 + v_y^2 = (v_0 + bx)^2 + (by)^2$$

So, I can find out the square of the magnitude of the velocity. So, what we want in the right hand side is

$$v_0^2 - v^2 = v_0^2 - \left[(v_0 + bx)^2 + (by)^2\right]$$

So, let us simplify that let us expand the square term. So,

$$v_0^2 - v^2 = v_0^2 - \left[v_0^2 + 2v_0bx + b^2x^2 + b^2y^2\right] = -(2v_0bx + b^2x^2 + b^2y^2)$$

So, now, let us substitute back. So,

$$p = p_0 - \frac{\rho}{2}\left[\left(2v_0bx + b^2x^2 + b^2y^2\right)\right]$$

So, this way becomes much simpler you can straight away apply the Bernoulli's equation to find out the pressure field. Of course, same expression which we have obtained from the Euler equation; I was a little more that was involved this is much more a simpler. So, two different ways.

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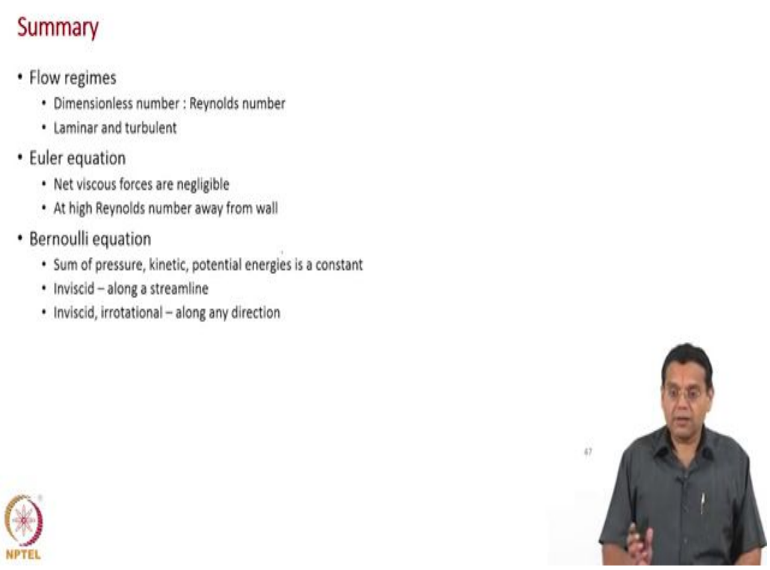

So, let us summarize and whatever we have discussed under a inviscid flows. We started with the discussion on flow regimes, they tell about the characteristics of the flow in terms of our dimensionless number namely Reynolds number which is the ratio of the inertial forces to viscous forces. We looked at two regimes one at low Reynolds number namely laminar regime laminar flow regime and one at high Reynolds number namely turbulent regime in between of course, you can have transition.

While, laminar was order turbulent was highly disordered then we discussed the Euler equation and Euler equations applicable for inviscid flows which means that net viscous forces are negligible and which happens at high Reynolds number away from the wall. And, then we derive the Bernoulli’s equation which tells you that the sum of pressure kinetic potential energy is of course, per unit mass is a constant. We derived that starting from the steady state Euler equation and we derived two forms of the Bernoulli’s equation; one of the flow is inviscid the Bernoulli equation is valid along a streamline. If it is inviscid and irrotational, then the Bernoulli equation is valid along any direction.