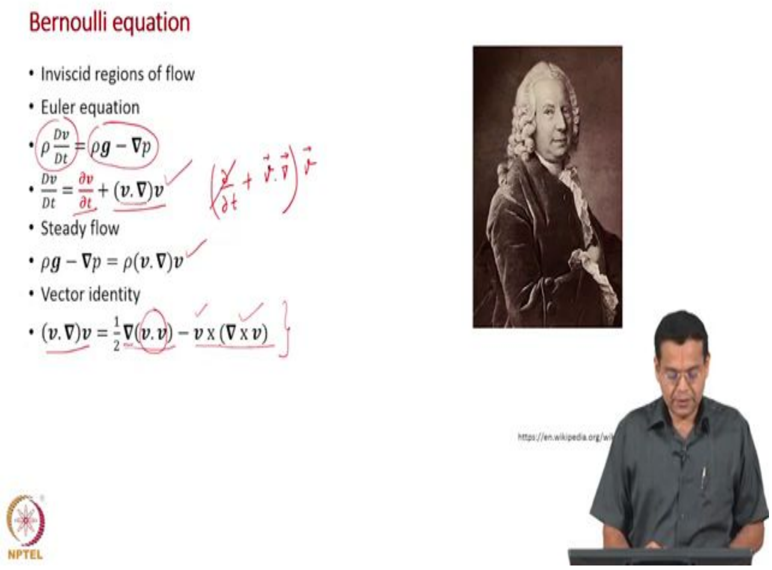
# Continuum Mechanics And Transport Phenomena

Prof. T. Renganathan
Department of Chemical Engineering
Indian Institute of Technology, Madras

## Lecture – 86
## Bernoulli Equation : Inviscid Flow

(Refer Slide Time: 00:13)




We are going to derive another equation of engineering relevance and engineering application. Lot of applications of this equation called the Bernoulli equation and he is Bernoulli and this equation relates pressure, velocity, elevation under inviscid flow condition; that is why we are discussing this. Because the main topic of discussion is inviscid flow under that we discussed flow regimes. So, that we know the scope of our discussion namely high Reynolds number flows. And with first we discussed the Euler equation, now we are discussing the Bernoulli equation.

So, inviscid regions of flow, what we are going to now is, derive the Bernoulli's equation. We will start the Euler equation,

$$\rho \frac{Dv}{Dt} = \rho g - \nabla p$$

So that is the Euler equation which we have seen in the previous discussion. And the substantial derivative of velocity has two terms; the local term and the convective term or the total acceleration is in terms of the local acceleration and convective acceleration. We will restrict the discussion to steady flows.

$$\frac{Dv}{Dt} = \frac{\partial v}{\partial t} + (v.\nabla)\,v$$

So, I will consider the steady state Euler equation. So,

$$\frac{Dv}{Dt} = (v.\nabla)\,v$$

So, the Euler's equation simplified to

$$\rho g - \nabla p = \rho\,(v.\nabla)\,v$$

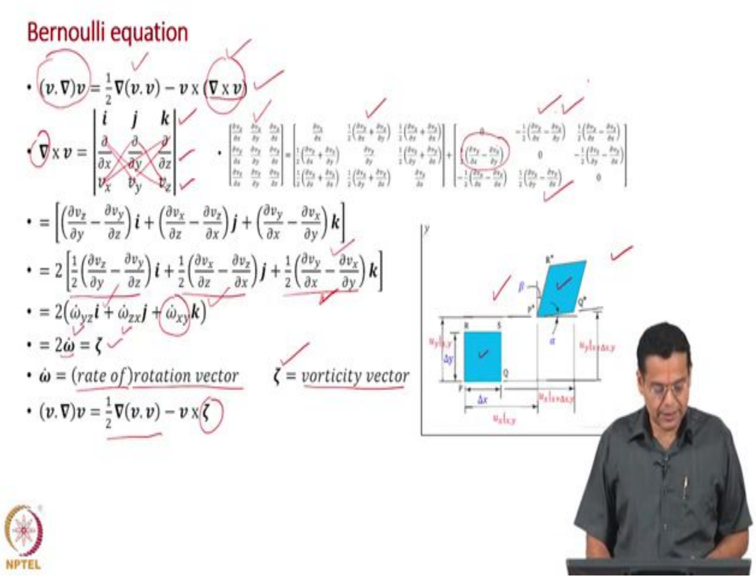
Now we will use an vector identity; what is the vector identity? You would have a come across several vector identities in your vector calculus course; for example, curl, gradient of a scalar is equal to 0, there is one vector identity. Similarly we are going to use this vector identity, we are not going to prove this is a little involved to prove this, even Fluid Mechanics book take this relationship. So, we are also taking this relationship without proving it.

$$(v.\nabla)v = \frac{1}{2}\,\nabla(v.v) - v\ x\ (\nabla\ x\ v)$$

What does it tell you, the left hand side we have $v.\nabla$ operating on velocity and $v.\nabla$ is the scalar operator, operating on a vector. So, left hand side is a vector, right hand side we have gradient of $v.v$. So, $v.v$ is a scalar, it is a magnitude of $v^2$ and gradient of a scalar will give you a vector.

Now coming to the last term, we have $\nabla xv$ which is a vector and once again v cross this vector gives you another vector. So, all the terms in this vector identity are vectors; obviously, a vector identity cannot have one term a scalar, other term is vector all of them should be of the same type. So, all the terms here are in terms of vector. So, the convictive acceleration term is going to be expressed in terms of the right hand side of this vector identity.

(Refer Slide Time: 04:36)




So, let us do that,

$$(v.\nabla)v = \tfrac{1}{2}\ \nabla(v.v) - v\ x\ (\nabla\ x\ v)$$

That is a vector identity. Now what we will do is, let us try to understand the $\nabla\ x\ v$ which is well known to you, it is the curl of the velocity vector which is a cross product of the gradient vector and the velocity vector ok; that is should be well known to you from a calculus course.

$$\nabla\ x\ v = \left| i\ j\ k\ \tfrac{\partial}{\partial x}\ \tfrac{\partial}{\partial y}\ \tfrac{\partial}{\partial z}\ v_x\ v_y\ v_z \right|$$

Now let us express that in terms of variables which have come across in this particular course. So, you know the cross product can be expressed in terms of the determinant. So, now, let us expand this determinant.

$$\nabla\ x\ v = \left[\left(\tfrac{\partial v_z}{\partial x} - \tfrac{\partial v_y}{\partial z}\right) i + \left(\tfrac{\partial v_x}{\partial z} - \tfrac{\partial v_z}{\partial x}\right) j + \left(\tfrac{\partial v_y}{\partial x} - \tfrac{\partial v_x}{\partial y}\right) k\right]$$

Let us multiply and divide by 2 and you will see something familiar to us.

$$\nabla\ x\ v = 2\left[\tfrac{1}{2}\left(\tfrac{\partial v_z}{\partial x} - \tfrac{\partial v_y}{\partial z}\right) i + \tfrac{1}{2}\left(\tfrac{\partial v_x}{\partial z} - \tfrac{\partial v_z}{\partial x}\right) j + \tfrac{1}{2}\left(\tfrac{\partial v_y}{\partial x} - \tfrac{\partial v_x}{\partial y}\right) k\right]$$

Now if you look at the terms here, it should sound something familiar to us especially the last term, because usually I have been working on the x y coordinate. What does it represent? That represents the rate of rotation in the x y plane; just to recall we express the velocity gradient tensor in terms of the strain rate tensor and the rotation rate tensor or rate of rotation tensor. And what we have here is one of the components of the rotation rate tensor; of course, we know that this is $\dot{\omega}_{xy}$ .

And so, what we have here is $\dot{\omega}_{xy}$ . Similarly the other terms also represent the rate of rotation in the different planes, $\dot{\omega}_{yz}$, $\dot{\omega}_{zx}$ . And that is what is shown here, this figure should be familiar to us which we discussed when we discussed strain rates in fluid mechanics, in fluids and we said this figure shows a fluid element at time t and then t + $\Delta$t. And we said a fluid element can undergo normal strain rate, shear strain rate and rotation rate. And that is rotation rate is what we are discussing now, this component this term represents that rate of rotation.

So, let us use that,

$$\nabla \; x \; v = 2(\dot{\omega}_{yz} i + \dot{\omega}_{zx} j + \dot{\omega}_{xy} k)$$

$$\nabla \; x \; v = 2\dot{\omega} = \xi$$

So, you have a vector now; how is the vector formed? The vector is formed with the components as the rate of rotation in the y z plane, z x plane and x y plane. So, let us give names to them,

$\dot{\omega} =$ the rate of rotation vector; $\xi =$ *vorticity vector*

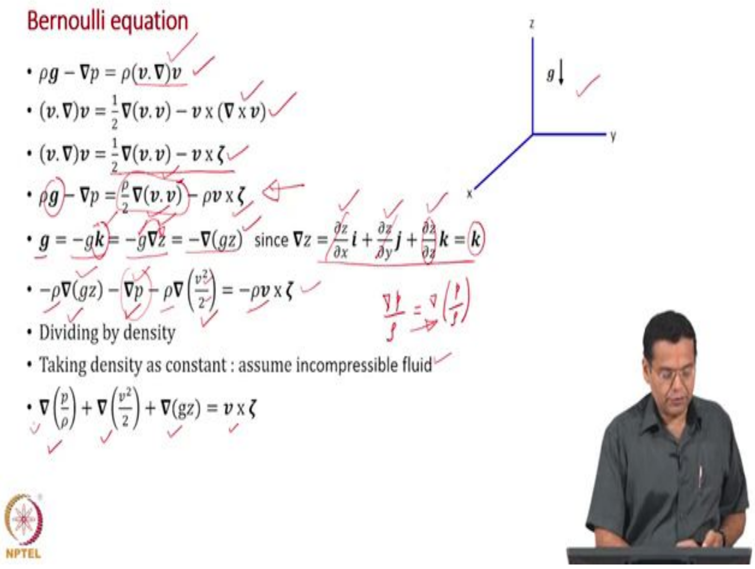
So, vorticity vector is twice the rate of rotation vector; in short what we are discussing to summary is what I discussed so far, what we did was, we use this vector identity and express the convective acceleration term as some of two terms. In that we had one $\nabla \; x \; v$ term, we have we know that $\nabla \; x \; v$ is the curl of v and we started with the determinant expression, expanded that and found that the components of the $\nabla \; x \; v$ vector are nothing, but the rate of rotation of the fluid elements in the three planes. And that $\nabla \; x \; v$ has now got a significance of twice the rate of rotation vector. So, we denote that by the vorticity vector.

So, let us do that,

$$(v.\nabla)\, v = \tfrac{1}{2} \; \nabla \; (v.v) - v \; x \; \xi$$

So, whenever you come across $\nabla \; x \; v$ or rate of rotation vector or vorticity vector this tensor should come to your mind; where we have discussed all the components as the rate of rotation in the respective planes. And this diagram should also come in come to your mind which includes of course, normal strain rate, shear strain rate and rotation. But now this discussion is confined to the rigid body rotation; remember the fluid, this rotation tells about the rigid body rotation of the fluid element.

(Refer Slide Time: 11:25)




Now let us proceed further.

$$\rho g - \nabla p = \rho (v.\nabla)\, v$$

So, this is the steady state Euler equation, where we are taken only the convective acceleration. And we discuss the vector identity,

$$(v.\nabla)\, v = \frac{1}{2}\, \nabla\, (v.v) - v\ x\ \xi$$

We are using this vector identity and just now we discussed that the $\nabla x v$ can be represented in terms of the vorticity vector. So, now, let us substitute this vector identity in terms of vorticity vector in the right hand side of this steady state Euler equation.

$$\rho g - \nabla p = \frac{\rho}{2}\, \nabla\, (v.v) - \rho v\ x\ \xi$$

Now, let us take this gravity vector and let us try to express that in terms of a gradient.

$$g = -gk = -g\nabla z = -\nabla(gz)$$

$$\nabla z = \frac{\partial z}{\partial x} i + \frac{\partial z}{\partial y} j + \frac{\partial z}{\partial z} k = k$$

So, let me just quickly repeat; what we are doing here is, expressing the g vector in terms of gradient of some variable, g vector is $-gk$, k is gradient of z, g is a constant you can bring inside. So, it becomes $-\nabla(gz)$.

Now let us substitute this in the above equation. So,

$$-\rho\nabla(gz) - \nabla p - \rho\nabla\left(\frac{v^2}{2}\right) = -\rho v \; x \; \xi$$

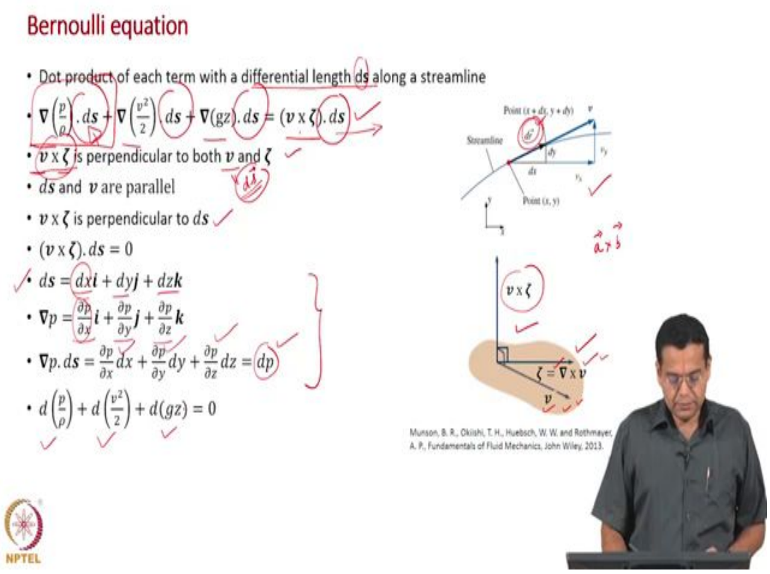
So, taken this equation, two things as have been done, g has been expressed in terms of gradient of $gz$ and $v.v$ has been expressed as velocity squared.

You may be wondering why express g in terms of a gradient of g z, because you have gradient of pressure, we have gradient of $\frac{v^2}{2}$ and so we also like to express g vector in terms of gradient of g z. Now what we will do, we will divide by density. But when I do, I can divide by density; but I make another assumption, that density is constant which means that, it is incompressible fluid.

$$\nabla\left(\frac{p}{\rho}\right) + \nabla\left(\frac{v^2}{2}\right) + \nabla(gz) = v \; x \; \xi$$

So, we have gradient of p by $\rho$, then we have gradient of $\frac{v^2}{2}$, then we have gradient of g z and right hand side we have $v \; x \; \xi$. So, moment I do this, or this can be done only if I assume incompressible fluid. And that is what has been done here, we have written as gradient of $\frac{p}{\rho}$.

(Refer Slide Time: 17:21)



So, now let us proceed further.

$$\nabla\left(\frac{p}{\rho}\right) + \nabla\left(\frac{v^2}{2}\right) + \nabla\,(gz) = v\ x\ \xi$$

Now what we are going to do now is, the previous equation take a dot product of each term with a differential length d s vector along a streamline. What is shown here? This figure is from our discussion on streamlines, same diagram has been shown here; where if you recall back streamline was defined such a way that, at any point if you draw a tangent you get the velocity vector.

Now we are considering a differential length $ds$ vector and in our earlier discussion and streamlines that differential length was shown as $dr$ vector. Now we are using $ds$ vector, because I am following derivation from this book Munson and $s$ can also mean, $s$ vector along this streamline. This $dr$ vector and present $ds$ vector both have a same physical significance, only nomenclature is different.

Now, because the, it is the differential length, it is tangent to the stream line. So, the velocity vector is along the $ds$ vector. So, what is it we are going to do now, take the earlier equation, take a dot product of each term with the differential length $ds$ vector along a streamline. Let us do that and remember that; $ds$ vector is along the velocity vector.

$$\nabla\left(\frac{p}{\rho}\right).ds + \nabla\left(\frac{v^2}{2}\right).ds + \nabla\,(gz)\,.ds = (v\ x\ \xi)\,.ds$$

So, $ds$ vector is parallel to the velocity vector. Now let us see what happens to the right hand side, having dotted with the $ds$ vector. Let us see what happens to the right hand side. To understand the right hand side, we will have to focus on the bottom figure.

A velocity vector is shown here and then the vorticity vector; they can be in two different directions, because one is velocity vector, other is the curl of velocity vector. So, this velocity vector and this is vorticity vector. Now we have one $v\ x\ \xi$ here and that is vector is shown here; we know that when we take cross product of two vectors, the resulting vector is perpendicular to both the vectors. So, in this case $v\ x\ \xi$ that resulting vector, is perpendicular to both v vector and vorticity vector.

So, let us write down that, $v\ x\ \xi$ is perpendicular to both v and $\xi$. Why is that? Suppose I have $a$ vector and then $b$ vector and if you take $a\ x\ b$ that vector is perpendicular to both $a$

vector and $b$ vector; instead of $a$ and $b$ here we have $v$ vector and $\xi$ vector. So, the cross product between $v$ vector and vorticity vector is perpendicular to both v vector and the $\xi$.

So, now we have seen the $ds$ vector and v vector are parallel, just now we have discussed that. So, what does it mean? This $v \, x \, \xi$ is perpendicular to v vector, but this v vector is parallel to the $ds$ vector; which means, that this $v \, x \, \xi$ is perpendicular to $ds$ vector and that is what we write here, $v \, x \, \xi$ is perpendicular to $ds$ vector.

Which means that, the right hand side is 0, $v \, x \, \xi$ that vector is perpendicular d s vector, which means the right hand side is 0.

$$(v \, x \, \xi).ds = 0$$

That is why we took a dot product with a differential length along a stream line. So, now, the right hand side vanishes. Now let us take the left hand side and see what happens. Now here we express that $dr$ vector in terms of $dxi + dyj + dzk$; similarly

$$ds = dxi + dyj + dzk$$

Now let us try to evaluate the first term here. First term on the left hand side;

$$\nabla p = \frac{\partial p}{\partial x} i + \frac{\partial p}{\partial y} j + \frac{\partial p}{\partial z} k$$

Now let us take the dot product of gradient of p with $ds$ vector;

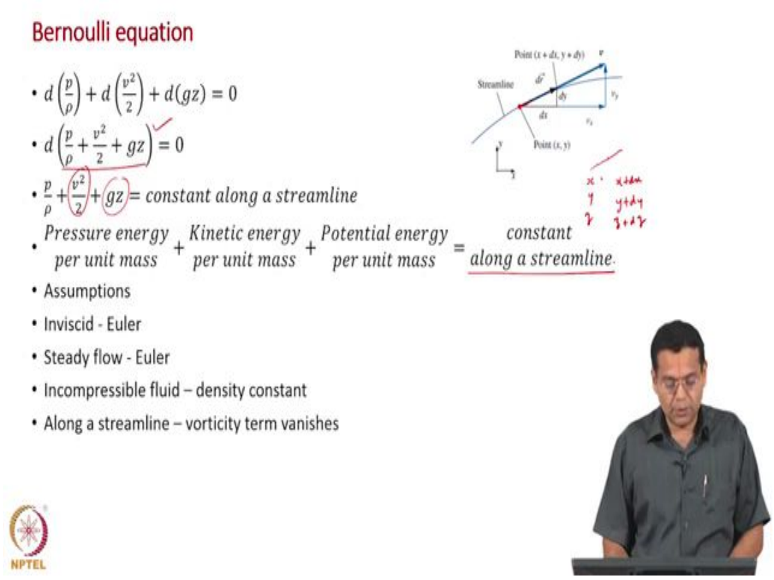
$$\nabla p.ds = \frac{\partial p}{\partial x} dx + \frac{\partial p}{\partial y} dy + \frac{\partial p}{\partial z} dz = dp$$

So, left hand side first term become $dp$; similarly, we can write for other terms as well. So, the equation is

$$d\left(\frac{p}{\rho}\right) + d\left(\frac{v^2}{2}\right) + d(gz) = 0$$

So, now of course, it becomes much simpler to read.

(Refer Slide Time: 24:09)




Now, let us rewrite that

$$d\left(\frac{p}{\rho}\right)+d\left(\frac{v^2}{2}\right)+d(gz)=0$$

And we can write this as

$$d\left(\frac{p}{\rho}+\frac{v^2}{2}+gz\right)=0$$

What does it mean?, we have taken a small length differential length along the streamline; which means that, we are at two points, the two points are along the stream line slightly separated on the streamline. We are not considering any two points, please keep that in mind, we are taking two points.

So, this tells you that the difference of this quantity; when I say quantity, sum of these three terms is equal to 0 between those two points. So, if you are on a streamline and if you consider two points adjustment to each other, the difference of this value between the two points is 0; what does it mean, that particular summation is a constant along the stream line.

$$\frac{p}{\rho}+\frac{v^2}{2}+gz=constant$$

So, $\frac{p}{\rho}+\frac{v^2}{2}+gz$ is equal to constant along a streamline. Those terms should be familiar to you; the first term is pressure energy per unit mass and then the second term is kinetic energy per unit mass very familiar to us; third term is the potential energy per unit mass is equal to constant along a streamline.

What is this pressure energy? This pressure energy represents the work done by the pressure force in moving the fluid through a distance. So, also called as flow energy; either you can you tell us pressure energy or flow energy because that represents the work done by the pressure in moving fluid over distance. We will come across this term, once again energy balance; of course, all these are different forms of mechanical energy. When you say mechanical energy what does it mean, complete conversion to mechanical work should be possible.

If you have thermal energy that cannot be completely converted to mechanical work because of the second law of thermodynamics; but these forms of energy are called mechanical energy, because complete conversion to mechanical work is possible. So, that is why the Bernoulli's equation is a mechanical energy balance.

Sometimes this equation is divided by g and written as the pressure head and the velocity head, and then the potential head that is also done. And so, the Bernoulli's equation tells you that, the sum of these energy is pressure energy, kinetic energy, potential energy; of course, all per unit mass is a constant along the stream line, not anywhere between any two point in the flow field.

What are the assumptions? Let us recall the assumptions,

- Inviscid flow, because we started with the Euler equation.
- We started with the steadied form of Euler equation, so we have also assumed a steady flow.
- Then during the derivation, we assume density to be constant, so which means; we assumed incompressible fluid. In fact, not all incompressible fluid, remember when we discussed continuity equation it is applicable for incompressible flow also. Which means that, the equation what we have derived is not necessarily, for let us say water it is also for flow of air at low velocities that is why we define incompressible flow and
- Along the stream line.

So, the Bernoulli's equation tells about the sum of these three energy is a constant along a streamline steady inviscid flow of a incompressible fluid. And this along a streamline where it that come from, remember we wanted to make the right hand side 0 where we had a vorticity term, where the rotation term. To make that 0, we have to consider this balance along the stream line and that is why this restriction comes from, that is where this restriction comes from.