Continuum Mechanics And Transport Phenomena

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Lecture – 85
Euler Equation

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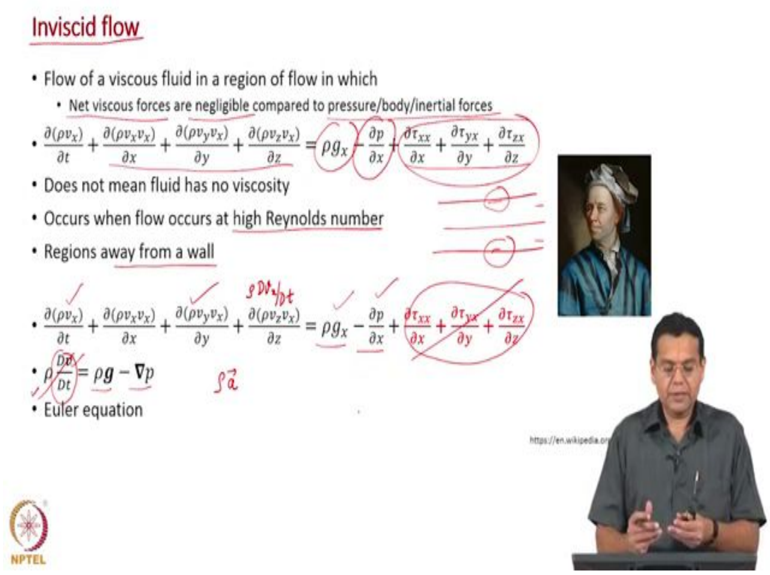


What is the relevance of the discussion of flow regime for the present case, it will tell the scope of Inviscid flow. Let us first define the inviscid flow, then you will understand why we really discussed the flow regimes. How do we define inviscid flow? Flow of a viscous fluid, when I say viscous fluid any fluid has some viscosity inherently.

So, flow of a viscous fluid, this viscous fluid need not be a very viscous fluid like oil could be even water, could be air as well. So, flow of a fluid with viscosity, in a region of flow in which net viscous forces are negligible compared to pressure, body, inertial forces.

$$\frac{\partial(\rho v_x)}{\partial t}+\frac{\partial(\rho v_x v_x)}{\partial x}+\frac{\partial(\rho v_y v_x)}{\partial y}+\frac{\partial(\rho v_z v_x)}{\partial z}=\rho g_x-\frac{\partial p}{\partial x}+\frac{\partial \tau_{xx}}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z}$$

Let us analyze this with the help of the linear momentum balance and that is a linear momentum balance; the left hand side we have the transient and convection term; right hand side we have the body force and the surface force due to pressure and viscous stresses. Now what does this tell you, the net viscous forces are negligible compared to pressure body inertial forces. Now, the last terms, $\frac{\partial \tau_{xx}}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z}$, are the net viscous forces, these terms all of them put together are negligible compared to this pressure force, body force, and inertial force as well.

So, when we say inviscid flow does not mean that; does not mean that the fluid has no viscosity that is not the correct definition ok; certainly we cannot have fluid which is which has no viscosity, any fluid it is a property of a fluid. So, will have a, any fluid will have a viscosity. When we say inviscid flow, look at the terminology it says inviscid flow; it tells about flow not about the fluid.

So, when we say flow is inviscid, when the net viscous forces are negligible compared to the pressure, body, inertial forces that is what we mean by inviscid flow. Now what is the relationship between this inviscid flow and the classification of flow in terms of flow regimes. Now, this inviscid flow occurs when the Reynolds number is high; that is why we discussed about flow regimes, we classified flow regimes based on Reynolds number; at low Reynolds number we have a laminar flow, at high Reynolds number we have turbulent flow.

So, inviscid flow occurs under high Reynolds number condition. Now next, when we have high Reynolds number, inviscid flow is not present through the entire region. Like let us give an example, let us say we have a flow between parallel plates or it could be flow in a pipe at high Reynolds number; when you are near the wall, the viscous forces have a significant contribution, you cannot neglect the viscous forces compared to the pressure, body and inertial forces.

So, you do not have inviscid flow in the region near the wall; but in the middle of the pipe are far away from the wall, you can neglect the viscous forces compared to the pressure, body, inertial forces. So, when does inviscid flow occur? First condition is high Reynolds number, second condition is the region should be away from the wall; that is why regions away from the wall at high Reynolds number you have inviscid flow.

That is why we discussed about the flow regime, so that we understand; I said we have discussed flow regime, so that we understand the scope of the present discussion or scope of inviscid flow. So, the inviscid flows occur at high Reynolds number condition and not known high Reynolds number that you should also, you can consider flow to be inviscid somewhere in the region away from the wall, not near the wall; near the wall viscous forces will contribute and you cannot neglect them.

Now, the equation simplify as we just neglect the viscous,

$$\frac{\partial(\rho v_x)}{\partial t} + \frac{\partial(\rho v_x v_x)}{\partial x} + \frac{\partial(\rho v_y v_x)}{\partial y} + \frac{\partial(\rho v_z v_x)}{\partial z} = \rho g_x - \frac{\partial p}{\partial x}$$

So, the terms which will remain to be solved for the inviscid flow or the transient term, the convection term on the left hand side; the right hand side you have the body force term and the surface force term due to pressure only that is a difference. The surface force due to viscous stresses play a negligible role and we are going to neglect those terms.

$$\rho \frac{Dv}{Dt} = \rho g - \nabla p$$

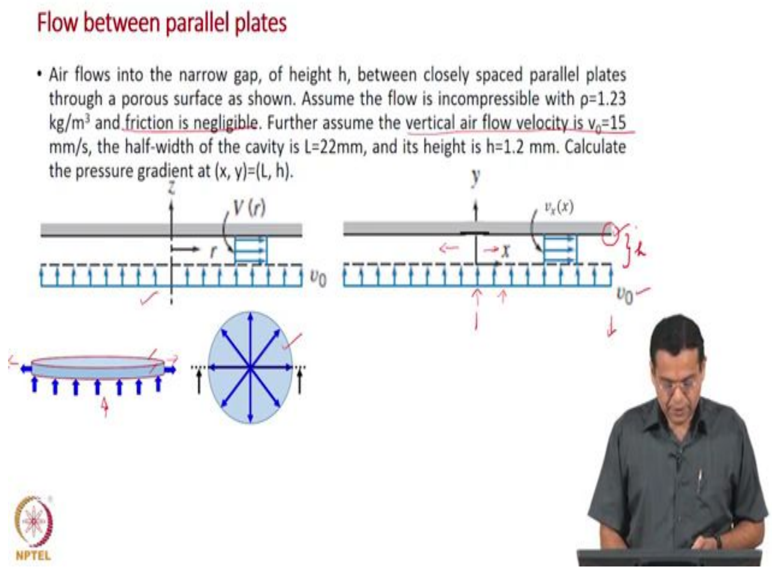
This is called the Euler equation and that is a photograph shows Euler there. So, what is Euler equation? The linear momentum balance without the viscous force term on the right hand side or to be more formally net viscous forces per unit volume terms on the right hand side is the Euler equation. When is it valid? At high Reynolds number; where is it valid? Away from the wall. So, that is should be clear the scope of application of the Euler equation.

Also like to recall our earlier equation we have, look at that we are gradually evolved in our equation. What do we mean by that? We derived the all encompassing linear momentum balance and then Navier Stoke equation as well. In terms of application we are slowly evolving; what do we mean by that, the first case, first level of application the fluid was at rest. So, you did not have the left hand side that was 0 and you had only gravity and then gradient of pressure.

Even then, even that could simplify to only the direction in the vertical direction. Second when the fluid moved as a rigid body, you have the left hand side; but it was one acceleration term for the entire fluid body. So, left hand side we had ρ an $\frac{Dv}{Dt}$ was represented as *a* vector; right hand side we had the both the pressure gradient term and the body force.

Now, we are not considering the fluid to move in as a rigid body, it can have velocity gradients as well and that is why we have the, I would say full form of the substantial derivative namely $\rho \frac{Dv}{Dt}$ term on the left hand side. That is a gradual evolution from our simple fluids at rest case, to fluid under rigid body motion, to inviscid flow.

Example: (Refer Slide Time: 08:36)



So, let us take one simple application of the Euler equation. This example is not new to us; we have come across this example as an application of differential form of total mass balance or the continuity equation. And this example in fact, we used both the integral mass balance and the differential mass balance to arrive at the velocity profiles.

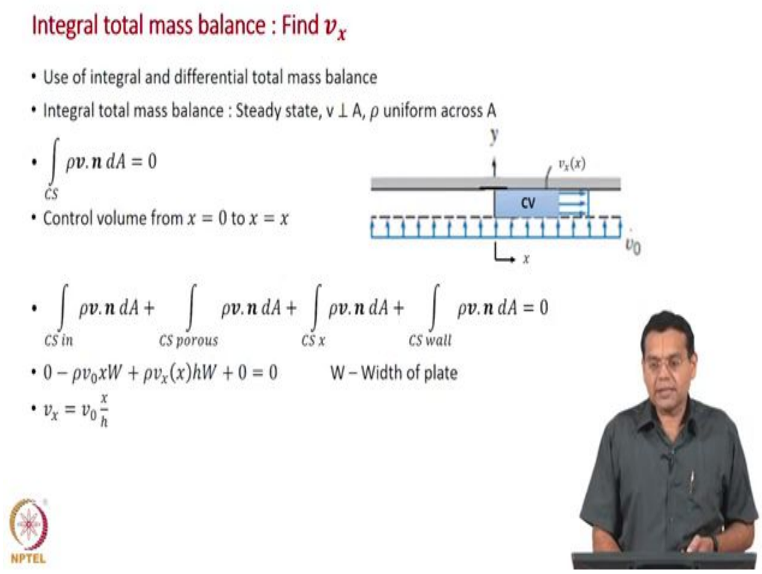
Let me just quickly recall the geometry. We have two plates, actual plates are circular and the bottom plate is porous; of course, the top plate is a solid plate. And air enters the bottom plate, the porous plate and flows out through the sides; if you look at the top view, this is how it looks. So, flow takes place radially and then leaves a plate.

And if you look at the front view, this is how it looks; but this geometry involves cylindrical coordinates. So, we said, we will consider flow between just two parallel plates; but the difference is that the bottom plate is porous and air enters through the bottom plate and flows out through the sides.

And now, let us read the example, air flows into the narrow gap of height h, between closely spaced parallel plates through the porous surface as shown. Assume the flow is incompressible with the ρ equal to 1.23 kg per meter cube density of air and friction is negligible that is a case we are discussing now. When we say friction is negligible, net viscous force is negligible.

Further assume the vertical airflow velocity is 15 millimeter per second, and the half width of the cavity length is given as 22 millimeters, the height is given as 1.2 millimeters it is a narrow gap. And we are asked to calculate the pressure gradient at (x, y) = (L, h).

Solution: (Refer Slide Time: 11:44)



So, these are recall slides, we applied a integral mass balance, we took a control volume from x = 0 to x = x, did integral mass balance and found out the expression for the x component of velocity as

$$v_x = v_0 \frac{x}{h}$$

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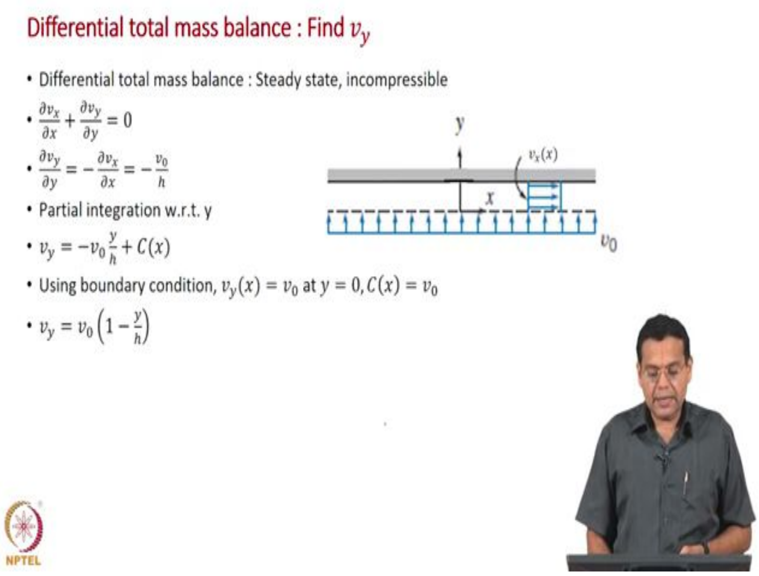

And then we did a differential mass balance, found out the y component of velocity as

$$v_y = v_0\left(1 - \frac{y}{h}\right)$$

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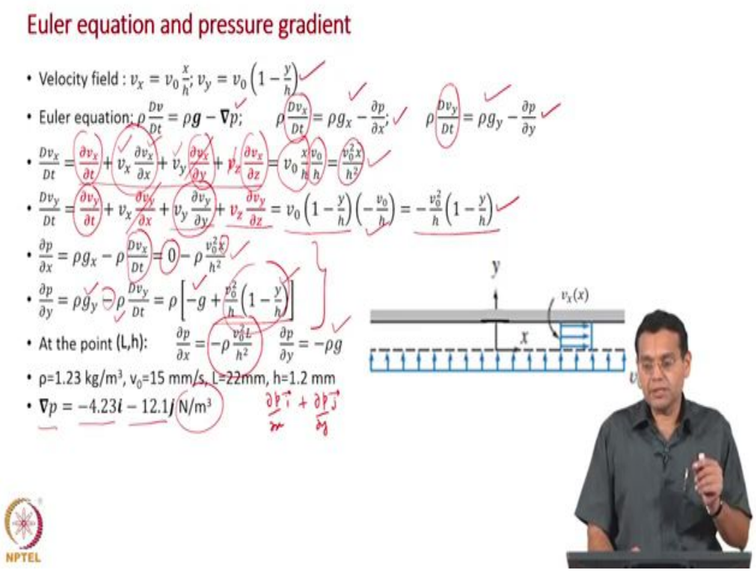

So, let us take those velocity components. So, velocity field was determined using the integral balance, integral mass balance and the differential mass balance;

$$v_x = v_0\frac{x}{h}; \qquad v_y = v_0\left(1 - \frac{y}{h}\right)$$

And these are the velocity components that we have done already. Now, let us take the Euler equation and see how do we use that to find out the pressure gradient. So, we will write the Euler equation the component form,

$$\rho\frac{Dv}{Dt} = \rho g - \nabla p$$

This is in the x direction, and the y direction;

$$\rho\frac{Dv_x}{Dt} = \rho g_x - \frac{\partial p}{\partial x}; \qquad \rho\frac{Dv_y}{Dt} = \rho g_y - \frac{\partial p}{\partial y}$$

To evaluate the x and y component we need to evaluate the $\frac{Dv_x}{Dt}$ term and $\frac{Dv_y}{Dt}$ term on the left hand side.

So, let us take $\frac{Dv_x}{Dt}$ expand and right as

$$\frac{Dv_x}{Dt} = \frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_y}{\partial y} = v_0\frac{x}{h}\frac{v_0}{h} = \frac{v_0^2 x}{h^2}$$

This term tells the acceleration of the fluid particle in the x direction. Now let us evaluate $\frac{Dv_y}{Dt}$ which is the acceleration of the fluid particle in the y direction.

$$\frac{Dv_y}{Dt} = \frac{\partial v_y}{\partial t} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} = v_0\left(1 - \frac{y}{h}\right)\left(-\frac{v_0}{h}\right) = -\frac{v_0^2}{h}\left(1 - \frac{y}{h}\right)$$

So, now we are ready to substitute in the x component, y component of the Euler equation. Let us do that, and while writing we will bring the pressure gradient term to the left hand side, because we are interested in finding out the pressure gradient.

$$\frac{\partial p}{\partial x} = \rho g_x - \rho\frac{Dv_x}{Dt} = 0 - \rho\frac{v_0^2 x}{h}$$

We are considering gravity along the y direction. So, $g_x = 0$. So, now, let us evaluate the pressure gradient the y direction using the y component of Euler equation. So, let us rearrange and write

$$\frac{\partial p}{\partial y} = \rho g_y - \rho\frac{Dv_y}{Dt} = \rho\left[-g + \frac{v_0^2}{h}\left(1 - \frac{y}{h}\right)\right]$$

So, this two equations tells you the, gradient of pressure in the x and y direction and their functions of x and y respectively. Whereas to evaluate at the point (L, h), so, we will substitute $x = L$ and $y = h$.

$$\frac{\partial p}{\partial x} = \rho\frac{v_0^2 x}{h} = \rho\frac{v_0^2 L}{h}$$

$$\frac{\partial p}{\partial y} = \rho\left[-g + \frac{v_0^2}{h}\left(1 - \frac{h}{h}\right)\right] = -\rho g$$

So, let us find the numerical values, we are given the density, we are given v_0, we are given the half width, we are given the distance between the two plates. So, let us substitute all in SI units; if you do that, you get the pressure gradient

$$\rho = 1.23\frac{kg}{m^3};\quad v_0 = 15\frac{mm}{s};\quad L = 22\ mm;\quad h = 1.2\ mm$$

$$\frac{\partial p}{\partial x} = -4.23\frac{N}{m^2};\quad \frac{\partial p}{\partial y} = -12.1\frac{N}{m^2}$$

So, the pressure gradient is

$$\nabla p = \frac{\partial p}{\partial x}i + \frac{\partial p}{\partial y}j = -4.23i - 12.1j\ \frac{N}{m^3}$$

So, what is seen as a simple application of the Euler equation to find out the pressure gradient under inviscid flow condition, not in general; when the flow is inviscid then our discussion is valid.