Continuum Mechanics And Transport Phenomena
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Lecture – 08
Substantial Derivative Part 1

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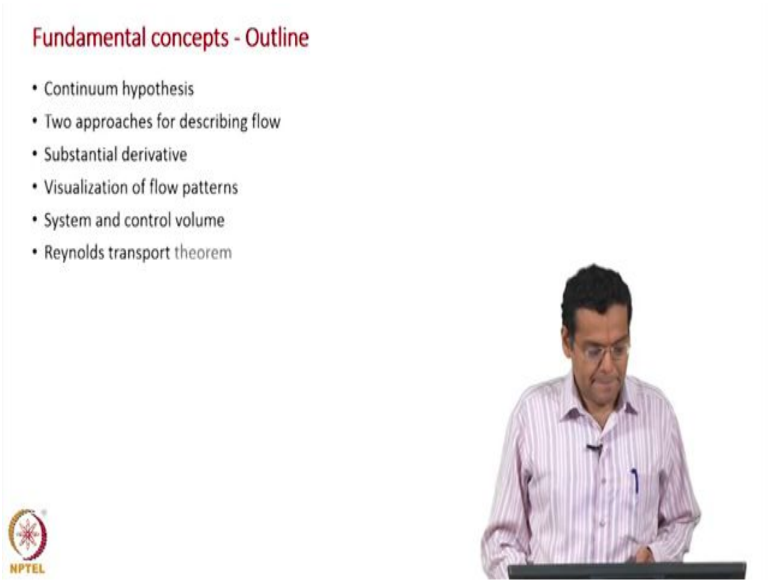

We have been discussing the fundamental concepts. In that we have discussed the continuum hypothesis and then the two different approaches for describing fluid flow. Now, we are going to discuss special derivative, derivative which is characteristic of continuum mechanics and transfer phenomena, what is called as substantial derivative. Now of course, other topics follow later on.

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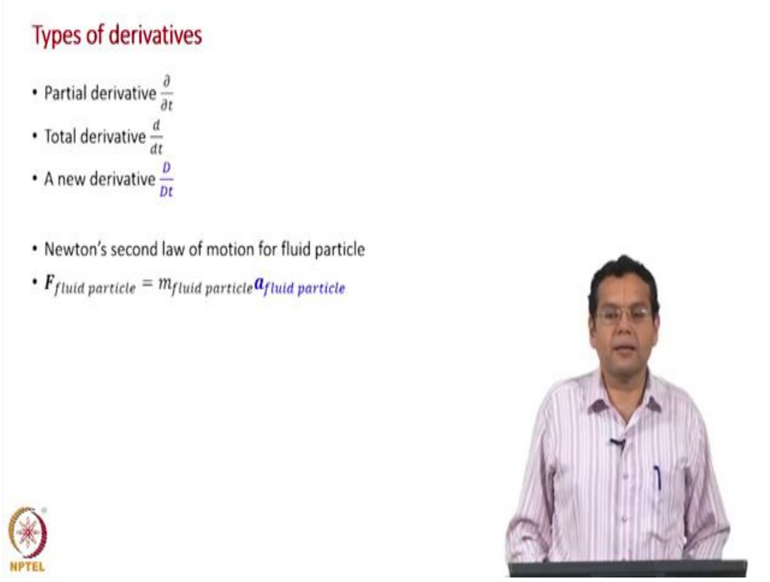

To begin with, we will look at the types of derivatives which are familiar to you,

The partial derivative = $\frac{\partial}{\partial t}$

The total derivative = $\frac{d}{dt}$ this should be familiar to you from your calculus course.

Now, a new derivative which is represented by $\frac{D}{Dt}$, which will have a new physical significance. Now, as you go along the course you will understand where is it used, how is it used, we will have a better feel with the physical significance, but right now, to give a quick motivation.

Suppose, if we extend Newton's, second law of motion for a fluid particle, we know Newton's second law for a solid particle. Suppose, we extend for a fluid particle, then we would write the Newton second law as the force acting on the fluid particle is equal to the mass of the fluid particle multiplied by the acceleration of fluid particle.

$$F_{fluid\ particle} = m_{fluid\ particle}\ a_{fluid\ particle}$$

What I have done is expressed the well-known Newton's second law of motion for a fluid particle, which we would have written so far many times for a solid particle.

Now, just to tell you that this acceleration of fluid particle can be expressed in terms of this new derivative which you are going to call a substantial derivative, right now there is a motivation. As we go along the course, we will use this derivative several times and several places you will understand as we go along.

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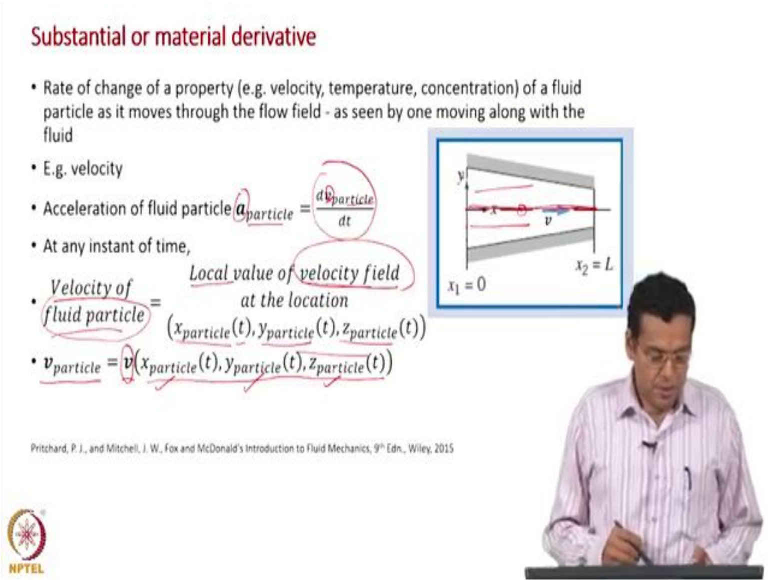

So, as I told you the name of the derivative substantial or the material derivative. Let us define it first before discussing it and derive an expression for that. So, how do you define? It is rate of change of a property when I say property, it could be velocity, temperature, concentration of a fluid particle. By this time you are familiar with what a fluid particle is, we have discussed in detail, we will be able to visualize it, as it moves through the flow field and as seen by one moving along with the fluid. To have an example I have shown here the converging nozzle, which we have seen earlier (see above refer slide).

So, now, we have considered several fluid particles and they move along the length of the converging nozzle. Now, suppose if you follow, take one fluid particle and then follow its motion, what is the rate of change of its property? So, rate of change of a property of a fluid particle as it moves through the flow field. How do you imagine? Suppose, if you sit on the fluid particle and go through the region of flow, what is the rate of change of the property, which you would experience; could be velocity, this particular case velocity, it could be temperature, it could be concentration.

We will take two examples first we will take velocity as an example for property, then we will quickly analogously do for temperature, just to show you that the substantial and metal derivative can be applied for any property. Usually, what is discussed is velocity and extended for other properties as well. We will discuss both velocity and temperature.

So, why this particular example of converging nozzle as we have seen earlier as the fluid flows along the length of the converging nozzle, because of the reduction in the cross section area there is an increase in velocity. So, from the fluid point of view it experiences as an acceleration. So, if you are sitting on the fluid particle and travelling along with it you will experience the increase in velocity and that is why this particular geometry, this configure has been chosen here.

What you are going to discuss is applicable for any general case. So, as a property we will take velocity as an example. Now, we want to find out the rate of change of velocity of the fluid particle which is of course, the acceleration of the fluid particle. And how do you define, because the particle just extend where particle mechanics knowledge of expressing the acceleration of fluid particle, in terms of derivative of velocity of the particle.

$$Acceleration\ of\ fluid\ particle\ a_{particle} = \frac{dv_{particle}}{dt}$$

This velocity is a vector here and acceleration is a vector here. So, we have expressed acceleration of the particle as the time derivative of the velocity of the particle. This exactly what you would have come across in a particle mechanics course only difference is here the particle refers a fluid particle, as I have been discussing so far. Now, we use a physical principle at any instant of time.

Now, the velocity of the fluid particle at a position at a particular instant is equal to the velocity of the fluid in that particular location.

$$Velocity\ of\ fluid\ particle = Local\ value\ of\ velocity\ field\ at\ the\ location$$
$$(x_{particle}(t),\ y_{particle}(t),\ z_{particle}(t))$$

So, what is the implication of this? The left-hand side is from a Lagrangian viewpoint, we are telling from particle viewpoint velocity of a fluid particle. On the right hand side, we have the velocity field, we have already discussed that moment I say field, it represents an Eulerian frame of reference.

So, why is it local value? We are at a particular position, at a particular instant, the fluid particle is that particular location. It’s velocity should be equal to the velocity of the fluid at that particular location, that is what is represented here as an expression x, the particle position of the x coordinate of the particle at that particular instant similarly, y particular similarly, z particle. So, looks like a very simple statement. It has very key importance, connects a Lagrangian view point to an Eulerian view point. Why is it Lagrangian? Talking about a fluid particle at left hand side, but talking about a velocity field at right hand side. Simple intrusion will tell you the particle is representative with the fluid. So, at a particular instant, at a particular position, the velocity of the fluid particle should be same as that of the fluid ok, given in terms of velocity field.

If you want to write this in expression, velocity of the particle is equal to the velocity in terms of Eulerian field.

$$v_{particle} = v(x_{particle}(t),\ y_{particle}(t),\ z_{particle}(t))$$

Left hand side is velocity of the particle is equal to velocity given in the Eulerian field not at any location at the location and where the particle is present given by the coordinates x particle, y particle and z particle.

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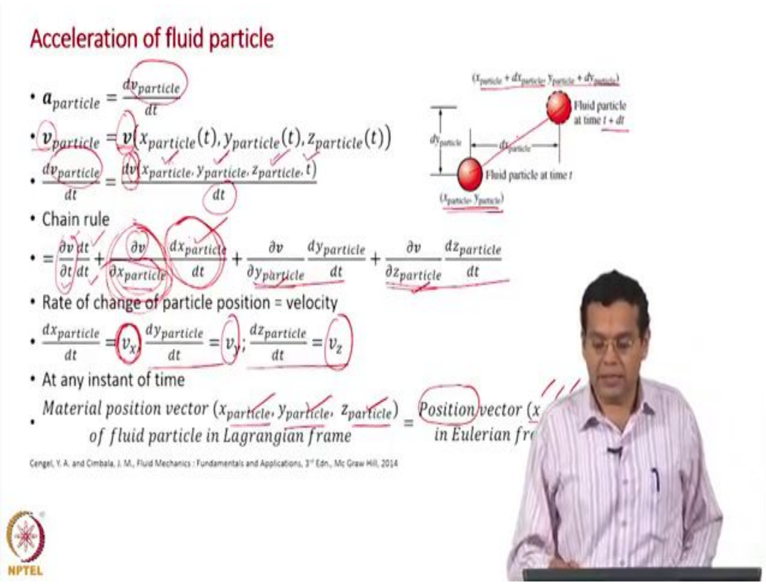

Now, we said the acceleration of the particle is given by the derivative of the velocity of the particle.

$$a_{particle} = \frac{dv_{particle}}{dt}$$

Now, we have used a physical principle and said that the velocity of the particle is equal to the velocity of the fluid at the position of the particle. This is what we have seen earlier.

$$v_{particle} = v(x_{particle}(t),\ y_{particle}(t),\ z_{particle}(t))$$

Now, I will substitute in this place instead of velocity of the particle, I will substitute this velocity of the in terms of Eulerian field. So,

$$\frac{dv_{particle}}{dt} = \frac{dv(x_{particle}, y_{particle}, z_{particle}, t)}{dt}$$

So, left hand side what we have is velocity of the particle, right hand side we have velocity in terms of Eulerian field and it is a function of x particle, y particle, z particle and time. What you have on the right hand side is a total derivative and of course, we know

by chain rule

$$\frac{dv_{particle}}{dt} = \frac{\partial v}{\partial t}\frac{dt}{dt} + \frac{\partial v}{\partial x_{particle}}\frac{dx_{particle}}{dt} + \frac{\partial v}{\partial y_{particle}}\frac{dy_{particle}}{dt} + \frac{\partial v}{\partial z_{particle}}\frac{dz_{particle}}{dt}$$

Right hand side you have a total derivative and it is a function of four variables x y z of the particle and time and using chain rule, we express in in terms of the partial derivatives. So, the independent variables are x y z and t. So, first we differentiate v with respect to time and next, we have we partial differentiate. What do we mean by partial differentiation?

For example here when I differentiate with respect to time x y z, all the three coordinates are kept constant, which remains at a particular location. So, when I say differential with respect to time and partially all the other three special coordinates are fixed. So, this represents the derivative at a particular position. And now, when I differentiate with respect to x particle I keep the time constant and the y particle and z particle constant; so, whereas, we are considering variation of velocity in the x direction only.

Now, what is shown here is a fluid particle at a location x particle and y particle. In a time instant from t to t+dt, it moves to another location where x particle + dx particle, a small displacement dx particle and y particle + dy particle. Now, if you take the rate of change of x coordinate with respect to time, you will get the velocity of the particle,

$$Rate\ of\ change\ of\ particle\ position = velocity$$

$$\frac{dx_{particle}}{dt} = v_x \;\; ; \quad \frac{dy_{particle}}{dt} = v_y \;\; ; \quad \frac{dz_{particle}}{dt} = v_z$$

Now, we have already seen that the velocity of the particle at a particle instant, particular location is equal to the Eulerian velocity and that is why I have used here, vx gives the velocity of the particle in the x direction. We have seen that at any instant the velocity of the particle is same as the velocity of the fluid, but now, I use the x component of the velocity of fluid given by the Eulerian description; similarly, in the y direction and similarly, in the z direction. So, in these expressions, we are going to replace $\frac{dx_{particle}}{dt}$ in terms of the velocity field, x component of velocity of the fluid given by the Eulerian description similarly, v_y and then similarly, v_z.

$$Material\ position\ vector\ \left(x_{particle},\ y_{particle},\ z_{particle}\right)\ of\ fluid\ particle\ in$$

$$Lagrangian\ fram = Position\ vector\ \ (x, y, z)\ in\ Eulerian\ fram$$

The material position vector denoted by x particle, y particle and z particle. So, material position vector of fluid particle, in Lagrangian frame is equal to the position vector. Why do I say position vector? Position vector is just like our usual x y z coordinate in the Eulerian frame.

When I say x y z it is an Eulerian frame when I say x particle, y particle, z particle it is in a Lagrangian frame. So, once again we replace this x particle, y particle, z particle with x y z. So, I will make two replacements; one is that these derivates will be expressed in terms of the velocity of the particles. And hence, in terms of velocity of the fluid in the respective directions and then similarly, these coordinates of the particles, they are material coordinates. Why do I say material coordinates as I told you they are position vector of the particle as it flows to the flow field and what replace that in terms of the Eulerians spatial locations. Yeah let us do that.

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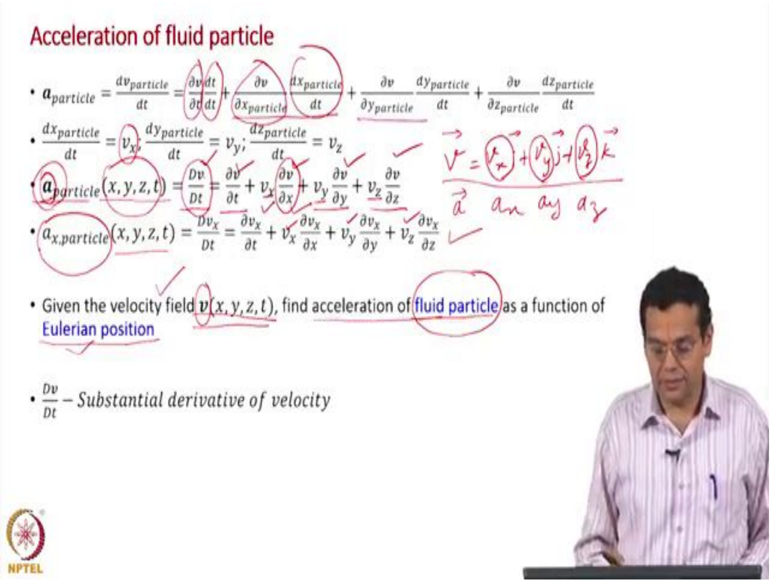

So,

$$a_{particle} = \frac{dv_{particle}}{dt}$$

$$= \frac{\partial v}{\partial t}\frac{dt}{dt} + \frac{\partial v}{\partial x_{particle}}\frac{dx_{particle}}{dt} + \frac{\partial v}{\partial y_{particle}}\frac{dy_{particle}}{dt} + \frac{\partial v}{\partial z_{particle}}\frac{dz_{particle}}{dt}$$

This is the expression which we have written for acceleration of the particle and then we wrote in terms of the total derivative of the expression in terms of the partial derivatives. Now, we have seen the time derivative of the x direction coordinate is the vx and similarly, the y direction we have vy and in the z direction we have vz. So, we will make these substitutions acceleration of a particle.

$$\frac{dx_{particle}}{dt} = v_x \;\; ; \quad \frac{dy_{particle}}{dt} = v_y \;\; ; \quad \frac{dz_{particle}}{dt} = v_z$$

After replacement,

$$a_{particle}\,(x,y,z,t) = \frac{Dv}{Dt} = \frac{\partial v}{\partial t} + v_x\frac{\partial v}{\partial x} + v_y\frac{\partial v}{\partial y} + v_z\frac{\partial v}{\partial z}$$

Now, that acceleration of a particle is a function of x y z and time So, moment whenever you come across capital D by Dt it represents the substantial derivative. Why substantial derivative? We are following a material or a substance and hence, called substantial derivative or material derivative, that is the reason for the name substantial derivative, material derivative.

Now, left hand side, I would like to repeat the acceleration of the particle remember it's a vector equation, acceleration is a vector, the velocities are also all vectors and these are components of velocity in the x y z directions (v_x, v_y, v_z). So, if you write v vector is equal to v_x i and then v_y j and then v_z k, where i, j, k are the unit vectors, the x component of the velocity field, y component of the velocity field, z component of the velocity field are v_x, v_y and v_z.

And remember, this has a good significance, acceleration of the particles from a Lagrangian view point, but that is expressed in terms of the Eulerians spatial location x, y, z are the usual coordinates, but this is the rate of change of velocity as you follow the fluid particle, which means it has a Lagrangian meaning attached to it, but it is expressed in the function of x y z in the flow field. What I have done next is, because the vectroial equation, you can write two components of it.

$$a_{x,particle}(x, y, z, t) = \frac{Dv_x}{Dt} = \frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z}$$

Similarly, you can write the a vector in terms of ay and then az and of course, i j k, etcetera.

So, ax particle is the x component of the acceleration particle, function of x, y, z and time and I just written instead of v vector vx. Similarly, vx here I have replaced everywhere the vector with the x , corresponding x component. So, this gives the acceleration of the particle in the x direction as the function of x y z and time. Now, what is the most important significance of this expression. Given the velocity field which we can easily measure, it is in Eulerian description. We already seen that the Eulerian measurement, Eulerian description is more practical.

So, given the velocity field $v(x, y, z, t)$, we are finding acceleration of a fluid particle as a function of the Eulerian position. This nicely combines the Lagrangian view point and the Eulerian view point, acceleration of fluid particle, why does Lagrangian, because the rate of change of velocity as you sit on the particle and follow it. That is why acceleration of fluid particle represents a Lagrangian variable, v is a Eulerian field, because the function of x y z and time.

So, if you have a velocity field, if you measure velocity at different locations you can find out what is the acceleration experience by a fluid particle at different spatial locations. That is the biggest advantage of this expression. Remember, we said the independent variables for

Lagrangian description are the initial position and time r_0 and t. So, if you have a velocity as a function of r_0 and t you can differentiate that and then get the acceleration, but usually we do not have access to that information, because that requires Lagrangian measurement.

What a access is to the velocity field information, but using that we are able to get the same meaning as a Lagrangian derivative. So, $\frac{Dv}{Dt}$ as I told you is called the substantial derivative, also called as material derivative. This particular case, it is the substantial derivative of velocity and, because it is substantial derivative velocity it is also called as the acceleration , because substantial derivative of other variable is do not have any special name, because it is velocity we call it as acceleration.