Continuum Mechanics And Transport Phenomena
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Lecture – 74
Hooke's Law – Stress-strain Relation

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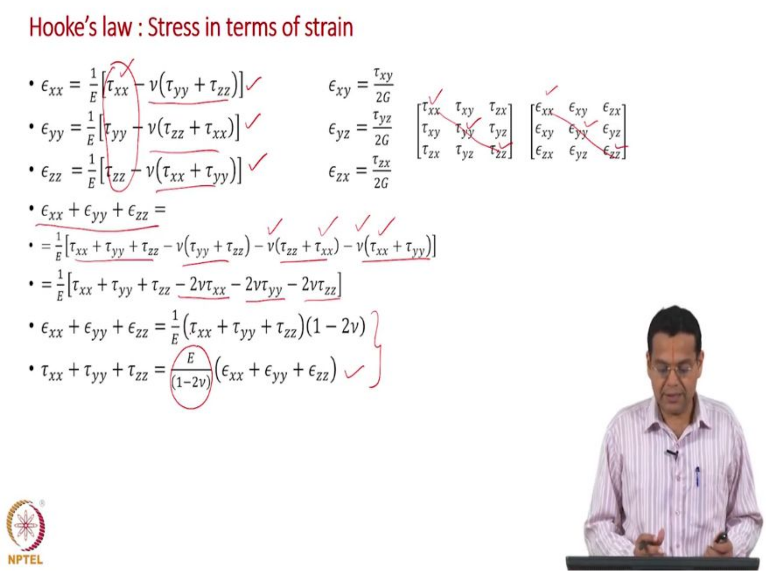

We have discussed so far are the assumptions behind Hooke's law namely homogeneous, isotropic, linear, elastic solid. So, under these assumptions Hooke's laws valid. We introduced material properties namely Young's modulus, shear modulus, Poisson's ratio and then we expressed the Hooke's law in the form of a strain as a function of stress. And we also saw that there are three material properties they are dependent on each other and we proved that only two independent material properties are there. Now what is that we are going to do now? Express Hooke's law in the form of stress as a function of strain.

So, conceptually nothing new it just involves some mathematical rearrangement. So, let us start doing that. So, let us list down the equations for Hooke's law expressing strain in terms of stress all the six equations

$$\varepsilon_{xx} = \frac{1}{E}\left[\tau_{xx} - \nu(\tau_{yy} + \tau_{zz})\right] \qquad \varepsilon_{xy} = \frac{\tau_{xy}}{2G}$$

$$\varepsilon_{yy} = \frac{1}{E}\left[\tau_{yy} - \nu(\tau_{xx} + \tau_{zz})\right] \qquad \varepsilon_{yz} = \frac{\tau_{yz}}{2G}$$

$$\varepsilon_{zz} = \frac{1}{E}\left[\tau_{zz} - \nu(\tau_{yy} + \tau_{xx})\right] \qquad \varepsilon_{zx} = \frac{\tau_{zx}}{2G}$$

Then for starting the mathematical rearrangement we will sum all the normal strain $\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}$ which means summing all the equations listed here. So,

$$\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = \frac{1}{E}[\tau_{xx} + \tau_{yy} + \tau_{zz} - \nu(\tau_{yy} + \tau_{zz}) - \nu(\tau_{xx} + \tau_{zz}) - \nu(\tau_{yy} + \tau_{xx})]$$

Now let us simplify this

$$\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = \frac{1}{E}[\tau_{xx} + \tau_{yy} + \tau_{zz} - 2\nu\tau_{xx} - 2\nu\tau_{yy} - 2\nu\tau_{zz}]$$

Now if you look at the sum of normal stresses which is common so, let us take that out

$$\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = \frac{1}{E}(\tau_{xx} + \tau_{yy} + \tau_{zz})(1 - 2\nu)$$

And we will write this for sum of normal stresses the previous equation is for sum of normal strain we will right this as an expression for some of normal stresses just simple rearrangement we will give,

$$\tau_{xx} + \tau_{yy} + \tau_{zz} = \frac{E}{(1-2\nu)}(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz})$$

These two equations either of them or both of them nice relationship why because they relate sum of normal stresses to the sum of normal strains in terms of property E and Poisson ratio, ν. So, that way a good relationship under these assumptions whatever we have done these equations both the equations relate sum of normal stresses to sum of normal strain that is a physical viewpoint of this equation, let us proceed further.

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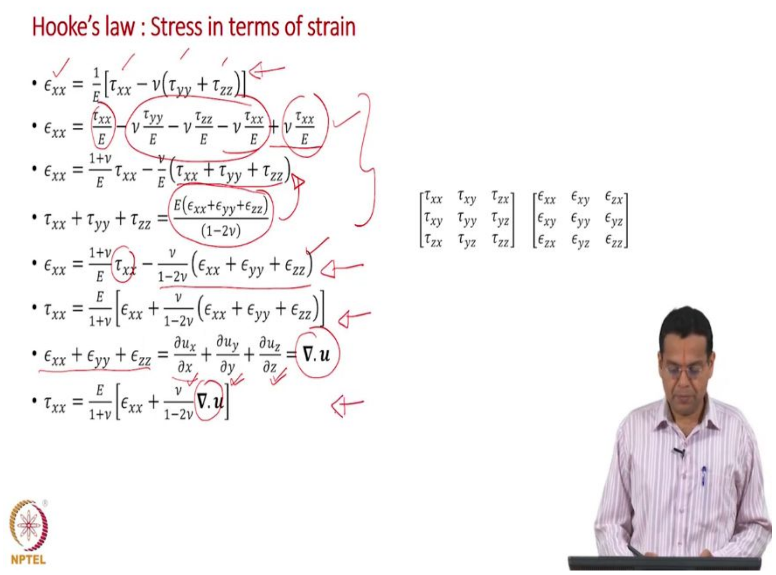

So, now let us write down the equation for the normal strain along x direction

$$\varepsilon_{xx} = \frac{1}{E}\left[\tau_{xx} - \nu(\tau_{yy} + \tau_{zz})\right]$$

Let us expand the equation

$$\varepsilon_{xx} = \frac{\tau_{xx}}{E} - \nu\frac{\tau_{yy}}{E} - \nu\frac{\tau_{zz}}{E}$$

And then, I will subtract $\nu\frac{\tau_{xx}}{E}$ and add $\frac{\tau_{xx}}{E}$,

$$\varepsilon_{xx} = \frac{\tau_{xx}}{E} - \nu\frac{\tau_{yy}}{E} - \nu\frac{\tau_{zz}}{E} - \nu\frac{\tau_{xx}}{E} + \nu\frac{\tau_{xx}}{E}$$

So, let us simplify this,

$$\varepsilon_{xx} = \frac{1+\nu}{E}\tau_{xx} - \frac{\nu}{E}(\tau_{xx} + \tau_{yy} + \tau_{zz})$$

So, we have already seen the relationship between sum of normal stresses and sum of normal strain in the previous slide which is written down here,

$$\tau_{xx} + \tau_{yy} + \tau_{zz} = \frac{E}{(1-2\nu)}(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz})$$

So, that I will use this relationship here,

$$\varepsilon_{xx} = \frac{1+\nu}{E}\tau_{xx} - \frac{\nu}{(1-2\nu)}(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz})$$

What we have done essentially is that in this expression, we had a normal strain on the left hand side and then three normal stresses were present on the right hand side. We want one equation in one unknown we want only one of the normal stresses by doing these steps in this equation there is only one normal stress present.

So, that we can rearrange and get an expression for normal stress that is was that was the idea behind all these steps, well. Let us do a simple rearrangement

$$\tau_{xx} = \frac{E}{1+\nu}\left[\varepsilon_{xx} + \frac{\nu}{(1-2\nu)}(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz})\right]$$

Now the sum of normal strains, we can express that in terms of the gradients of displacement field we can express as

$$\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = \nabla . u$$

So, let us in a more little more simpler way,

$$\tau_{xx} = \frac{E}{1+\nu}\left[\varepsilon_{xx} + \frac{\nu}{(1-2\nu)}\nabla .u\right]$$

Whenever you come across this divergence of displacement field in the future slides it is nothing, but either in terms of gradients the sum of these gradients or sum of these normal strains. So, we said we are going to express normal stress in terms of strains and that is what we have done here and of course, another way of representing that.

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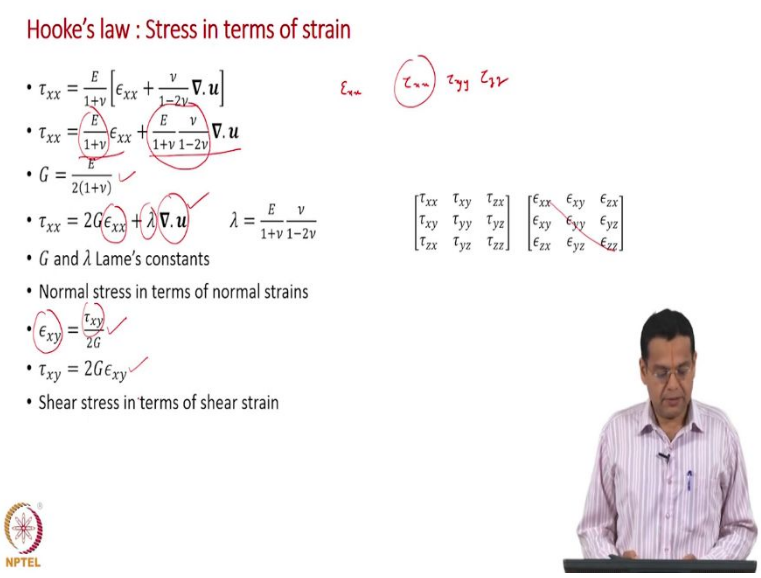

Now, we will do some more rearrangement,

$$\tau_{xx} = \frac{E}{1+\nu}\left[\varepsilon_{xx} + \frac{\nu}{(1-2\nu)}\nabla .u\right]$$

$$\tau_{xx} = \frac{E}{1+\nu}\varepsilon_{xx} + \frac{E}{1+\nu}\frac{\nu}{(1-2\nu)}\nabla .u$$

Then we have seen the relationship between G, E and ν which we have derived already

$$G = \frac{E}{2(1+\nu)}$$

So,

$$\tau_{xx} = 2G\varepsilon_{xx} + \lambda\nabla .u$$

We put all terms $\lambda = \frac{E}{1+\nu}\frac{\nu}{(1-2\nu)}$ as one other property , why is it a property because E, ν are properties. So, λ is a combination of those properties so, λ is another property. So, of course, this looks much simpler and we are going to use this form of the stress strain relationship for further discussion even when you later on go to Newton's law of viscosity it looks similar to this only.

The G and λ are call lame's constants and what we have done is, normal stress in terms of normal strains. On the right hand side you have ε_{xx} and $\nabla.u$ has the sum of all the normal strains. So, on the right hand side you have all the normal strains and for the case of a shear strain shear stress relationship we have seen

$$\varepsilon_{xy} = \frac{\tau_{xy}}{2G}$$

Now, it becomes very simple just rearrange and write

$$\tau_{xy} = 2G\varepsilon_{xy}$$

In the case of normal strain and normal stress it involved some steps the reason is the expression for normal strain ε_{xx} involved all the three normal stresses. That is why we have to do some rearrangement and express τ_{xx} in terms of the normal strain. But in this case epsilon for the case of shear strain shear strain depends on the shear stress in that plane only. So, simple rearrangement is sufficient to express the shear stress in terms of shear strain.

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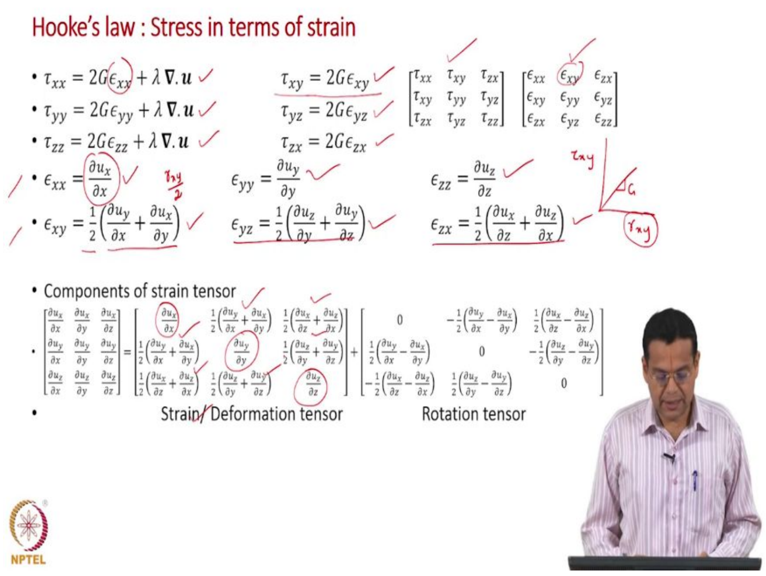

So, now let us extent these two other directions that is what we have seen now. So, let us write this analogously for other directions.

$$\tau_{xx} = 2G\varepsilon_{xx} + \lambda\nabla.u \qquad \tau_{xy} = 2G\varepsilon_{xy}$$

$$\tau_{yy} = 2G\varepsilon_{yy} + \lambda\nabla.u \qquad \tau_{yz} = 2G\varepsilon_{yz}$$

$$\tau_{zz} = 2G\varepsilon_{zz} + \lambda\nabla.u \qquad \tau_{zx} = 2G\varepsilon_{zx}$$

And we should know that ε_{xx} is the normal strain, but if you say as normal strain the purpose is not solved because we will have to express in terms of displacement gradient then only it is complete. So,

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x}; \qquad \varepsilon_{yy} = \frac{\partial u_y}{\partial y}; \qquad \varepsilon_{zz} = \frac{\partial u_z}{\partial z}$$

And what about, ε_{xy}, remember it is

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right); \qquad \varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z}\right); \qquad \varepsilon_{zx} = \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right)$$

Why do we express this, we do not express τ_{xy} in terms of γ_{xy} we express τ_{xy} in terms of ε_{xy} what is the reason? Our objective is to relate components in this stress tensor to the components in the strain tensor. What is appearing in the strain tensor is ε_{xy} not γ_{xy} that is why in this relationship relate τ_{xy} and then ε_{xy}. Though the actual in terms of measurement is γ_{xy}.

Now to connect to what we have done earlier, we can identify that the terms written here in the right handed expressions in terms of gradients all these six terms are nothing, but the components of the strain tensor which you have discussed earlier.

Now, to just to connect and then to have a recall as some few recall slides here this was a strain tensor this was the rotation tensor, what we will do now is we will connect this and recall also what we have discussed earlier.

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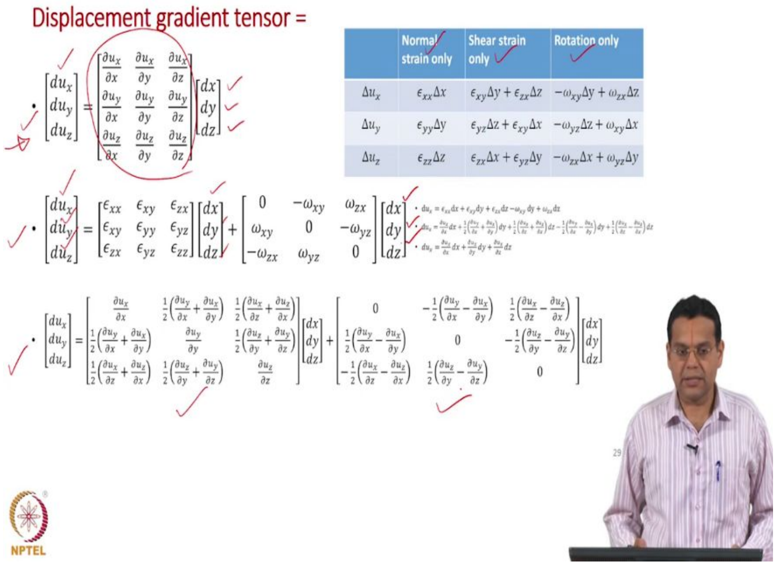

This was discussed when we discussed strain under solid mechanics, what did we do we derived a relationship between d_{u_x}, d_{u_y}, d_{u_z} and dx, dy, dz in two ways. First way is this in more mathematically which resulted in the displacement gradient tensor, the second one was more analytically more geometrically where we considered displacement due to a difference in displacement due do normal strain, shear strain, rotation only and wrote these two equation.

These two expressions are also relationship between d_{u_x}, d_{u_y}, d_{u_z} and dx, dy, dz these were written in a more analytical way geometrical way. Now, based on the first equation and the second or third equation we said that this displacement gradient tensor is equal to sum of these two tensors that is what we discussed.

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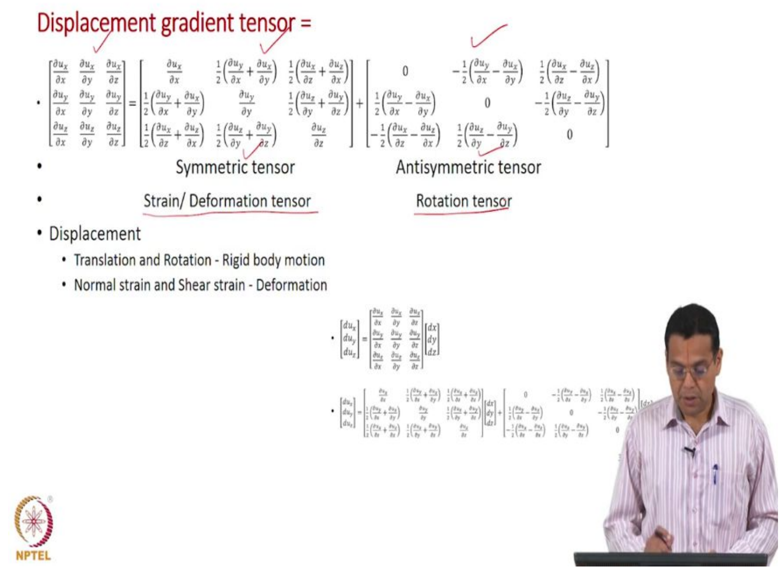

The displacement gradient tensor is sum of the strain tensor and the rotation tensor, what is the significance of the first tensor that includes the normal strain and shear strain, the second tensor includes the rotation contribution. And we said strain tensor is symmetric, rotation tensor is anti-symmetric and the names given where strain or deformation tensor, because it involves contribution from normal strain, shear strain. Second tensor was called as rotation tensor because it involves contribution from rotation. And we concluded that translation and rotation which are rigid body motion and normal strain, shear strain which are deformation all contribute to displacement.

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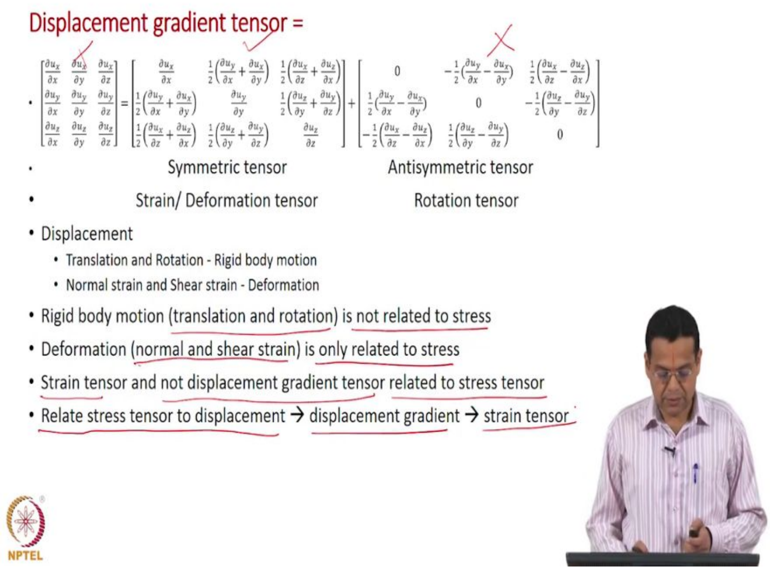

Next what we discussed was more importantly this particular slide. In fact, this was a concluding slide then what we said was rigid body motion which is translation and rotation it is not related to stress. Deformation which is normal and shear strain is only related to stress. Strain tensor are not displacement gradient tensor related to stress tensor, what did we say stress has to be related to the strain tensor not to the displacement gradient tensor this also is not connected to the stress.

Look at the last line, relate stress tensor to displacement not displacement gradient, but strain tensor and that is what exactly we are done now. After several classes we put forth the question then saying that we have to relate stress to the components of strain tensor that is what exactly we have done in today’s lecture, related the stress left hand sides are all stress right hand sides are all components of the strain tensors.

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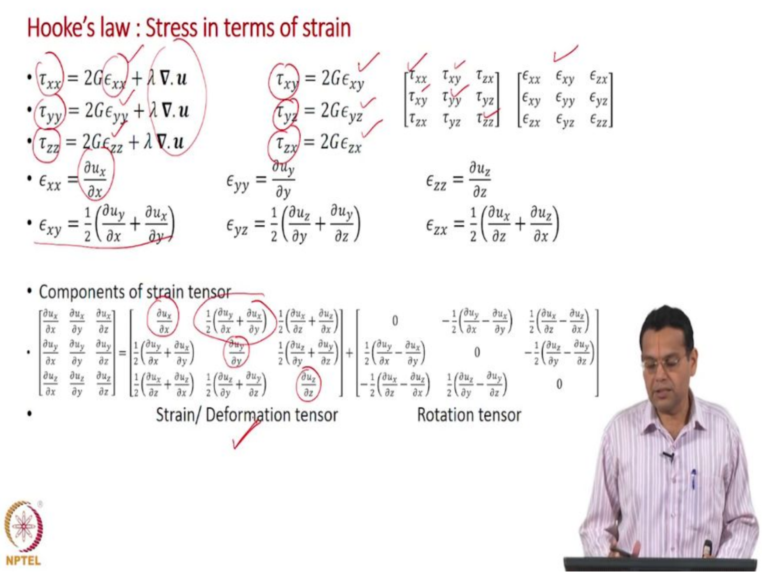

So, let us look at it again now it becomes very clear this slide the, what we have on the left hand side are all the components of stress tensor and as we have seeing here the normal stresses and the shear stresses. In the right hand side what you have are all components of the strain tensor components of the strain tensor alone, not the displacement gradient tensor, not the rotation tensor, only the strain tensor.

$$\tau_{xx} = 2G\varepsilon_{xx} + \lambda\nabla.u \qquad \tau_{xy} = 2G\varepsilon_{xy}$$

$$\tau_{yy} = 2G\varepsilon_{yy} + \lambda\nabla.u \qquad \tau_{yz} = 2G\varepsilon_{yz}$$

$$\tau_{zz} = 2G\varepsilon_{zz} + \lambda\nabla.u \qquad \tau_{zx} = 2G\varepsilon_{zx}$$

So, whatever appears on the right hand side, we know $\nabla.u$ is nothing, but sum of all the normal strains. So, whatever you have on the right hand side are all components of the strain tensor.

So whatever was our objective we have achieved here relating stress to strain more precisely components of stress tensor to the components of strain tensor or deformation tensor, only the normal strain, shear strain result in stresses that is why we have done this.

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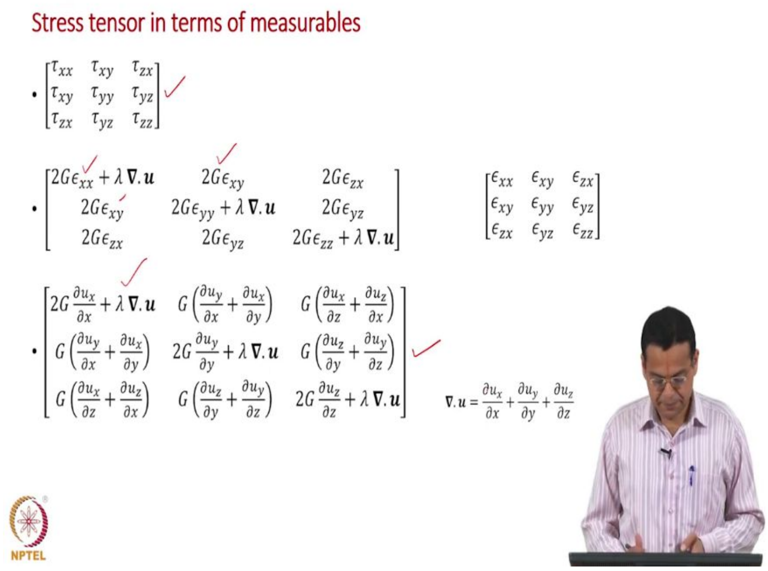

So, let us put them more compactly in the form of a tensor this is the stress tensor,

$$\left[\tau_{xx}\ \tau_{xy}\ \tau_{zx}\ \tau_{xy}\ \tau_{yy}\ \tau_{yz}\ \tau_{zx}\ \tau_{yz}\ \tau_{zz}\ \right]$$

At this point when we introduced stress tensor it was a very much physically meaningful quantity. We said if you give me the stress tensor I can find out stress vector acting on any plane, just take do matrix multiplication of this n vector with the stress tensor you get stress vector acting on any plane so, very physically meaningful quantity and useful quantity as well.

But it is not measurable, it is a variable of physical interest physical meaning, but moment we write this as where in the previous slide it becomes expressed in terms of measurable.

$$\left[\tau_{xx}\ \tau_{xy}\ \tau_{zx}\ \tau_{xy}\ \tau_{yy}\ \tau_{yz}\ \tau_{zx}\ \tau_{yz}\ \tau_{zz}\ \right] = \left[2G\varepsilon_{xx} + \lambda\nabla .u\ 2G\varepsilon_{xy}\ 2G\varepsilon_{zx}\ 2G\varepsilon_{xy}\ 2G\varepsilon_{yy} + \lambda\nabla .u\ 2G\varepsilon_{yz}\ 2G\varepsilon_{zx}\ 2G\varepsilon_{yz}\ 2C\right.$$

All the terms ε_{xx}, ε_{xy} where all in terms of at the components of the strain tensor they are all expressible in terms of displacement gradient and hence stress tensor has been expressed in terms of measurable. Remember that was one of the main point we started off saying that we are going to express an immeasurable in terms of a measurable and that is what we have done here.

That is a key point here as long as this is written it is a nice physical quantity where which physical meaning as well, but it is immeasurable, moment you express in this way it becomes a measurable and of course, you have the property G and then λ they have to come from experiments. That is why we said stress strain relationship is empirical we cannot theoretically get G and λ, we are only going to make some measurements and find out, what is G and λ?

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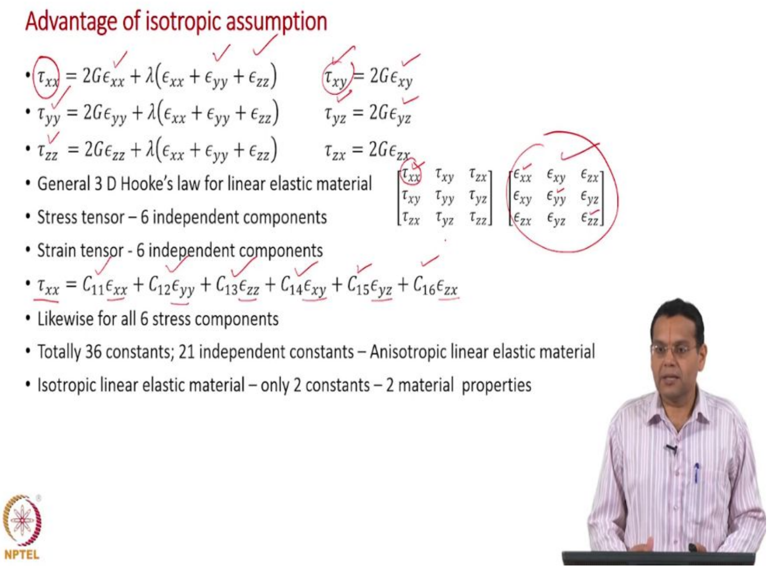

Just like to mention what is the advantage of the isotropic assumption we said we have derived the Hooke's law for homogeneous, isotropic, linear elastic solid. We will keep all the assumptions suppose we say it is non - isotropic, what would I be in the situation we are not going to derive just discuss that. So, that we will appreciate the really significant advantage of assuming isotropic condition.

$$\tau_{xx} = 2G\varepsilon_{xx} + \lambda\nabla.u \qquad \tau_{xy} = 2G\varepsilon_{xy}$$

$$\tau_{yy} = 2G\varepsilon_{yy} + \lambda\nabla.u \qquad \tau_{yz} = 2G\varepsilon_{yz}$$

$$\tau_{zz} = 2G\varepsilon_{zz} + \lambda\nabla.u \qquad \tau_{zx} = 2G\varepsilon_{zx}$$

Now this is the equation of course, relating normal stress to the normal strain and shear stress to the shear strain. Now if you look at τ_{xx}, the way in which you have derived depends only on ε_{xx}, ε_{yy}, ε_{zz}. Similarly if you take τ_{yy}, τ_{zz} it depend only on ε_{xx}, ε_{yy}, ε_{zz}, if you take τ_{xy} depends only on ε_{xy}, τ_{yz} depends only on ε_{yz}.

Now, what would be the more general scenario, what can be the general scenario? Right now this τ_{xx} depends only on ε_{xx}, ε_{yy}, ε_{zz} but in general it can depend on all these components when I say all these components, 6 components there are 9, but 6 are independent. So, τ_{xx} can dependent all these 6 components similarly τ_{xy} can also depend on all these 6 components, that is what we are going to write here. So, stress tensor as 6 independent components, strain tensor as 6 independent components. So, every independent component in the stress tensor can depend on all the 6 independent components in the strain tensor.

Let us write one equation for a sample. So,

$$\tau_{xx} = C_{11}\varepsilon_{xx} + C_{12}\varepsilon_{yy} + C_{13}\varepsilon_{zz} + C_{14}\varepsilon_{xy} + C_{15}\varepsilon_{yz} + C_{16}\varepsilon_{zx}$$

Now look at the constants, you have, 6 constants here, now likewise I can write for all the 6 independent stress tensor components which means I will have totally 36 constant or 36 properties will be there. When I say constants remember these constants are nothing but properties, but you can prove that 15 are dependent and only 21 are independent constants, out of 36 you can prove that 21 independent constants are there.

So, if you are saying you are going to derive a stress strain relationship for a homogeneous anisotropic linear elastic material then we would end up in 21 material properties, because in all our relationship been into write the properties depending on direction, but moment you take isotropic condition as we have done here we will result only in 2 constants or 2 material properties that is the luxury I would say by assuming isotropic condition. That is why we discussed assumptions to begin with itself the whole scope of our stress strain relationship is limited to those assumptions homogeneous, isotropic, linear elastic solid.