Continuum Mechanics and Transport Phenomena
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Lecture – 73
Relation between Material Properties

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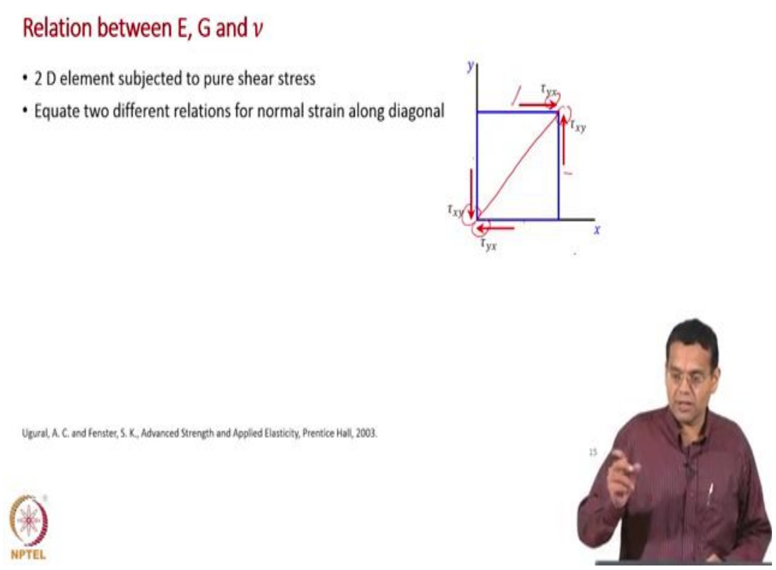

Now, we will derive relationship between the Young's modulus E, the shear modulus G and the Poisson ratio ν from this different books follow different ways of deriving it following from this book by Ugural and Fenster, Advanced Strength and Applied Elasticity. Only for this part I have referred this book because it was in line with what we have discussed. First I will give an overview of what you are going to; however, we are going to proceed, we will take a two-dimensional element, it could be a plate or region inside a solid and subject to pure shear stress. So, to subject only shear stress that is a point to be noted.

The two-dimensional elements subjected to pure shear stress is shown here and our usual sign convention is adopted, on a positive face positive force. So, all shear stress components are in the positive sense and negative face negative direction of force. What we are going to do is, consider the diagonal and then derive expression for normal strain along the diagonal in two different ways and equate. What do we mean by that, when you apply this shear stress that the diagonal elongates. So, which mean there is a normal strain along the diagonal and we are going to derive expression for this normal strain in two different ways and equate it that is overall idea.

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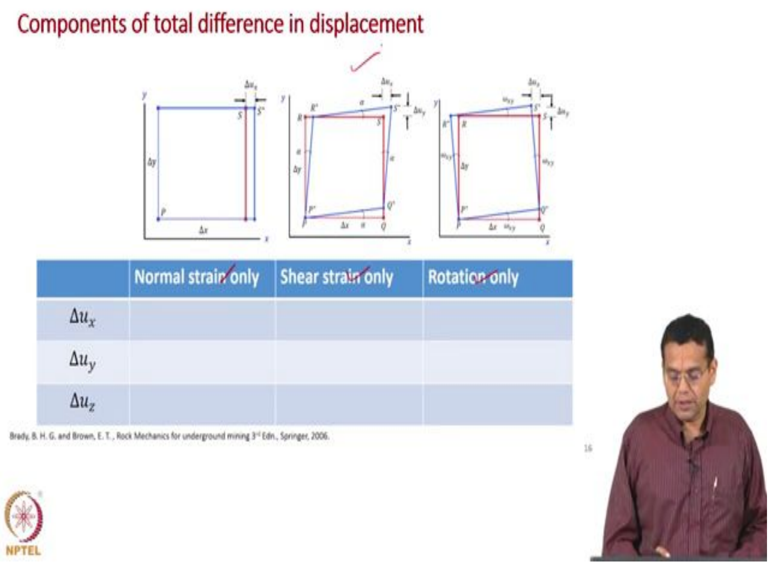

For deriving the first relation, I would suggest you to recall these slides which I discussed earlier, I just shown here for a purpose of recall. We discussed this, when we discussed the components of difference in displacement due to normal strain, shear strain and rotation. Our attention our relevance right now is this case where they are component, where we discussed components of displacement because of shear strain only.

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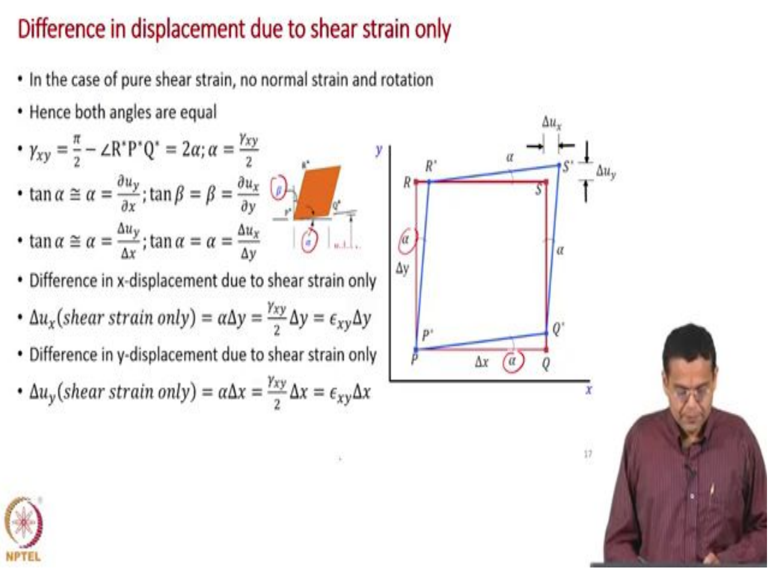

So, just show the slides just for recall we considered a plate subjected to only shear strain, and in the case of shear strain these angles are same, ok. Earlier they were just α and β, but when you subjected to pure shear strain they become equal and they are just α.

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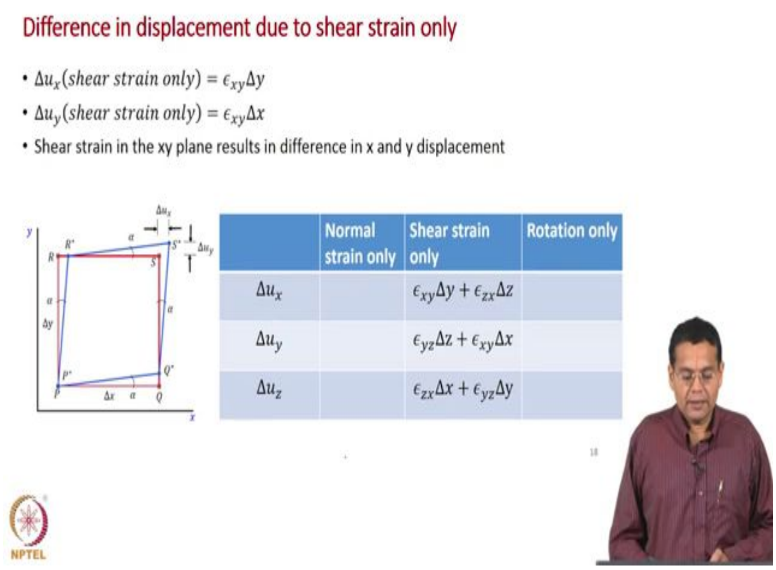

And when they are discussed what are the difference in displacement in x direction, y direction etcetera.

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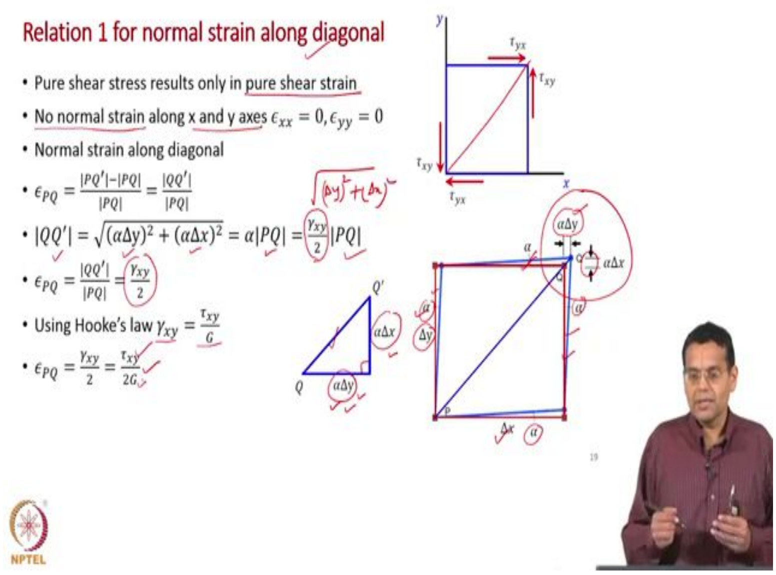

Now, as I told you we consider a two dimensional element subject to pure shear stress and pure shear stress results only in pure shear strain. That is the first statement, pure shear stress, when I say pure shear stress there are no normal stresses acting only shear stresses are acting results only in pure shear strain so, there is no normal strain. So, no normal strain along x and y axis we are considering only two-dimensional case.

Now, if you look at the title of this slide and this pure shear strain it may look contradicting. This title says normal strain along diagonal, first line says pure shear strain, when we say a pure shear strain it means that there are no normal strain not along any direction along the x and y direction. So, when I say pure shear strain I mean there is no normal strain along x and y axis not along any direction.

$$\varepsilon_{xx} = 0; \quad \varepsilon_{yy} = 0$$

There can be normal strain along other directions for example, there is normal strain along the diagonal that is the meaning of these statements ok. Pure shear strain means no normal stain along x direction and y direction can be along let us say the normal direction and there is normal strain, Now this diagram is similar to the diagram which you have seen earlier in the previous slide in the recall slide.

The red boundary shows the initial state and subjected to a pure shear stress and results in the final state given with the blue boundary and as we have discussed earlier, it is symmetric and the angles are same they are both α.

Now, the diagonal PQ becomes PQ'. So, which means there is normal strain along the diagonal, let us write an expression

$$\varepsilon_{PQ} = \frac{|PQ'|-|PQ|}{|PQ|} = \frac{|QQ'|}{|PQ|}$$

Usual definition of normal strain change in length by original length, what are the changes in length, length of a PQ' minus length of PQ divided by length of PQ.

Now how to find length of QQ'. So, that let us consider the triangle as we have discussed earlier also when α is very very small. So, it can be approximated to a triangle, when you consider that triangle and take $\tan(\alpha)$ approximately equal to α.

So, the whatever is magnified and shown, the increase in length QQ', that increased length is shown here and the vertical length is $\alpha\Delta x$ and the horizontal length is $\alpha\Delta y$. Now we can find out the length; the increase in length QQ' is

$$\left|QQ'\right| = \sqrt{(\alpha\Delta y)^2 + (\alpha\Delta x)^2} = \alpha\,|PQ| = \frac{\gamma_{xy}}{2}|PQ|$$

Now, what we want is, length of $\frac{QQ'}{PQ}$ that is what exactly we have derived here and bring to the left hand side

$$\frac{|QQ'|}{|PQ|} = \frac{\gamma_{xy}}{2}$$

So, physical significance of this statement is the normal strain along the diagonal is equal to the shear strain by 2, $\frac{\gamma_{xy}}{2}$.

Now, we will use Hooke's law, when I say Hooke's law there are 6 equations we are using one of the equations, what is that

$$\gamma_{xy} = \frac{\tau_{xy}}{G}$$

So,

$$\varepsilon_{PQ} = \frac{\gamma_{xy}}{2} = \frac{\tau_{xy}}{2G}$$

So, we have related normal strain along the diagonal to the applied shear stress τ_{xy} and the material property shear modulus G.

So, we have derived one relationship for normal strain along the diagonal. We will have to derive one alternate relationship and equate both of them, because let us say a plate subjected to shear stress, that normal strain should be same whatever way you look at it, we derive alternate expression and equate both of them.

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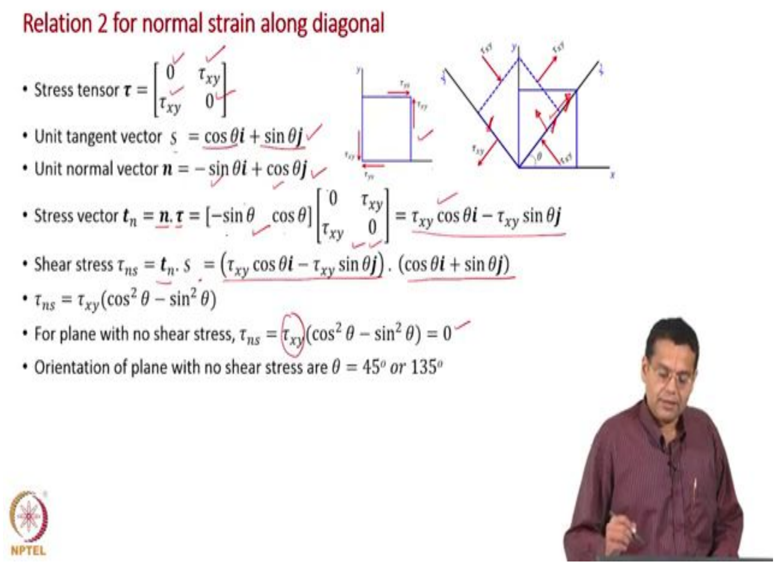

What is the second relationship? Now we have a recall our discussion on stress element and then stress tensor, we represent the condition in terms of a stress element subjected to only pure shear stress. So, in terms of a stress tensor the diagonal components are 0.

$$Stress\ tensor = \tau = [0\ \tau_{xy}\ \tau_{xy}\ 0]$$

Now I want to consider a plane which is at an angle θ with the horizontal and I consider a vector along this plane and that is the tangent vector to that particular plane and which I denote it as S based on our usual nomenclature and the vector is

$$S = cos\theta i + sin\theta j$$

So, we are considered a plane at an angle θ with the x axis and the, you have a plane and the unit vector tangential to the plane is given above. Now I consider a unit normal which in our nomenclature is n vector and we know that this vector should be perpendicular to the tangent vector. So, what is the expression for this vector is,

$$n = -sin\theta i + cos\theta j$$

If you simply take a dot product n with S you get 0. So, we know that normal vector should be perpendicular to the tangent vector. So, what is it we have done, we are represented the condition of pure shear stress in terms of a stress tensor and wrote down expression for a plane at an angle theta with the x axis wrote an expression for the tangent vector an expression for the normal vector.

Now we will find the expression for a stress vector. We have seen that it is the dot product of the unit normal vector with the stress tensor or in simple terms matrix multiplication of the normal vector and the stress tensor.

$$t_n = n.\tau = [-sin\theta \; cos\theta \;] \left[0 \; \tau_{xy} \; \tau_{xy} \; 0 \;\right] = \tau_{xy} cos\theta i - \tau_{xy} sin\theta j$$

A simple matrix multiplication will give us this vector. Now, once you have found out the stress vector we have already seen examples we can you know how to find out, what is the shear stress, what is the normal stress etcetera acting on the plane. So, let us find out what is the shear stress, what do you do, it is adjust the projection of this stress vector along the tangential direction. So, take a dot product of the stress vector with the unit tangent vector

$$\tau_{nS} = t_n.S = (\tau_{xy} cos\theta i - \tau_{xy} sin\theta j) .(cos\theta i + sin\theta j)$$

Now, do a simple dot product we get

$$\tau_{nS} = \tau_{xy} (\theta \;\; - \theta \;)$$

This is the expression for the shear stress. Now, I want to find out a plane where there is no shear stress, shear stress free plane you will understand why do you do that. To find out that plane what should I do, I should equate the shear stress to 0 and that is what I have done here

$$\tau_{xy} (\theta \;\; - \theta \;) = 0$$

Now τ_{xy} cannot be 0 because we are applying that it is a given constant value. So, what is the condition that depends on θ ; so,

$$\theta \;\; - \theta \;\; = 0; \qquad \theta = 45^0 or\ 135^0$$

So, $\theta = 45^0$ is diagonal, $\theta = 135^0$ is perpendicular to the diagonal, that is why we are interested in a plane on which there is no shear stress. Because remember where are we proceeding we are going to proceed we are proceeding towards getting another alternate expression for normal strain along diagonal that is why we said we will identify a plane along which there is no shear stress.

So, along this plane there is no shear stress, there is only a normal stress and then on this plane also which is 90^0 along that plane also there is no shear stress there is only normal stress.

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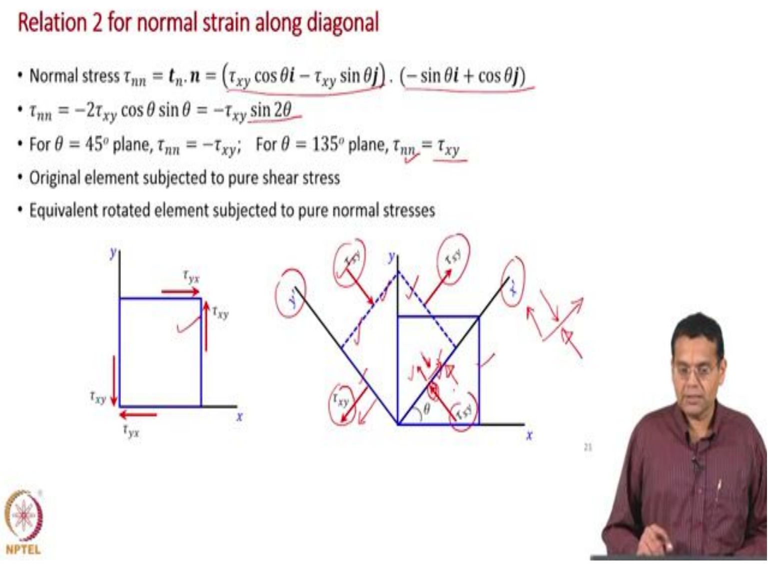

Now, let us find out what are the normal stress values along those planes; how do you find normal stress? Just take dot product of the stress vector with the unit normal vector

$$\tau_{nn} = t_n.n = (\tau_{xy}cos\theta i - \tau_{xy}sin\theta j).(-sin\theta i + cos\theta j)$$

And if you take the dot product you get

$$\tau_{nn} = -2\tau_{xy}cos\theta sin\theta = -\tau_{xy}sin2\theta$$

Now, let us evaluate further two planes.

$\theta = 45^0$ you get $\tau_{nn} = -\tau_{xy}$

$\theta = 135^0$ you get $\tau_{nn} = \tau_{xy}$

Now, this represents the rotated stress element, why rotated because one plane is along the diagonal other plane is 90 degrees left of the diagonal. Now we have considered this plane and now we are found out the normal stress to be $-\tau_{xy}$ which means it is acting along this direction and now because I am indicating the force on the other side opposite side I am indicating in this direction. Remember we discussed, if you have a surface the normal stress acting on opposite sides are equal in magnitude opposite in direction.

So, what we have done is that, we had a stress element subjected to pure shear stress when you rotate it 45 degrees to the left that is what we have done rotate 45 degrees to the left then it is equivalent to a stress element subjected to only normal stresses. There are no shear stresses that is what you see here when you look at this a dashed boundary which represents the rotated stress element there are only normal stresses. Those normal stresses are been expressed in terms of τ_{xy} values, but in this rotated configuration there are only normal stresses and the new axis are denoted as x' axis and y' axis.

So, along the x' direction it is a tensile and along the y' axis it is compressive that is what you see here, τ_{xy} in one direction is tensile and in other direction τ_{xy} is the normal stresses which are compressive. So, original element subjected to pure shear stress equivalent rotated element subjected to pure normal stresses that is what we have done now. Why are we doing all this our final objective is to get an expression for normal strain along the diagonal, what is the axis now x dash axis that is what we will do now.

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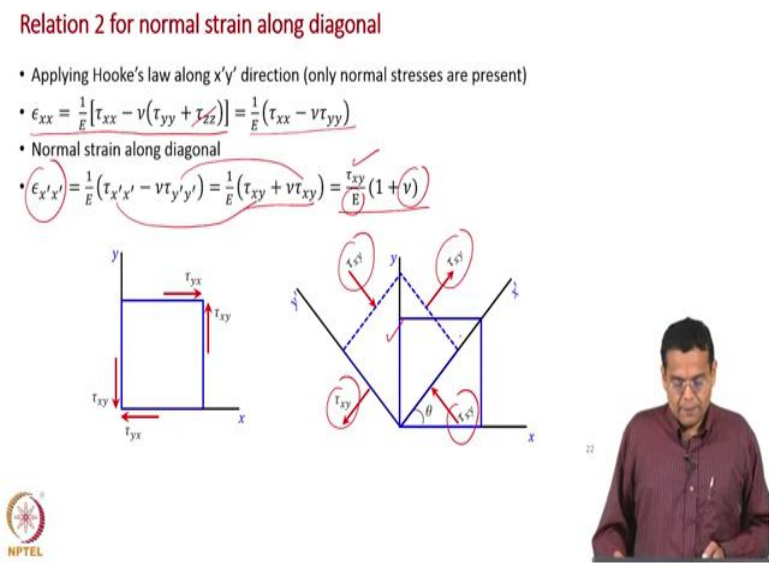

Now we will apply Hooke's law in this rotated plane x'y' plane, where only normal stresses are present. Remember we wrote 6 equations for Hooke's law I am writing here one of the expressions for normal strain with along x direction.

$$\varepsilon_{xx} = \frac{1}{E}\left[\tau_{xx} - \nu\left(\tau_{yy} + \tau_{zz}\right)\right] = \frac{1}{E}\left[\tau_{xx} - \nu\tau_{yy}\right]$$

We are considering 2 dimensional case so, there is no τ_{zz} . So, this is the simple expression for the present case. Now we have to apply this along the x'. So, normal strain along the diagonal now the x' axis and diagonal are coinciding along with each other. So,

$$\varepsilon_{x'x'} = \frac{1}{E}\left[\tau_{x'x'} - \nu\tau_{y'y'}\right] = \frac{1}{E}\left[\tau_{xy} + \nu\tau_{xy}\right] = \frac{\tau_{xy}}{E}(1+\nu)$$

So, we have got another expression for the normal strain along diagonal a second relationship, which once again remember relates to the applied pure shear stress and material properties ν and then Young's modulus. So, just to summarize this slide first what we did was in the previous slide, we had a stress elements subjected to pure shear stress we proved that that is equivalent to an element which is rotated by 45 degrees subjected to pure normal stress.

So, we considered that rotated element and applied the Hooke's law along x dash axis which coincides with the diagonal of the original element, and then applied the Hooke's law along that diagonal.

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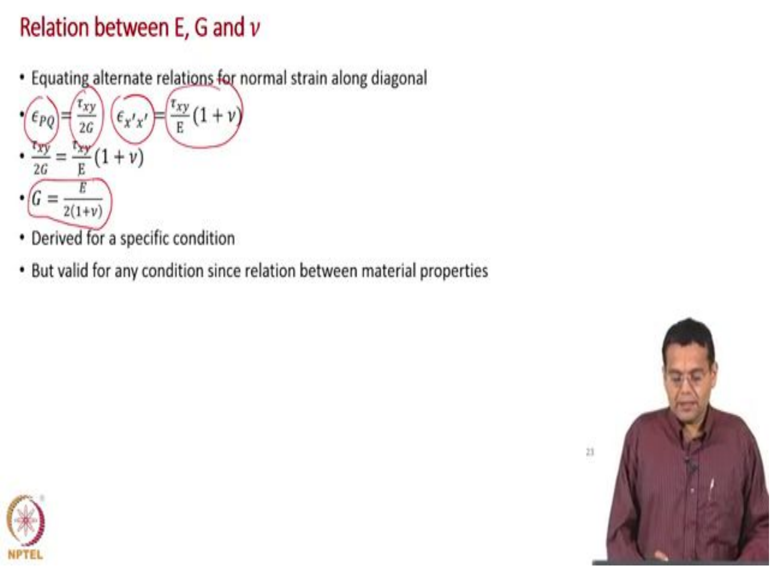

So, now becomes very simple equating alternate relations for normal strain along diagonal,. First we found out

$$\varepsilon_{PQ} = \frac{\tau_{xy}}{2G}$$

Second time we got

$$\varepsilon_{x'x'} = \frac{\tau_{xy}}{E}(1+\nu)$$

Though two different nomenclatures for the two relation significance is same normal strain along in the diagonal, first time $\frac{\tau_{xy}}{2G}$, second time $\frac{\tau_{xy}}{E}(1+\nu)$ both should be same.

$$\frac{\tau_{xy}}{E}(1+\nu) = \frac{\tau_{xy}}{2G}$$

So, when you equate these two you get the relationship as

$$G = \frac{E}{2(1+\nu)}$$

That is what we were proceeding towards, we wanted a relationship between E, G and ν, and they are not independent, the middle properties are dependent. Just like to mention we have derived this for a specific condition take a plate two dimensional case subject to pure shear stress, but remember these are relationship between material properties. So, they should be valid under any general condition, we cannot take a very general condition rather that becomes difficult. So, we took a very simplified geometry and derived the relationship, but they are valid for any condition since their relationship between material properties.

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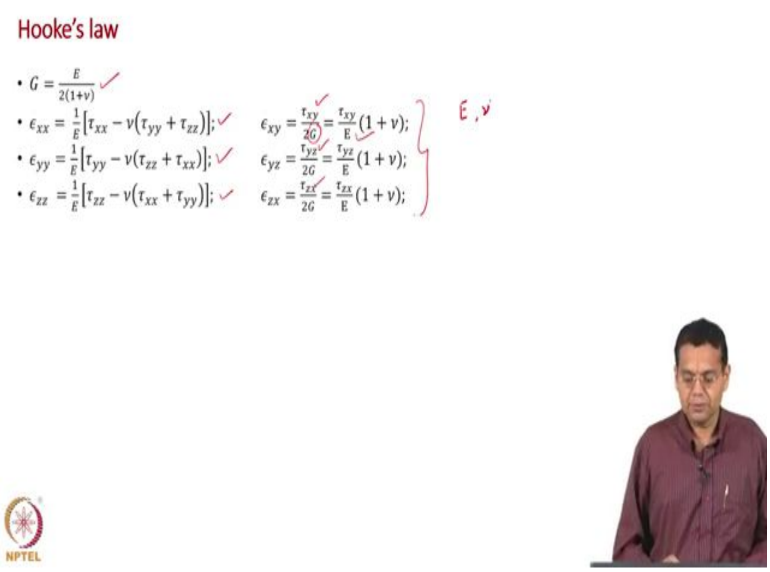

So, now we can write the Hooke's law in terms of two material properties. So, the first line represents the relationship between G, E and ν, which I derived in the last slide and the six equations which I have written earlier are written here again, but now instead of G, I can write in terms of E and $1+\nu$. So, now, these set of equations have only two material properties which are E and ν, it is for us to replace any one material property in terms of the other two, here I have replaced G in terms of E and ν. So, these set of equations I have only two material properties namely E and then ν.