Continuum Mechanics And Transport Phenomena
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Lecture – 72
Hooke's law – Strain-stress Relation

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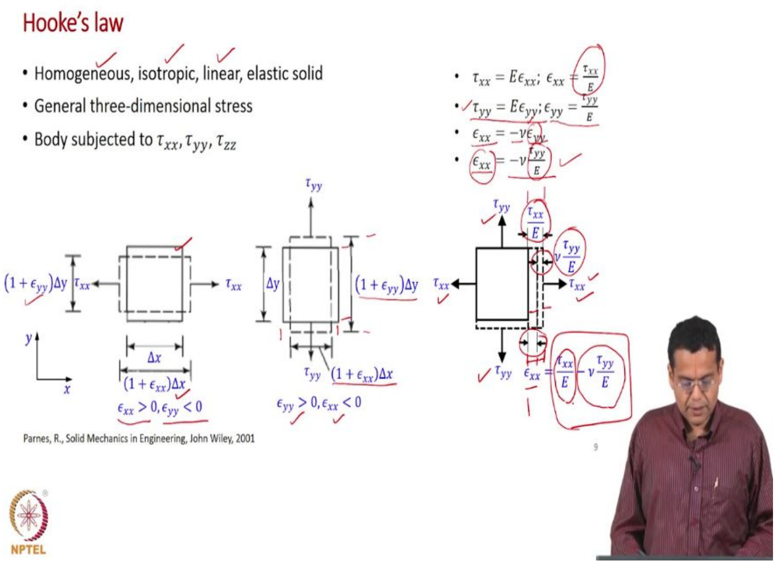

Now, we will start deriving the Hooke's law. Look at the first bullet, it lists all the assumptions we are discussed so far, it summarizes all the assumptions we are discussed so far. So, when I say Hooke's law, it is valid only under these assumptions that is why we discussed. When we talk of a relationship between stress and strain, then assumptions are to be discussed. Hooke's laws is such a relationship and so, this limits the scope of Hooke's law.

So, the assumptions are:

- Homogeneous; properties are the same at any point now we know what properties are. We are seen looked at three material properties Young's modulus, shear modulus, Poisson ratio
- Isotropic; at the point material properties are same in all directions
- Linear elastic solid; elastic a different relationship between stress and strain. When we say stress and strain applies both for normal stress, shear stress and then normal strain

shear strain. And linear tells you the relationship between them between the stress and strain is linear.

And what we do is we are consider a body subjected to a general three dimensional stress. What do I mean by that? I have a body subjected to normal stress along x direction and y direction and z direction. Now to for easy discussion, we will consider a two dimensional case and that is what is shown here. We have a plate in the undeformed configuration before the force is being applied before the stress is being applied.

And the length of the sides are Δx and Δy. And the first figure what is shown is it is subjected to normal stress along x direction τ_{xx}, and it is shown to be ten tensile here. Now the changed length or the changed length of the sides are also shown $(1+\tau_{xx})\Delta x$ and the length of along the y direction has changed to $(1+\tau_{yy})\Delta y$.

For this condition there is elongation along the x axis so, $\tau_{xx} > 0$; there is contraction along the y axis so, $\tau_{yy} < 0$. Now what is shown in the second diagram is the same plate subjected to normal stress along y direction. Please note the point we are subjecting the plate to both normal stress along x direction and normal stress along y direction. And so, the second figure shows the plate subjected to normal stress along y direction. Now there is elongation along the y direction so, the original length along y direction is Δy. The new length increased length is $(1+\tau_{yy})\Delta y$.

Now, there is contraction along x axis which is the perpendicular direction and the original length is Δx and the new length is $(1+\tau_{xx})\Delta x$. So, for this condition $\tau_{yy} > 0$ elongation along y axis and contraction along x axis so, $\tau_{xx} < 0$. Now what is that the third figure represents? We are applying both the normal stresses together that is what is shown here; τ_{xx} and then τ_{yy}.

We are applying both the tensile stresses together the first figure shows the plate subjected to normal stress along x axis, second figure shows normal stress along y direction, third represents the plate subjected to normal stress along x direction and y direction.

Now, let us say there is no normal stress along y direction momentarily; let us take there is no normal stress along y direction. Then because of normal stress along x direction what would be the normal strain along x direction? This given by $\frac{\tau_{xx}}{E}$ this would have been the let us say increase in length along x direction. If that two dimension element has been subjected to normal stress along x direction only. But because there is normal stress along y direction there is some decrease in length along the x direction, there is some nor negative or contraction along the x direction what is that value, how do you find out that.

For the normal stress along y direction, the relationship between connecting τ_{yy} and ε_{yy} is given by the equation, $\tau_{yy} = E\varepsilon_{yy}$ that is like we are seen earlier for x direction. So, this relationship relates normal stress along y direction to normal strain along y direction by the Young's modulus. Remember once again we are taking the same Young's modulus independent of direction because it is a isotropic. I write that relationship for normal strain along y direction $\varepsilon_{yy} = \frac{\tau_{yy}}{E}$.

Now my direction of stress is along y direction so, x becomes the perpendicular direction. So how do we write normal strain along the perpendicular direction is $-\nu$ times is normal strain along the direction of stress. So, when we substitute for ε_{yy} in terms of $\frac{\tau_{yy}}{E}$, we get the expression for normal strain along x direction due to normal stress along y direction is that. So, the this is to be a negative there is a decrease in length which is $-\nu\frac{\tau_{yy}}{E}$.

So, the contraction is $-\nu\frac{\tau_{yy}}{E}$. If there were no normal stress along y direction, this would have been the increase in length. But because there is normal stress along y direction, there is some contraction. So, the net increase in length is only so much that is what is shown here.

The total is shown here $\frac{\tau_{yy}}{E}$ and the contraction because of τ_{yy} we shown here the net is shown here. So, in terms of equation there is shown geometrically in terms of equation, the let us call the effective normal strain along x direction is equal to $\frac{\tau_{xx}}{E}$. If there were no normal stress along y direction only this would be the value, but because they are normal stress along y direction it cause the contraction along the x direction which is $-\nu\frac{\tau_{yy}}{E}$. And so, this two terms put together is the net normal strain along the x direction due to τ_{xx} and τ_{yy}.

The first term represents contribution from τ_{xx} which is let us say along which is elongation. Second term tells a contribution because of τ_{yy} which is contraction and for the sum together you get the net normal strain. So, what does it we have done? We have considered a body subjected to all the three normal stresses for the present case, we are considered only two dimensional case subjected to only τ_{xx} and τ_{yy} and seen what is the normal strain along x direction.

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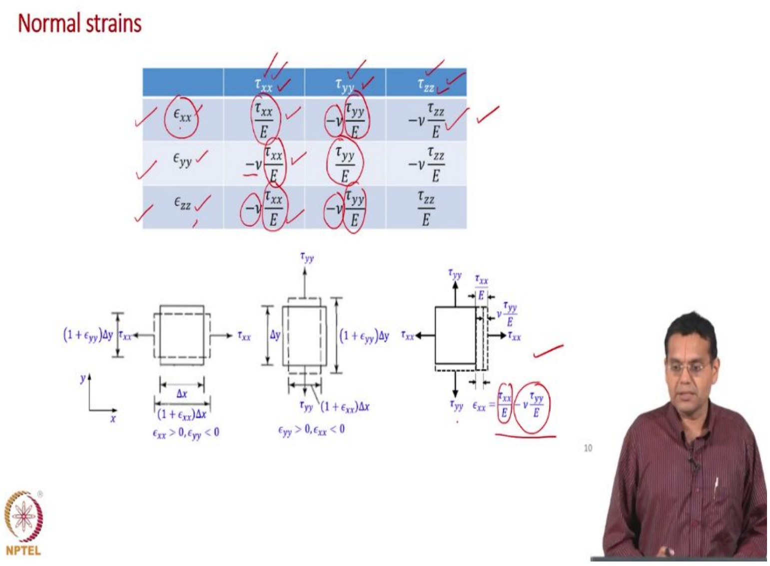

Now, based on this let us come discuss this table. What this table shows us that the column headings are the normal stresses along x direction, y direction, z direction. What are the row headings? The normal strain along x direction, y direction and z direction.

	τ_{xx}	τ_{yy}	τ_{zz}
ε_{xx}	$\frac{\tau_{xx}}{E}$	$-\nu\frac{\tau_{yy}}{E}$	$-\nu\frac{\tau_{zz}}{E}$
ε_{yy}	$-\nu\frac{\tau_{xx}}{E}$	$\frac{\tau_{yy}}{E}$	$-\nu\frac{\tau_{zz}}{E}$
ε_{zz}	$-\nu\frac{\tau_{xx}}{E}$	$-\nu\frac{\tau_{yy}}{E}$	$\frac{\tau_{zz}}{E}$

Let us take the first column. What is the first column? The direction of stresses along the x axis, direction of strain along the same x axis and they are related by simply $\frac{\tau_{xx}}{E}$. Now for when the stress is along x direction y and z are the perpendicular directions and they are related by $-\nu\frac{\tau_{xx}}{E}$. Similarly the strain along the z direction which another perpendicular direction is $-\nu\frac{\tau_{xx}}{E}$.

So, what is the significance of the entries in the first column? You applied a normal stress along x direction and the three rows represent normal strain along x direction, normal strain along y direction and normal strain along z direction. This is what we are seen in the slide previous to the earlier slide. Similarly if we apply normal stress along y direction, the normal strain along y direction is $\frac{\tau_{yy}}{E}$ because it is the same direction as the stress. Now the perpendicular directions are x and then z. How do we represent $-\nu\frac{\tau_{yy}}{E}$. Similarly for the z direction $-\nu\frac{\tau_{yy}}{E}$. Similarly we can write for the case where you apply normal stress along z direction.

Now let us analyze particular row what we have seen is column wise, let us analyze one particular row, let us take the first row. Now what is the significance of the first row? It says normal strain as the row heading. It tells the contribution to normal strain because of normal stress in x direction, normal stress in y direction, normal stress in z direction and this is what we have seen in the previous slide and that is what is shown here.

We have seen the first two terms here. Look at the first row in the first row, the first column entries $\frac{\tau_{xx}}{E}$. Second column entries $-\nu\frac{\tau_{yy}}{E}$ and because it is three dimensional case, you have one more entry $-\nu\frac{\tau_{zz}}{E}$. The significance of the different terms in the first row are contribution to normal strain along x direction because of normal stress in x direction, normal stress in y direction, normal stress in z direction is it. So, now if you sum up all of them, you get the normal strain in a x direction because of normal stresses in x, y and z direction that is the significance.

So, what does this table about? We can discuss both column wise and row wise. The entries in the column are the effect of stress in one direction on strain in three directions. If you consider row wise, it is effect of stress in three directions on strain in one direction ok. So, similarly we can discuss for ε_{yy} and ε_{zz}.

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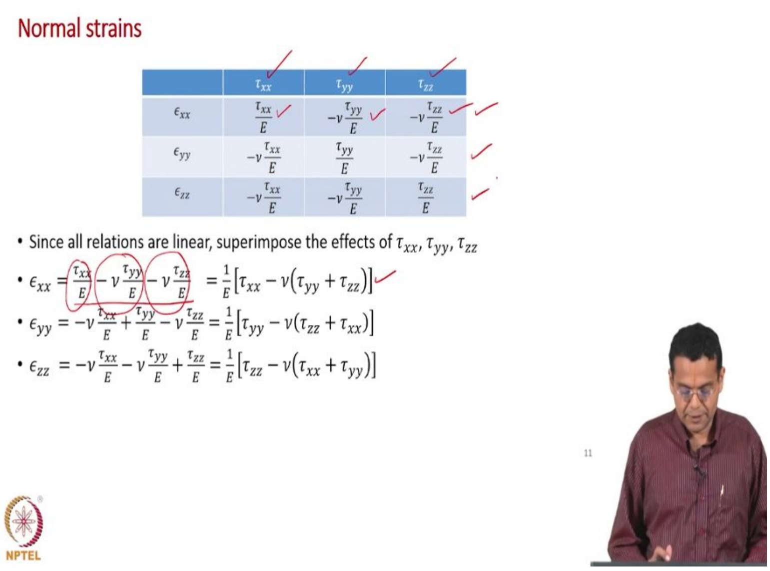

Because all the relationships are linear we just superimpose the effects of the three normal stresses. What we have written first column is because of normal stress in x direction second because of normal stress in y direction. Similarly, normal stress in z direction and each individually tells you what is the normal strain in the x direction. Now we are just going to add when is it possible because they are linear relationship and that is why we are able to just add them or more formally, we can say superimposed. So, that is why since all relations are linear, superimposed effects of τ_{xx} , τ_{yy} , τ_{zz} that is the meaning of this addition. So, the ε_{xx} represents the normal strain

$$\varepsilon_{xx} = \frac{\tau_{xx}}{E} - \nu\frac{\tau_{yy}}{E} - \nu\frac{\tau_{zz}}{E} = \frac{1}{E}\left[\tau_{xx} - \nu\left(\tau_{yy} + \tau_{zz}\right)\right]$$

This addition is possible or super position is possible because the relationships are linear. When you look at that expression what should come to your mind the way in or the way in which we should we interpret that is the first term represents the effect on normal strain due to normal stress in the same direction. Other two the effect on normal strain because of normal stress in the two other perpendicular directions that is what we should understand. So, the effect of normal stress in all three directions are brought into this equation or summed up in this equation.

Similarly you can write for other directions as well which means I am summing up the entries in the second row and summing up entries in the third row as well. So, second row tells the normal strain in the y direction because of once again three normal stresses

$$\varepsilon_{yy} = -\nu \frac{\tau_{xx}}{E} + \frac{\tau_{yy}}{E} - \nu \frac{\tau_{zz}}{E} = \frac{1}{E}\left[\tau_{yy} - \nu\left(\tau_{xx} + \tau_{zz}\right)\right]$$

and third row tells normal strain in z direction because of normal stress in three directions.

$$\varepsilon_{zz} = -\nu \frac{\tau_{xx}}{E} - \nu \frac{\tau_{yy}}{E} + \frac{\tau_{zz}}{E} = \frac{1}{E}\left[\tau_{zz} - \nu\left(\tau_{xx} + \tau_{yy}\right)\right]$$

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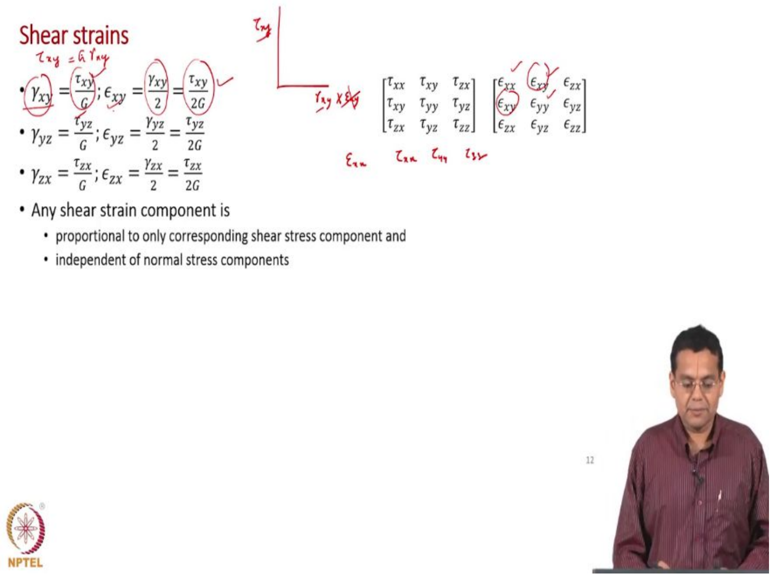

So, for we have discussed about normal strain. So, now, let us discuss about shear strain. We introduced the shear modulus based on the linear relationship between τ_{xy} and γ_{xy}. We said τ_{xy} versus γ_{xy} is linear and the proportionality constant is shear modulus.

Same expression is written, but for shear strain we have been the previous slide we wrote expressions for the normal strain. Now we are going to write expressions for the shear strain. So, the shear strain in the x y plane γ_{xy} is

$$\gamma_{xy} = \frac{\tau_{xy}}{G}$$

Now remember we discussed about the strain tensor where the components were ε_{xx}, ε_{yy} and then ε_{xy} was the component of the strain tensor.

So, we can write an expression for ε_{xy} which is the component of strain tensor. We have seen that

$$\varepsilon_{xy} = \frac{\gamma_{xy}}{2} = \frac{\tau_{xy}}{2G}$$

Remember the graph is between τ_{xy} and γ_{xy} not ε_{xy}, the experimentally measure is γ_{xy} not ε_{xy}. So, the G connects τ_{xy} and γ_{xy}. Why are we interested in ε_{xy} because that is the component which appears in the strain tensor and ε_{xy} is related to γ_{xy} as $\frac{\gamma_{xy}}{2}$.

So, we can relate γ_{xy} and τ_{xy} also which is $\frac{\tau_{xy}}{2G}$. Similarly we can write expressions for γ_{yz} as,

$$\gamma_{yz} = \frac{\tau_{yz}}{G}$$

You can write expression for ε_{xy} also as,

$$\varepsilon_{yz} = \frac{\gamma_{yz}}{2} = \frac{\tau_{yz}}{2G}$$

Similarly,

$$\gamma_{zx} = \frac{\tau_{zx}}{G}$$

$$\varepsilon_{zx} = \frac{\gamma_{zx}}{2} = \frac{\tau_{zx}}{2G}$$

Now, we should note one difference between the way in which you wrote normal strains and then shear strain. Now when we an expression for ε_{xx} on the right hand side, we took into effect of τ_{xx}, τ_{yy} and τ_{zz}. So, normal stresses acting on x along x direction, y direction, z direction had effect on normal strain along x direction.

But now look at the expression for shear strain γ_{xy} is returned to be depending only on τ_{xy}. So, one quick rough way of understanding is that you have a plate, you apply shearing stress the change in angle depends on the this shear stress alone. This shear strain along x y plane is not going depend on stress along y z plane and z x plane. So, that way these relationships are I would say simpler compared to the relationships for normal strain, they just depend on the shear stress in that plane only. So, any shear strain component is proportional to only corresponding shear stress component and independent of normal stress components also, we have not even considered effect of normal stress also.

So, any shear strain component let us say γ_{xy} is proportional to only corresponding shear stress competencies which is τ_{xy} and independent of normal stress components. We have not considered dependency have γ_{xy} on let us say τ_{xx}, τ_{yy}, τ_{zz}. That way these relationships are like the standalone relationship straight away, you can get shear strain in terms of shear stress and later on will express shear stress in terms of shear strain also.

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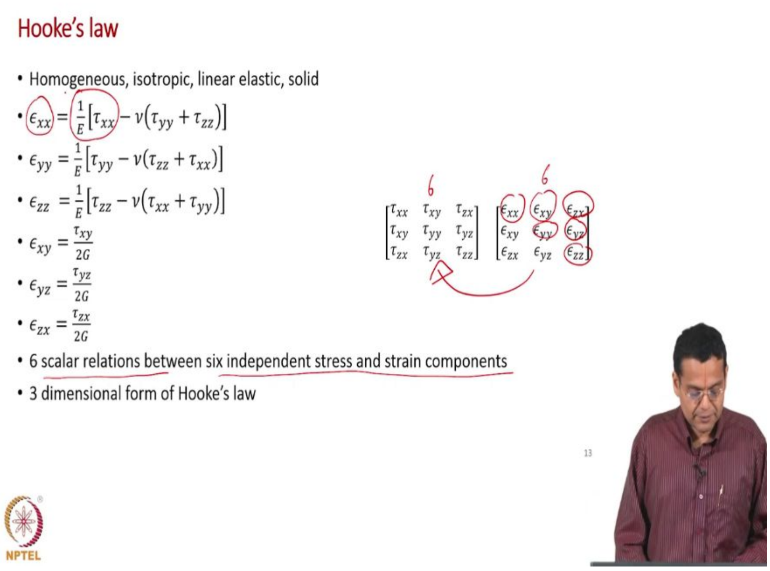

Those are the stress tensor and the strain tensor. So, those are the assumptions which have discussed homogeneous, isotropic, linear, elastic, solid. We can now summarize all the relationships which have discussed so far for normal strain and shear strain. So, we have seen as the expression for normal strain along x axis, analogously we can write an expression for normal strain along y axis; similarly normal strain along z axis.

$$\varepsilon_{xx} = \frac{\tau_{xx}}{E} - \nu\frac{\tau_{yy}}{E} - \nu\frac{\tau_{zz}}{E} = \frac{1}{E}\left[\tau_{xx} - \nu\left(\tau_{yy} + \tau_{zz}\right)\right]$$

$$\varepsilon_{yy} = -\nu\frac{\tau_{xx}}{E} + \frac{\tau_{yy}}{E} - \nu\frac{\tau_{zz}}{E} = \frac{1}{E}\left[\tau_{yy} - \nu\left(\tau_{xx} + \tau_{zz}\right)\right]$$

$$\varepsilon_{zz} = -\nu\frac{\tau_{xx}}{E} - \nu\frac{\tau_{yy}}{E} + \frac{\tau_{zz}}{E} = \frac{1}{E}\left[\tau_{zz} - \nu\left(\tau_{xx} + \tau_{yy}\right)\right]$$

We are also written expression for the off diagonal elements namely ε_{xy}, ε_{yz} and then ε_{zx}.

$$\varepsilon_{xy} = \frac{\gamma_{xy}}{2} = \frac{\tau_{xy}}{2G}$$

$$\varepsilon_{yz} = \frac{\gamma_{yz}}{2} = \frac{\tau_{yz}}{2G}$$

$$\varepsilon_{zx} = \frac{\gamma_{zx}}{2} = \frac{\tau_{zx}}{2G}$$

So, how did we write? We wrote them in terms of the components of the stress tensor. So, 6 scalar relations between 6 independent stress and strain components. Remember we discussed the stress tensor has 9 components, but only 6 are independent. Similarly for the strain tensor there are 9 components, but only 6 are independent, $\varepsilon_{xy} = \varepsilon_{yx}$ we are not even shown ε_{yx} here because we have taken ε_{yx} to be equal to ε_{xy}.

So, 6 independent stress tensor components 6 independent strain tensor components and there are 6 scalar relations between the 6 independent stress and strain components and that is the three dimensional form of Hooke's law.

Where is a Hooke's law known to you. That is $\varepsilon_{xx} = \frac{\tau_{xx}}{E}$ that is where what we already know very well known to is one simplified form of what we are discussed so far ok. We are rarely discussed more general form of Hooke's law a three dimensional version of Hooke's law even what we are discussed is under this assumption homogeneous, isotropic, linear elastic ok. What we already know very well known to us most of us are the Hooke's law given by $\varepsilon_{xx} = \frac{\tau_{xx}}{E}$.

What we are seen is a more generic version these are applicable for a three dimensional case and remember the Hooke's law is written for ε_{xy} and not for γ_{xy} . Why is it? Because ε_{xy} is the component of the strain tensor not γ_{xy} .

The reason is I am we are writing expressions for components of strain tensor in terms of components of stress tensor that is why, but the original experimentation is between τ_{xy} and γ_{xy} because there is some experiment. We measured straightaway change in angle that is why the x axis there is γ_{xy} , but we need a relationship for ε_{xy} that has to be kept in mind.

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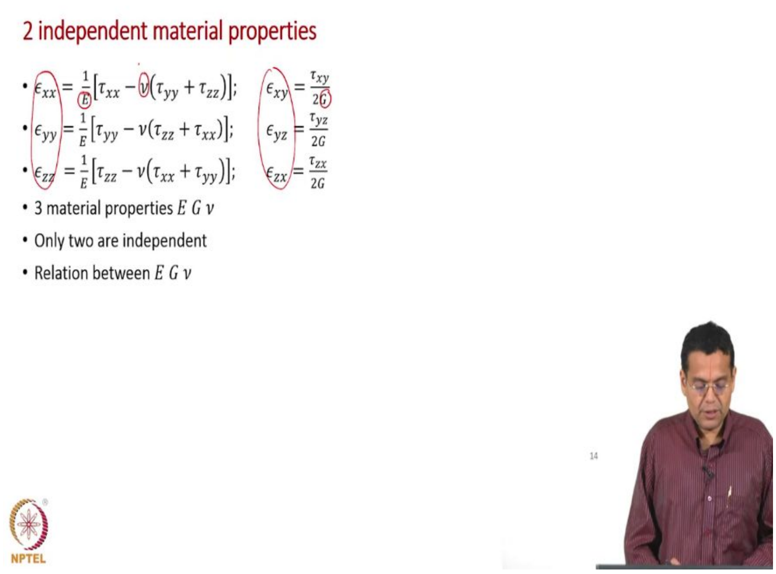

Now, if you look at the equations and even if you look at the way in which we are discussed, there are three material properties. What are they? The Young's modulus E and then the shear modulus G, the Poisson ratio ν; there are three material properties ok, but only two of them are independent. One depends on the other, one of them depends on the other, you cannot there cannot be three independent material properties.

We can show that there is a relationship between E, ν and G resulting in only two independent material properties and that is our next objective what is the relationship between the three material properties. So, we will derive the relationship between three material properties because the Hooke's law under these assumptions can have only two independent material properties. So, we have proceeding towards deriving a relationship between E, G and ν.