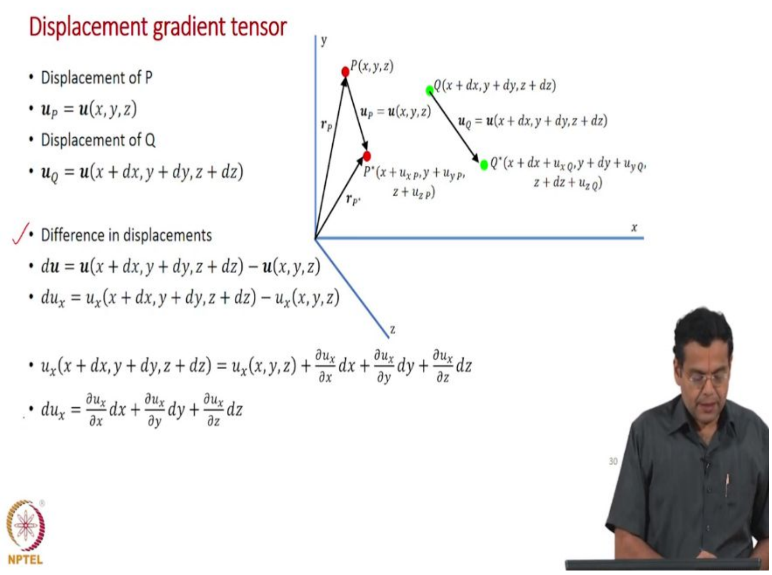
Continuum Mechanics And Transport Phenomena
Prof. T. Renganathan
Department of Chemical Engineering
Indian Institute of Technology, Madras

Lecture - 67
Velocity Gradient Tensor

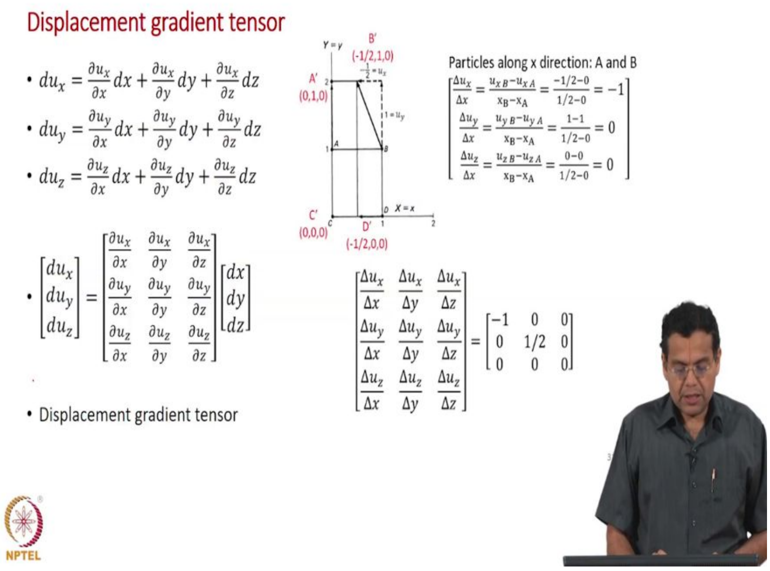
(Refer Slide Time: 00:15)



In case of solid mechanics, we moved on further to look at the components of displacement or look at the different components of total displacement that we did by getting an expression for displacement gradient tensor in two different ways. The first method was more mathematical, the second one the expression was derived by considering the different components of displacement. So, the next few slides are recall slides which we have discussed under solid mechanics.

So, this slide shows the derivation for displacement gradient tensor using the first method which is the mathematical method. So, we consider the two particles, looked at the difference in displacement and expressing the total derivative in terms of partial derivatives.

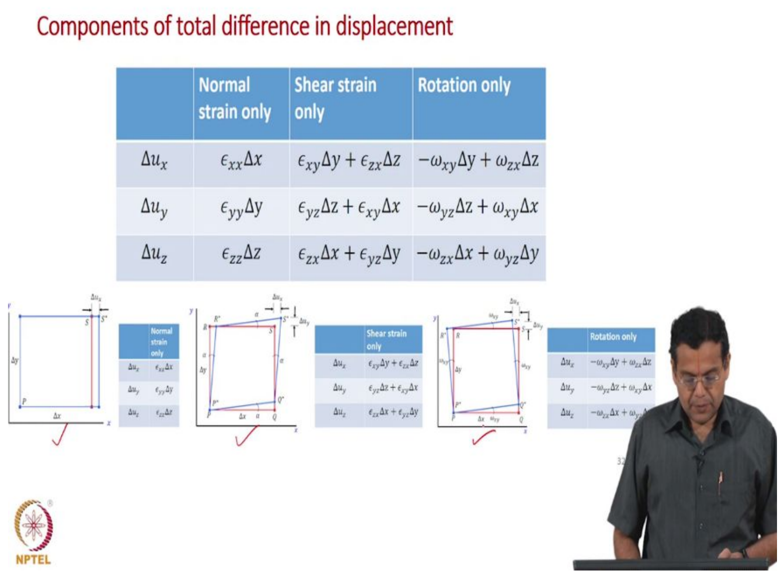
(Refer Slide Time: 01:25)



We could express the relationship between du_x, du_y, du_z in terms of dx, dy, dz and the tensor relating these two is the displacement gradient tensor. So, this is the first method in which we arrived at the displacement gradient tensor, the more mathematical way.

$$\left[du_x \; du_y \; du_z \right] = \left[\frac{\partial u_x}{\partial x} \; \frac{\partial u_x}{\partial y} \; \frac{\partial u_x}{\partial z} \; \frac{\partial u_y}{\partial x} \; \frac{\partial u_y}{\partial y} \; \frac{\partial u_y}{\partial z} \; \frac{\partial u_z}{\partial x} \; \frac{\partial u_z}{\partial y} \; \frac{\partial u_z}{\partial z} \right] [dx \; dy \; dz \,]$$

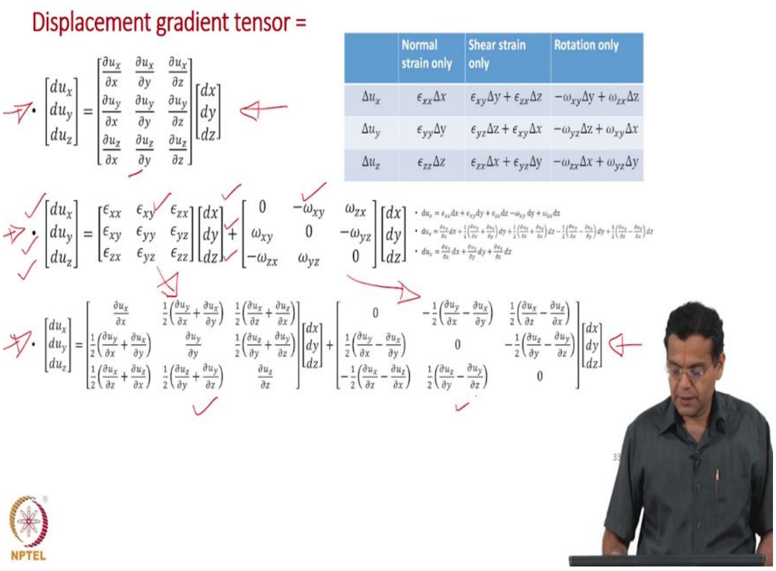
(Refer Slide Time: 01:59)



Second what we did was we looked at the components of total displacement. We consider normal strain only, shear strain only, rotation only.

	Normal strain only	Shear strain only	Rotation only
Δu_x	$\varepsilon_{xx}\Delta x$	$\varepsilon_{xy}\Delta y + \varepsilon_{zx}\Delta z$	$-\omega_{xy}\Delta y + \omega_{zx}\Delta z$
Δu_y	$\varepsilon_{yy}\Delta y$	$\varepsilon_{yz}\Delta z + \varepsilon_{xy}\Delta x$	$-\omega_{yz}\Delta z + \omega_{xy}\Delta x$
Δu_z	$\varepsilon_{zz}\Delta z$	$\varepsilon_{zx}\Delta x + \varepsilon_{yz}\Delta y$	$-\omega_{zx}\Delta x + \omega_{yz}\Delta y$

(Refer Slide Time: 02:21)



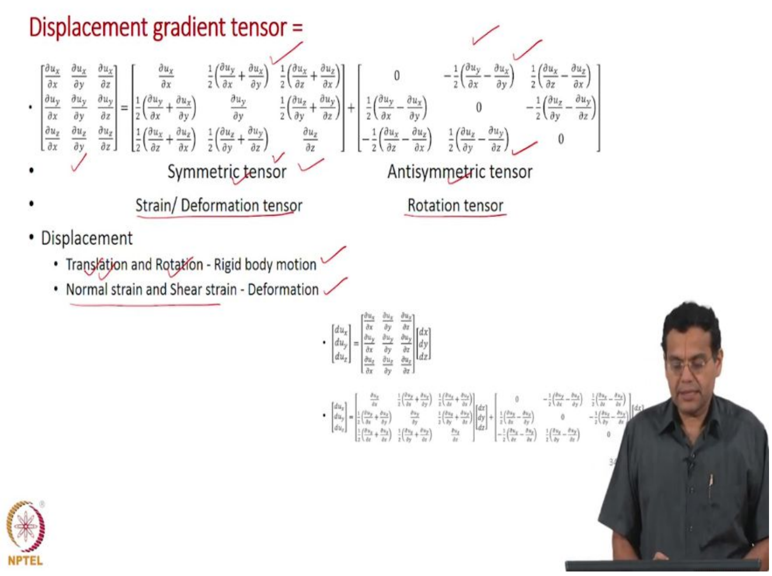
And, then once again got an expression between du_x, du_y, du_z and dx, dy, dz and the second way was more physical because we consider the normal strain, shear strain and then rotation.

$$[du_x\ du_y\ du_z\ ] = [\varepsilon_{xx}\ \varepsilon_{xy}\ \varepsilon_{xz}\ \varepsilon_{xy}\ \varepsilon_{yy}\ \varepsilon_{yz}\ \varepsilon_{zx}\ \varepsilon_{yz}\ \varepsilon_{zz}\ ]\,[dx\ dy\ dz\ ] + [0 - \omega_{xy}\ \omega_{xz}\ \omega_{xy}\ 0 - \omega_{yz} - \omega_{zx}\ \omega_{yz}\ 0\ ]\,[dx\ dy$$

$$[du_x\ du_y\ du_z\ ] = \left[\frac{\partial u_x}{\partial x}\ \frac{1}{2}\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right)\ \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right)\ \frac{1}{2}\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right)\ \frac{\partial u_y}{\partial y}\ \frac{1}{2}\left(\frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z}\right)\ \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right)\ \frac{1}{2}\left(\frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z}\right)\ \frac{\partial u_z}{\partial z}\ \right]\,[dx$$

So, this equation shows the relationship between du_x, du_y, du_z and dx, dy, dz obtain by the using the mathematical method, the first method. And, these two are the equations obtain using the second method where we considered the components of total displacement one at a time namely normal strain, shear strain and then rotation and here the equation is in terms of the component of the two tensors. Now, from these two equations we could write that the displacement gradient tensor is a sum of these two tensors.

(Refer Slide Time: 03:45)



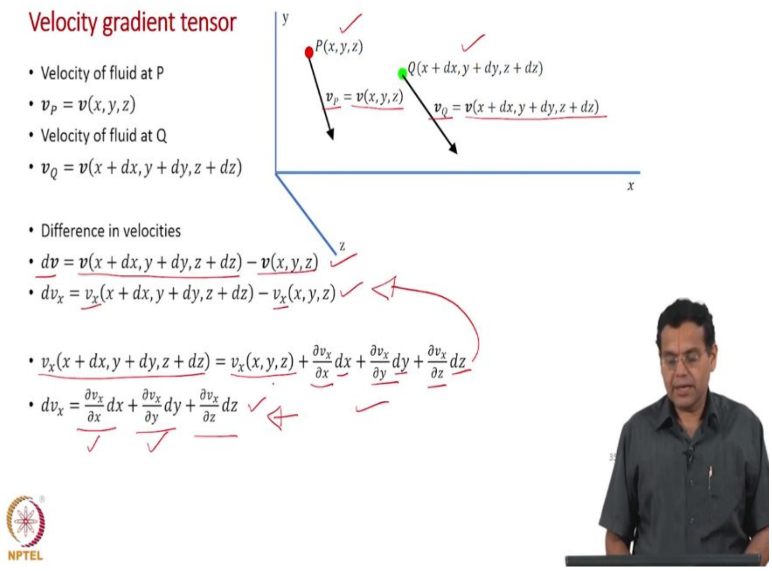
So, the displacement gradient tensor is a sum of a two tensors.

$$\left[\frac{\partial u_x}{\partial x}\ \frac{\partial u_x}{\partial y}\ \frac{\partial u_x}{\partial z}\ \frac{\partial u_y}{\partial x}\ \frac{\partial u_y}{\partial y}\ \frac{\partial u_y}{\partial z}\ \frac{\partial u_z}{\partial x}\ \frac{\partial u_z}{\partial y}\ \frac{\partial u_z}{\partial z} \right] = \left[\frac{\partial u_x}{\partial x}\ \frac{1}{2}\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right)\ \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right)\ \frac{1}{2}\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right)\ \frac{\partial u_y}{\partial y}\ \frac{1}{2}\left(\frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z}\right)\ \frac{1}{2}\left(\right.$$

The first one was symmetric, second one was antisymmetric. The first one was called as the strain or deformation tensor, second one we called as the rotation tensor. So, what we have done is split the displacement gradient tensor into two tensors; a strain tensor and a rotation tensor. Objective was to separate the rigid body motion from the deformation. We looked at difference in displacement so that translation is taken care. To remove the rotation part, we considered each of the component separately namely normal strain, shear strain, rotation separately, include the normal strain and shear strain into the strain tensor. And so, the other tensor just represents the rotation component.

So, by doing so we have separated the rotation component from the deformation component that is what is shown here displacement consists of rigid body motion and deformation. Rigid body motion consists of translation and rotation, translation has been taken care by taking difference in displacement and now rotation has been separated. And so, we are left with only normal strain and shear strain in the deformation or strain tensor.

(Refer Slide Time: 05:33)



Now, analogously we will express the velocity gradient tensor in terms of two tensors. Here again it is done by deriving an expression for the velocity gradient tensor in two different ways; one the mathematical way the other by splitting into components; the normal strain rate, shear strain rate and the rotation rate. The case of fluids, we will just rederive the first method the mathematical way alone, second one we will just analogously express it.

So, now what is it we are going to do now? We are going to derive the expression for velocity gradient tensor using the first method which is the mathematical way. Now what we do for that? We consider two points P and Q, the coordinates are of P are x, y, z coordinates of Q are $x + dx$, $y + dy$, $z + dz$. So, they are two neighboring points in the fluid domain, velocity of fluid at point P let us denote by v_P also equal to the velocity field at x, y, z.

$$v_P = v(x,\ y,\ z)$$

Now, if velocity of fluid at point Q will denote it as v_Q and which is equal to the velocity field at $x + dx$, $y + dy$, $z + dz$.

$$v_Q = v(x + dx,\ y + dy,\ z + dz)$$

Now, let us look at the difference in velocities. Difference between the velocities a two neighboring points which we denote as

$$dv = v\,(x + dx,\ y + dy,\ z + dz) - v(x,\ y,\ z)$$

So, to proceed further we will write in terms of the components; let us write it for the x direction

$$dv_x = v_x(x + dx,\ y + dy,\ z + dz) - v_x(x,\ y,\ z)$$

So, this tells the vectorial difference in velocities between two neighboring points. Now, let us take the $v_x(x + dx,\ y + dy,\ z + dz)$ and expand that in Taylor series, some multivariable Taylor series because the function of three variables x, y, z.

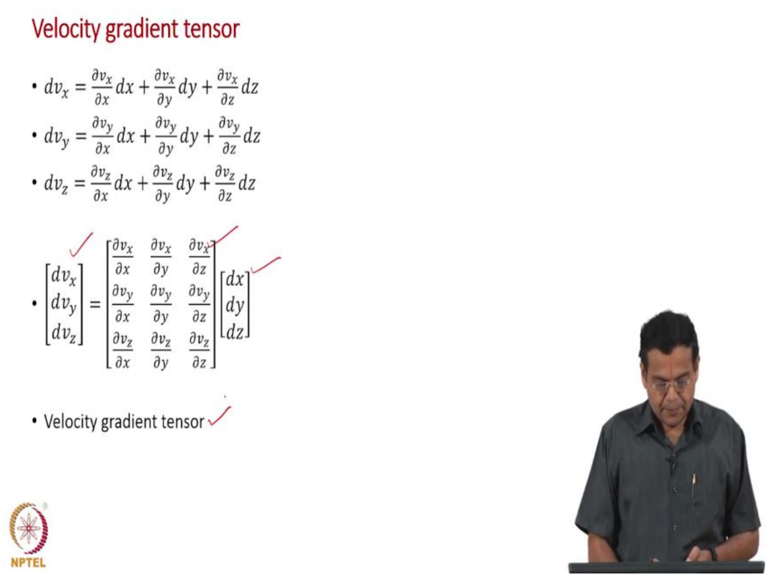
$$v_x(x + dx,\ y + dy,\ z + dz) = v_x(x,\ y,\ z) + \frac{\partial v_x}{\partial x}dx + \frac{\partial v_x}{\partial y}dy + \frac{\partial v_x}{\partial z}dz$$

So, if you substitute this Taylor series expansion in the previous equation,

$$dv_x = \frac{\partial v_x}{\partial x}dx + \frac{\partial v_x}{\partial y}dy + \frac{\partial v_x}{\partial z}dz$$

So, what this tells you is dv_x is the total difference in x component of velocity with in the two points that has been expressed in terms of the partial derivatives. Other way of looking at it is expressing the total derivative in terms of partial derivative; two ways of looking at it.

(Refer Slide Time: 10:35)



Let us repeat this for the other directions. Let us write for dv_x, dv_y and then dv_z,

$$dv_x = \frac{\partial v_x}{\partial x}dx + \frac{\partial v_x}{\partial y}dy + \frac{\partial v_x}{\partial z}dz$$

$$dv_y = \frac{\partial v_y}{\partial x}dx + \frac{\partial v_y}{\partial y}dy + \frac{\partial v_y}{\partial z}dz$$

$$dv_z = \frac{\partial v_z}{\partial x}dx + \frac{\partial v_z}{\partial y}dy + \frac{\partial v_z}{\partial z}dz$$

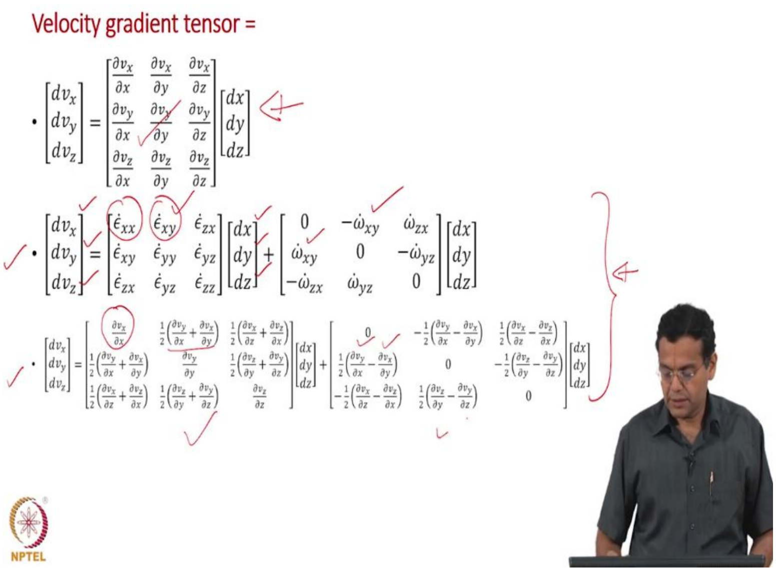
Now, express them in terms of a matrix,

$$\left[dv_x \; dv_y \; dv_z\right] = \left[\frac{\partial v_x}{\partial x} \; \frac{\partial v_x}{\partial y} \; \frac{\partial v_x}{\partial z} \; \frac{\partial v_y}{\partial x} \; \frac{\partial v_y}{\partial y} \; \frac{\partial v_y}{\partial z} \; \frac{\partial v_z}{\partial x} \; \frac{\partial v_z}{\partial y} \; \frac{\partial v_z}{\partial z}\right][dx \; dy \; dz]$$

So, left hand side, we have the column vector dv_x, dv_y and then dv_z, and right hand side, we have dx, dy, dz and then the matrix which relates these two vectors is the velocity gradient tensor. Why is it a tensor? First velocity has three directions. Now we are looking at the gradient of the velocity, this gradient can be along three directions. So, totally giving us 9 combinations of directions and hence it is a velocity gradient tensor.

So, what is that we have done now is derived an expression for the velocity gradient tensor by using the first method which is the more mathematical method. We are not going to derive the expression using the second method we will straight away write analogously from our knowledge in solid mechanics.

(Refer Slide Time: 11:55)



That is what is shown here though. So, this equation is from the previous slide expressing velocity gradient tensor using the first method.

$$\left[dv_x \; dv_y \; dv_z\right] = \left[\frac{\partial v_x}{\partial x} \; \frac{\partial v_x}{\partial y} \; \frac{\partial v_x}{\partial z} \; \frac{\partial v_y}{\partial x} \; \frac{\partial v_y}{\partial y} \; \frac{\partial v_y}{\partial z} \; \frac{\partial v_z}{\partial x} \; \frac{\partial v_z}{\partial y} \; \frac{\partial v_z}{\partial z}\right][dx \; dy \; dz]$$

Now, if you follow a approach similar to what we have done for solid mechanics as we had in the second method, what is it? Considering normal strain rate separately, shear strain rate separately, rotation rate separately, then we will be able to write this equation.

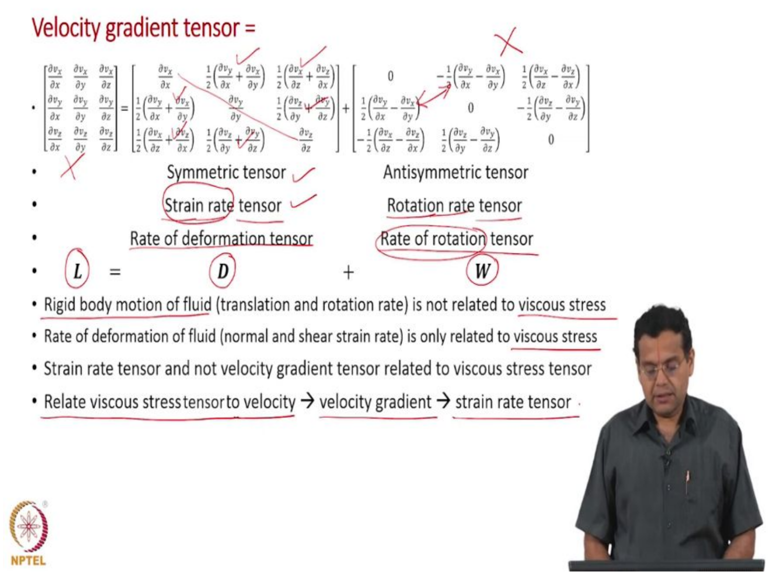
$$\left[dv_x\ dv_y\ dv_z\ \right] = \left[\dot{\varepsilon}_{xx}\ \dot{\varepsilon}_{xy}\ \dot{\varepsilon}_{xz}\ \dot{\varepsilon}_{xy}\ \dot{\varepsilon}_{yy}\ \dot{\varepsilon}_{yz}\ \dot{\varepsilon}_{zx}\ \dot{\varepsilon}_{yz}\ \dot{\varepsilon}_{zz}\ \right][dx\ dy\ dz\ ] + \left[0 - \dot{\omega}_{xy}\ \dot{\omega}_{xz}\ \dot{\omega}_{xy}\ 0 - \dot{\omega}_{yz} - \dot{\omega}_{zx}\ \dot{\omega}_{yz}\ 0\ \right][dx\ dy$$

This also relates dv_x , dv_y , dv_z in terms of dx, dy, dz, but splits into two tensors. And the next equation writes this equation in terms of components.

$$\left[dv_x\ dv_y\ dv_z\ \right] = \left[\frac{\partial v_x}{\partial x}\ \frac{1}{2}\left(\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y}\right)\ \frac{1}{2}\left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right)\ \frac{1}{2}\left(\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y}\right)\ \frac{\partial v_y}{\partial y}\ \frac{1}{2}\left(\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z}\right)\ \frac{1}{2}\left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right)\ \frac{1}{2}\left(\frac{\partial v_z}{\partial y} + \frac{\partial v}{\partial z}\right.$$

So, now based on this equation and these equations, we can write the velocity gradient tensor as sum of these two tensors and that is what is shown in the next slide.

(Refer Slide Time: 13:53)



The velocity gradient tensor that is in the left hand side is equal to sum of two tensors.

$$\left[\frac{\partial v_x}{\partial x}\ \frac{\partial v_x}{\partial y}\ \frac{\partial v_x}{\partial z}\ \frac{\partial v_y}{\partial x}\ \frac{\partial v_y}{\partial y}\ \frac{\partial v_y}{\partial z}\ \frac{\partial v_z}{\partial x}\ \frac{\partial v_z}{\partial y}\ \frac{\partial v_z}{\partial z}\ \right] = \left[\frac{\partial v_x}{\partial x}\ \frac{1}{2}\left(\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y}\right)\ \frac{1}{2}\left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right)\ \frac{1}{2}\left(\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y}\right)\ \frac{\partial v_y}{\partial y}\ \frac{1}{2}\left(\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z}\right)\ \frac{1}{2}\left(\right.$$

The first one is a symmetric tensor as we can see the off diagonal elements are same the symmetric tensor. And, the second tensor is antisymmetric because the off diagonal elements are say of same magnitude opposite in sign; and how do we name them? In the case of solids we call it as strain tensor because it contained information about deformation namely normal strain and shear strain.

Now, this tensor now contains information about rate of deformation namely normal strain rate and shear strain rate. All the diagonal elements represent the normal strain rate and the off diagonal elements represent the shear strain rate by 2. So, we call this strain rates the strain rate tensor. Of course, the second tensor was called as rotation tensor for a solids now for the case of fluids it is the rotation rate tensor. So, we have split the velocity gradient tensor into the strain rate tensor and the rotation rate tensor just like we expressed the displacement gradient tensor as some of strain tensor and rotation tensor. So, this slide is that way analogues to what we have discussed for solids.

Other names are there, we have seen deformation in solids is analogues to rate of deformation in fluids. So, other name is rate of deformation tensor and all this second tensor is also called as rate of rotation tensor. In terms of more popular terminology strain rate tensor is more popular and then rate of rotation tensor is more popular.

We will introduce some nomenclature so; that it will becomes handy for us later on. The velocity gradient tensor left hand side is L, right hand side that the rate of deformation tensor is D and rate of rotation tensor is W. D is used so, that represents deformation we use omega for rotation. So, we are using capital W for tensor.

$$L = D + W$$

Where,

$$L = Strain\ rate\ tensor = \left[\frac{\partial v_x}{\partial x}\ \frac{\partial v_x}{\partial y}\ \frac{\partial v_x}{\partial z}\ \frac{\partial v_y}{\partial x}\ \frac{\partial v_y}{\partial y}\ \frac{\partial v_y}{\partial z}\ \frac{\partial v_z}{\partial x}\ \frac{\partial v_z}{\partial y}\ \frac{\partial v_z}{\partial z} \right]$$

Similarly,

$$D = \left[\frac{\partial v_x}{\partial x}\ \frac{1}{2}\left(\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y}\right)\ \frac{1}{2}\left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right)\ \frac{1}{2}\left(\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y}\right)\ \frac{\partial v_y}{\partial y}\ \frac{1}{2}\left(\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z}\right)\ \frac{1}{2}\left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right)\ \frac{1}{2}\left(\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z}\right)\ \frac{\partial v_z}{\partial z} \right]$$

$$W = \left[0\ -\frac{1}{2}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\ \frac{1}{2}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\ \frac{1}{2}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\ 0\ -\frac{1}{2}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\ -\frac{1}{2}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\ \frac{1}{2}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\ 0 \right]$$

Now, whatever follows is once again analogues to what we discuss for solid mechanics. Rigid body motion of fluid is not related to viscous stress. What is that mean? Remember our objective was to relate the viscous stress tensor to the velocity gradient and then to understand velocity gradient, we went to solid mechanics understood displacement gradient. We come back to fluid mechanics to and then discussed about rate of deformation velocity gradient etcetera.

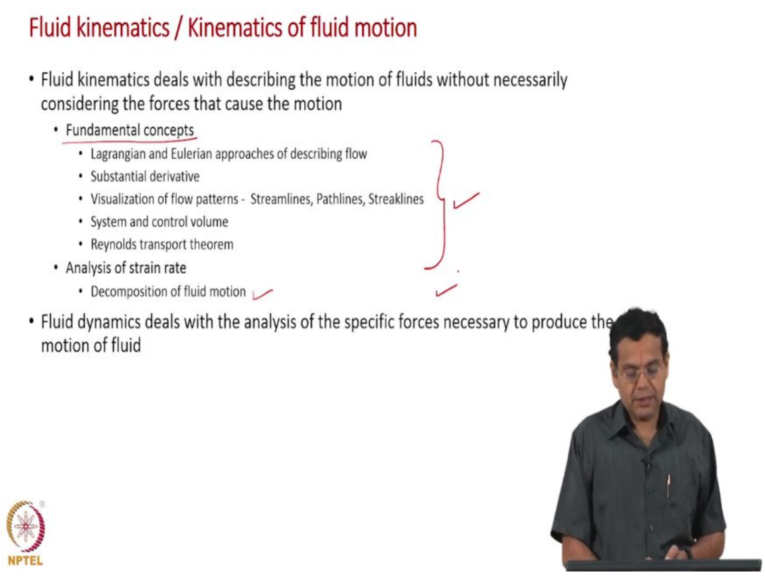
Now, what is that that has to be related to the viscous stress? Based on the discussion on solid mechanics and now for fluids also, the rigid body motion of fluid what is that the entire body just translating and then the entire body rotating is not related to the viscous stress. When a fluid body just translates and then just rotates there are no internal viscous stresses developed. So, that is not related to the viscous stress. So, this tensor is not related to the viscous stress.

Rate of deformation of fluid which is which includes normal and shear strain rate that only is related to the viscous stress. So, the first tensor is what is related to the viscous stress. Whenever there is normal strain rate, shear strain rate; then we have viscous stresses. So, we have to relate the rate of deformation of fluid and viscous stress which means that strain rate tensor and not velocity gradient tensor related to the viscous stress tensor. We said we need to relate the viscous stresses to velocity.

Then we said velocity gradient, but now we are saying it is not the entire velocity gradient only a part of it which is the strain rate tensor or the rate of deformation tensor. Why is that? Because, velocity gradient tensor also includes the rate of rotation tensor, but we have seen that the viscous stress tensor is not related to the viscous stresses are not related to the rate of rotation just like we discussed for solids. Remember we said if you have a solid under just translate just rotates, there are no internal stresses. But if you pull a solid and then and then change the angle of a solid, then internal stresses are developed ok. So, that is why the normal strain shear strain was related to the stress.

Similarly here whenever there is normal we have a fluid element that length keeps increasing angle between two line elements keeps changing, then viscous stresses are developed in fluids. So, the strain rate tensor not the velocity gradient tensor is related to the viscous stress tensor. And so, the last line we need to relate viscous stress tensor to velocity that is what we said. A better way of saying is to velocity gradient is a still more process way of saying is related to the strain rate tensor. So, we need to relate the viscous stress tensor to the strain rate tensor.

(Refer Slide Time: 20:59)



So, like to discuss a terminology which we have not mentioned so far namely fluid kinematics. If you look at fluid mechanics books, fluid kinematics is discussed among the first or second chapters. Let us look at the definition fluid kinematics deals with describing the motion of fluids without necessarily considering the forces that cause the motion. So, just talks about motion of fluids, does not give importance or pay attention to what causes the fluid motion or the forces which causes the fluid motion.

So, what are the topics which are discussed under fluid kinematics?

- The Lagrangian, Eulerian approaches of describing flow,
- Substantial derivative,
- Visualization of flow patterns; namely Streamlines, Pathlines and Streaklines,
- Distinction between system and control volume,
- Reynolds transport theorem and then
- Decomposition of fluid motion.

What is decomposition of fluid motion, that is expressing the velocity gradient tensor in terms of the strain rate tensor and then the rotation rate tensor which means that we are decomposing the fluid motion into deformation and then rotation.

The way in which we have discussed is we have discussed all these topics under that heading fundamental concepts at the beginning of the course just like any fluid mechanics book could do, but based on the organization of the course, we have discussed this decomposition of fluid motion just now under when we are discussing strain rate for fluids. So, but these topic put together all come under the heading of fluid kinematics or kinematics of fluid motion.

So, what is fluid dynamics? Fluid kinematics within focus on the forces; so, fluid dynamics deals with the analysis of specific forces necessary to produce the motion of fluid. Fluid kinematics we dealt only with motion of fluid without considering the forces. Fluid dynamics we pay attention to the forces which cause the motion of fluid.