Continuum Mechanics And Transport Phenomena
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Lecture - 66
Volumetric Strain Rate

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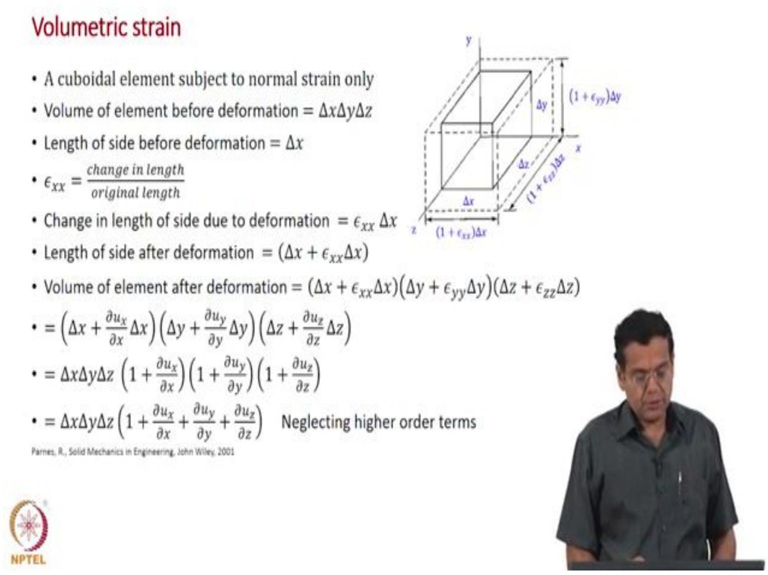

Then we discussed about volumetric strain in solid mechanics and then related the volumetric strain to the displacement gradients.

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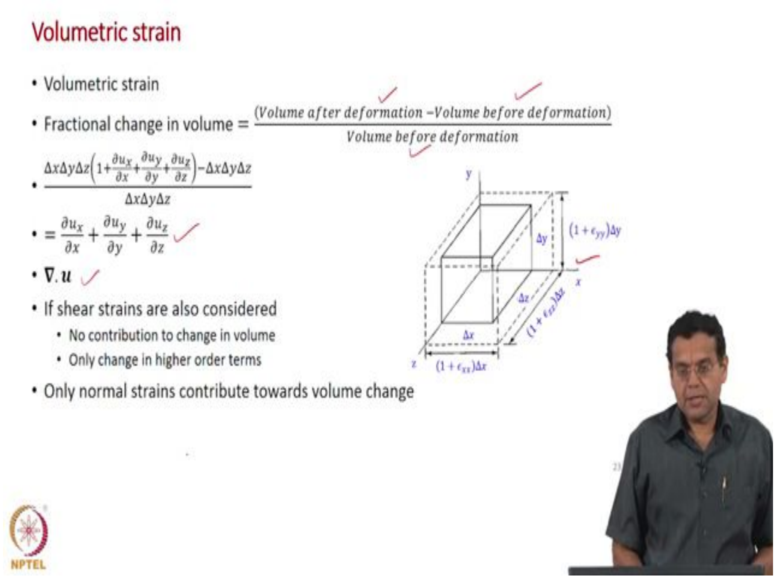

Volumetric strain is the fractional change in volume which is

$$V olumetric\ strain = \frac{V olume\ after\ deformation - V olume\ before\ deformation}{V olume\ before\ deformation}$$

Or change in volume by original volume, something similar to our definition for normal strain. And we took this cuboidal element and proved that the volumetric strain is the sum of the three displacement gradients which we expressed as divergence of the displacement field.

$$\nabla .u = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}$$

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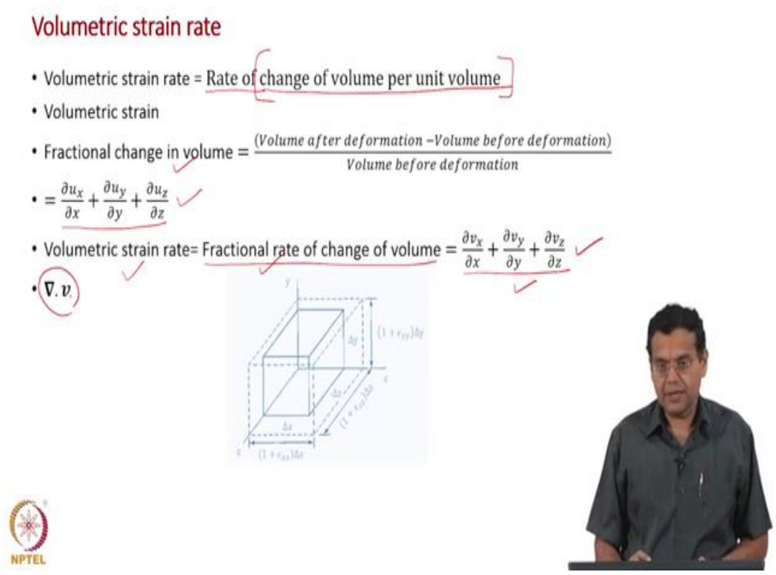

Now, analogously we can relate the volumetric strain rate to the velocity gradients, let us see how do we do that. First let us define volumetric strain rate. For volumetric strain, it was change of volume per unit volume and so the volumetric strain rate is rate of change of volume per unit volume.

$$Volumetric\ strain = \frac{Volume\ after\ deformation - Volume\ before\ deformation}{Volume\ before\ deformation}$$

This definition we have seen just now for a volumetric strain as fractional change in volume which we have seen as the sum of these three displacement gradients.

So, now, analogously for the case of fluids, the volumetric strain rate which is the fractional rate of change of volume. So, volumetric strain is change in volume per unit volume. And volumetric strain rate is rate at which this happens.

Just like we had some of the sum of three displacement gradients, we will now have sum of these three velocity gradients

$$Volumetric\ srain\ rate = \nabla.v = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Just like in the case of normal strain rate and shear strain rate, we had for the case of solids, for example, normal strain in terms of displacement gradient, left hand side normal strain became normal strain rate, right hand side became velocity gradient. Similarly, here we have volumetric strain in terms of displacement gradient and so, volumetric strain rate in terms of velocity gradient, same expression, replace the displacements with the corresponding velocities. And just like we expressed the fractional change in volume as divergence of displacement field, we can express the volumetric strain rate in terms of divergence of the velocity field.

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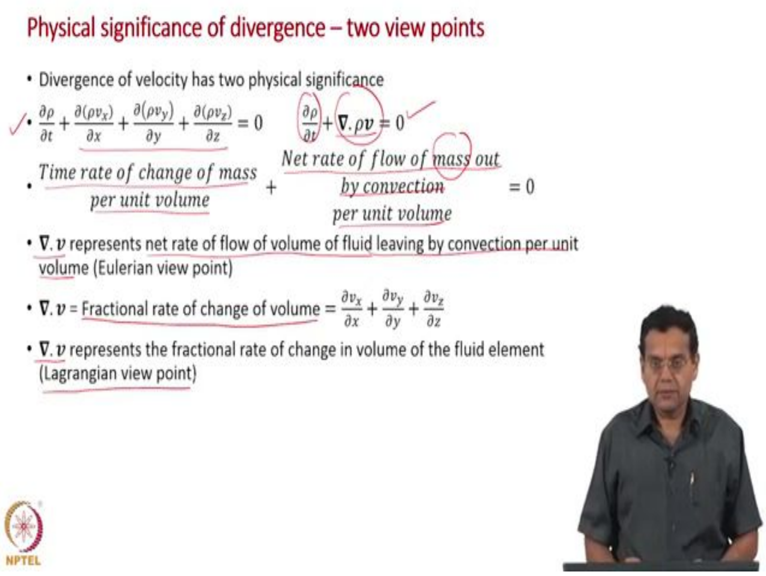

So, now, we have come across divergence of velocity earlier, once again we are coming across divergence of velocity. So, now, there are two viewpoints or two ways of discussing the physical significance of divergence. And so we are going to discuss the physical significance of divergence from two viewpoints that is the objective now. As we have said divergence of velocity has two physical significance, what are they? first, we derived the continuity equation and express that in terms of vectorial form in this way.

$$\frac{\partial \rho}{\partial t} + \nabla . \rho v = 0$$

What is the significance of the two terns in the continuity equation or the differential total mass balance? The first term represents time rate of change of mass per unit volume, second term, either vectorially as divergence of ρv vector, $\nabla . \rho v$ or sum of these three terms that represents net rate of flow of mass out by convection per unit volume. So, divergence of ρv vector, $\nabla . \rho v$ represents net rate at which mass flows out by convection per unit volume. So, if you take $\nabla . v$, that represents net rate of flow of volume of fluid.

In the case of $\nabla . \rho v$, it was mass, but in $\nabla . v$ same physical significance, instead of mass it is volume. Now, this gives physical significance of $\nabla . v$ from a Eulerian view point, we have a small region. And $\nabla . v$ represents net rate of flow of volume of fluid leaving by convection per unit volume. Now, we have seen in alternate view point for $\nabla . v$, the previous slide which is fractional rate of change of volume. So, we can interpret $\nabla.v$ also as fractional rate of change of volume.

What does it mean? If you have a fluid element and let us say if you are following the fluid element, the rate of change of volume of that fluid element of course per unit volume is represented by $\nabla.v$. So, another physical significance of $\nabla.v$ is that, the rate of change of volume of a fluid element as you follow the fluid element of course, normalized per unit volume and this significance of $\nabla.v$ is from a Lagrangian view point.

First one was in terms of an Eulerian view point; this physical significance is from a Lagrangian view point. So, $\nabla.v$ can be expressed both from a Eulerian view point which is what we already discussed when we discuss the continuity equation, the new view point is from a Lagrangian view point and that is the fractional rate of change of volume of a fluid element.

Why is it Lagrangian? Remember when we discussed about normal strain rate, shear strain rate, similarly volumetric strain rate, all the derivatives are substantial derivatives $\frac{D}{Dt}$. So, all the derivatives are following the fluid motion and that is why $\nabla.v$, the fractional rate of change of volume represents that we would observe if we follow a fluid element.

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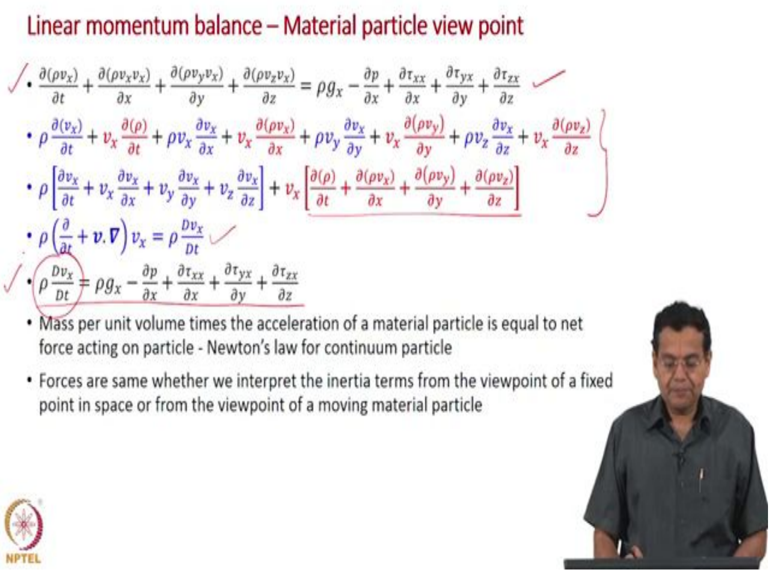

Now, having understood $\nabla.v$, the divergence of velocity vector in two different viewpoints. What we will do is express the continuity equation for a material particle view point, actually we have done this earlier for the linear momentum balance this slide is a recall slide. We derived the linear momentum balance from a Eulerian view point and then did some simple rearrangements for the terms in the left hand side and that is what is shown here.

$$\rho\frac{Dv_x}{Dt} = \rho g_x - \frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}$$

So, we re-express the linear momentum balance from a material particle point of view. Why is that, the left hand side is now in terms of mass per unit volume times the acceleration of the fluid particle. So, the left hand side now is in terms of a Lagrangian viewpoint. So, the same linear momentum balance, we expressed from a material particle view point and discuss the significance as well. And now what we are going to do is express the equation of continuity from a material particle view point.

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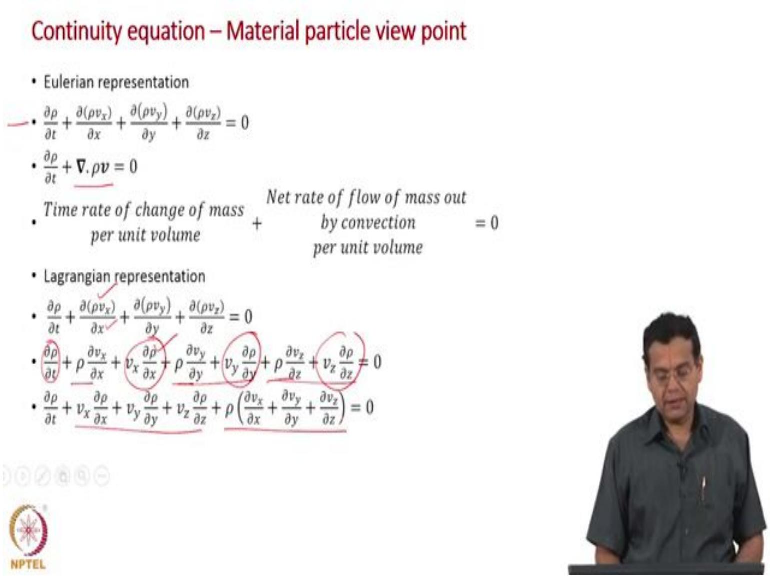

So, let us look at the Eulerian representation. So, this is the equation of continuity which we have derived when we derived the conservation equation for total mass, the differential form of that.

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v_x)}{\partial x} + \frac{\partial(\rho v_y)}{\partial y} + \frac{\partial(\rho v_z)}{\partial z} = 0$$

You can also express the continuity equation in terms of divergence.

$$\frac{\partial \rho}{\partial t} + \nabla.\rho v = 0$$

And what is the significance, the first term represents time rate of change of mass per unit volume, second term represents net rate of flow of mass out by convection per unit volume that we have seen again.

How do you represent the Lagrangian form? As we have done for the linear momentum balance, remember, when we do in material particle view point. When you express the linear momentum balance in material particle view point, we are we rearranged the left hand side only that transient term and then the convection term right hand side was still the sum of forces acting. So, let us take the equation of continuity

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v_x)}{\partial x} + \frac{\partial(\rho v_y)}{\partial y} + \frac{\partial(\rho v_z)}{\partial z} = 0$$

. Let us apply product rule.

$$\frac{\partial \rho}{\partial t} + \rho\frac{\partial(v_x)}{\partial x} + v_x\frac{\partial(\rho)}{\partial x} + \rho\frac{\partial(v_y)}{\partial y} + v_y\frac{\partial(\rho)}{\partial y} + \rho\frac{\partial(v_z)}{\partial z} + v_z\frac{\partial(\rho)}{\partial z} = 0$$

So, let us group terms together

$$\frac{\partial \rho}{\partial t} + v_x\frac{\partial(\rho)}{\partial x} + v_y\frac{\partial(\rho)}{\partial y} + v_z\frac{\partial(\rho)}{\partial z} + \rho\left(\frac{\partial(v_x)}{\partial x} + \frac{\partial(v_y)}{\partial y} + \frac{\partial(v_z)}{\partial z}\right) = 0$$

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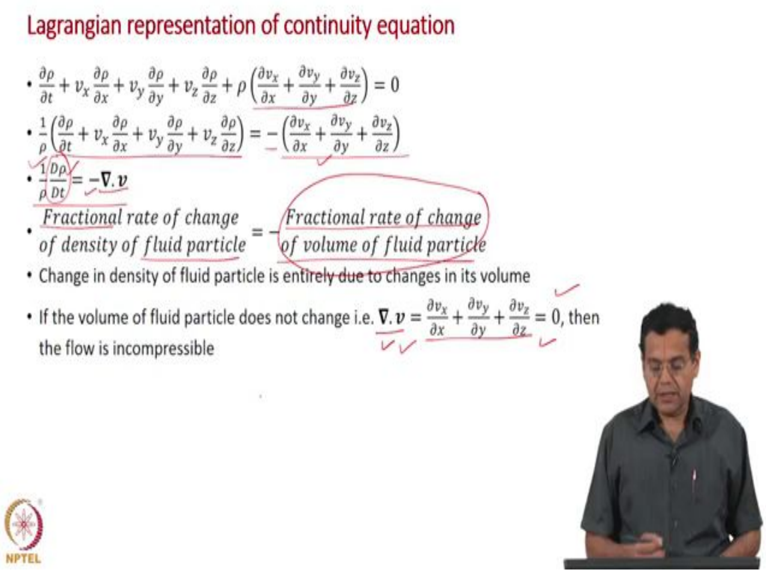

Let us rewrite that last equation here, and then we will be able to easily identify terms of physical significance.

$$\frac{\partial\rho}{\partial t} + v_x\frac{\partial(\rho)}{\partial x} + v_y\frac{\partial(\rho)}{\partial y} + v_z\frac{\partial(\rho)}{\partial z} + \rho\left(\frac{\partial(v_x)}{\partial x} + \frac{\partial(v_y)}{\partial y} + \frac{\partial(v_z)}{\partial z}\right) = 0$$

We will do a small rearrangement, so that we can write the physical significance of the terms.

$$\frac{1}{\rho}\left(\frac{\partial\rho}{\partial t} + v_x\frac{\partial(\rho)}{\partial x} + v_y\frac{\partial(\rho)}{\partial y} + v_z\frac{\partial(\rho)}{\partial z}\right) =- \left(\frac{\partial(v_x)}{\partial x} + \frac{\partial(v_y)}{\partial y} + \frac{\partial(v_z)}{\partial z}\right)$$

Now, we can easily identify that left hand side four terms together constitute the substantial derivative of density. So,

$$\frac{1}{\rho}\frac{D\rho}{Dt} =- \nabla.v$$

So, what we did is, took the continuity equation which we have derived, applied product rule then simple rearrangement and then we could get you could express the continuity equation from a Lagrangian point of view or a material particle point of view. Why is it Lagrangian, because we have $\frac{D\rho}{Dt}$ on the left hand side which is substantial derivative which is derivative following a fluid particle.

So, in terms of significance, what was the left hand side tell us, fractional rate of change of density of the fluid particle. Why is it fractional rate of change of density, we have $\frac{D\rho}{Dt}$ is rate of change of density of fluid particle, why is it, because it is $\frac{D\rho}{Dt}$ it is fractional because we have $\frac{1}{\rho}$. So, left hand side represents fractional rate of change of density of fluid particle. Right hand side we have seen just few slides back that $\nabla.v$ represents fractional rate of change of volume of fluid particle.

So, now, look at the significance, both the left hand side and right hand side corresponds to a fluid particle. Left hand side represents fractional rate of change of density, right hand side fractional rate of change of volume of both correspond to a fluid particle. So, as you follow a fluid particle, what is it rate of change of density, that is what is given by this equation of continuity expressed in this form.

So, what does it imply, change in density of fluid particle. So, when you look at this equation of continuity, you should imagine that we are following a fluid particle and looking at it is rate of change of density ok. So, change in density of a fluid particle is entirely due to changes in its volume, because of change in volume its density changes. Or density changes attributed to the change in volume.

Now, if the volume of fluid particle does not change, what does it mean, you are tracking the fluid particle and the volume of the fluid particle does not change. What can we conclude, if you look at the right hand side, the fractional rate of change of volume which is $\nabla . v = 0$. Then of course, the flow is incompressible.

So, this is the very physically meaningful definition of incompressible flow. What is that, if you follow a fluid particle and if its volume does not change, then we say the flow is incompressible and the condition is

$$\nabla . v = \frac{\partial(v_x)}{\partial x} + \frac{\partial(v_y)}{\partial y} + \frac{\partial(v_z)}{\partial z} = 0$$

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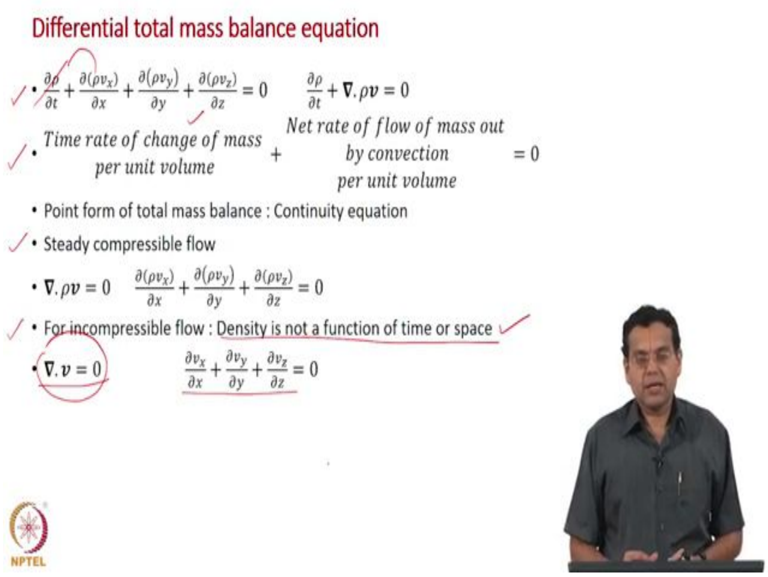

And we already discussed incompressible flow. So, let us compare both the conditions. This slide is a recall slide which we are discussed already. The continuity equation,

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v_x)}{\partial x} + \frac{\partial(\rho v_y)}{\partial y} + \frac{\partial(\rho v_z)}{\partial z} = 0$$

And then we discussed two special cases;

- One is for steady compressible flow,

$$\nabla.\rho v = \frac{\partial(\rho v_x)}{\partial x} + \frac{\partial(\rho v_y)}{\partial y} + \frac{\partial(\rho v_z)}{\partial z} = 0$$

- Another one is incompressible flow. We said for incompressible flow, density is not a function of time or space.

$$\nabla.v = \frac{\partial(v_x)}{\partial x} + \frac{\partial(v_y)}{\partial y} + \frac{\partial(v_z)}{\partial z} = 0$$

This same condition which we have discussed now as $\nabla.v = 0$ and we said density is not a function of space or time which means $\frac{D\rho}{Dt} = 0$. So, both the conditions are equivalent.

First we said the condition for incompressible flow in terms of density, as density not varying with time or space and hence arrived at $\nabla.v = 0$ and we said that is a incompressible flow. Now, once again we arrive at the same condition $\nabla.v = 0$ in a more physically meaningful way, because $\nabla.v$ represents fractional rate of change of volume of fluid particle. And if the volume does not change, then it is incompressible flow.

And when $\nabla.v = 0$ then $\frac{D\rho}{Dt} = 0$, which is in line with our earlier condition that density is not a function of time or space. So, we have discussed the condition for incompressible flow in two different ways.