Continuum Mechanics And Transport Phenomena

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Lecture - 60
Components of Total Displacement – Part 1

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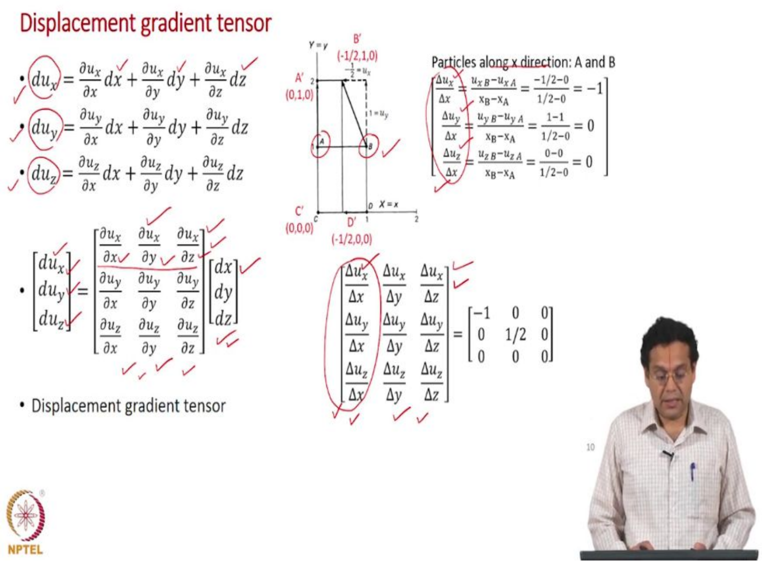

$$\left[du_x \; du_y \; du_z\right] = \left[\frac{\partial u_x}{\partial x} \; \frac{\partial u_x}{\partial y} \; \frac{\partial u_x}{\partial z} \; \frac{\partial u_y}{\partial x} \; \frac{\partial u_y}{\partial y} \; \frac{\partial u_y}{\partial z} \; \frac{\partial u_z}{\partial x} \; \frac{\partial u_z}{\partial y} \; \frac{\partial u_z}{\partial z}\right]\left[dx \; dy \; dz\right]$$

We have derived this relationship between du_x, du_y, du_z and dx, dy, dz and the displacement gradient tensor the first way which is more mathematical. Now, we will derive the same relationship in the more geometrical way in terms of the components of the total difference in displacement.

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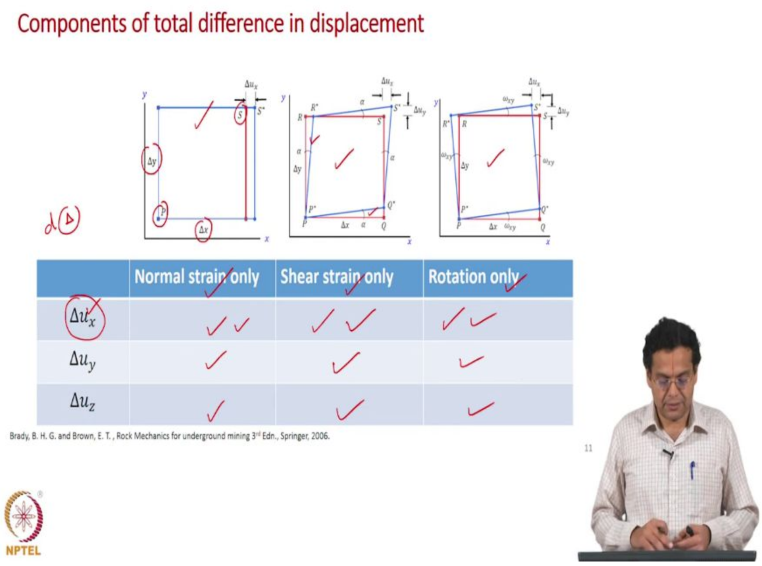

Like I mention the book which is being followed here by Brady and Brown in the Rock Mechanics for underground mining very peculiar reference they sense that this particular topic which are going discuss is done mostly mathematically in most of the books. I would appreciate certainly this book that has done more geometrically and that is why we following that approach.

If you are refer any typical book on solid mechanics or continuum mechanics, this particular step is done more mathematically, to be very specific this reference is being chosen. So, that we get a good geometrical interpretation. What is that we are going to do, derive the same relationship in alternate way in a more geometrical sense.

Now, also look at the components of the total difference in displacement. The title has to be explained first we had displacement we said we want to eliminate translation. So, we have looked at difference in displacement. Now, what is total difference in displacement, just tells you that whatever two points and their total difference in displacement which may be because of rotation, normal strain, shear strain, I do not include translation now because, we are looking at difference in displacement so translation is gone, but still remember this total difference includes that due to normal strain, shear strain and rotation; rotation is still there.

At the end of this, what we will see is remember what we said was we want to eliminate both the rigid body motions, namely translation and rotation. We are taken care of translation very easily just by difference. End of this discussion, we will separate out rotation also that is the overall picture let us go to the details. How are we going to do this, we are going to take plate or two-dimensional region, plate is little more physically imagine and visualize, actually it is a two-dimensional region in a solid, and we are going to consider points P and then S.

The x distance between them is Δx, y distance between them is Δy and that is what is shown in the figure (above referred slide image). Remember when you derived the relationship or displacement gradient, we consider two points P and Q and the difference in x coordinates was Δx, y was Δy, z was Δz. Remember we are going to arrive at the same expression. So, here also I am considering two points of course, in 2D, which are separated by Δx and Δy.

Now, three cases will be considered one by one. What are the three cases? The first case I consider that for the understanding will take up as a plate, the plate undergoes only normal strain. Look at the title here, in the figure, this figure shows only normal strain and this title shows normal strain. Second figure, I consider only shear strain. The lengths are same; and there is no rotation of the element, but there is only shear strain because there is change in angle. So, only shear strain and the second column heading is shear strain only. Third I just consider the rotation of the plate only, no normal strain, no shear strain, and that is the third configuration, only rotation.

Now, what are we going to do? we said displacement is because of translation, rotation, normal strain, shear strain. We are taken care of translation by difference in displacement which means that if you say difference in displacement along one direction let say x-direction, that has components from normal strain, shear strain and rotation. What do you mean by that?

If there are two points and then there is a difference in the x displacement which means that could be because of normal strain, could be because of shear strain, could be because of rotation. What you have observe is some effect of all these three put together in terms of difference. We are going to dissect it find out the components that is exactly we are going to do.

	Normal strain only	**Shear strain only**	**Rotation only**
Δu_x			

Δu_y			
Δu_z			

This table is a matrix where we are going to find out the components of the total difference in displacement, understand why total difference in displacement, we have a difference in displacement what you observe is total, it has its components that why I call as total difference in displacement.

Just want to mention that finally we will go back to differential because our relationship in terms of differentials just for understanding tentatively come to in terms of difference.

So, what is that you are going to do, we are going to find out the components of total difference in displacement along three directions. What do I mean by that? Take two points, you observed a difference in their displacement that is the total difference in displacement that is the this heading of this row, that we are going to split into that the difference due to normal strain, difference due to shear strain, difference due to rotation, difference in x displacement due to all this; similarly difference in y displacement, similarly difference in z displacement.

So, this table is going to be filled up in a more geometrical way. So, just to summarize they are going to take a plate or two-dimensional region, subjected to normal strain only, shear strain only, rotation only, find out what are the components of the total displacement along x-direction, y-direction, z-direction.

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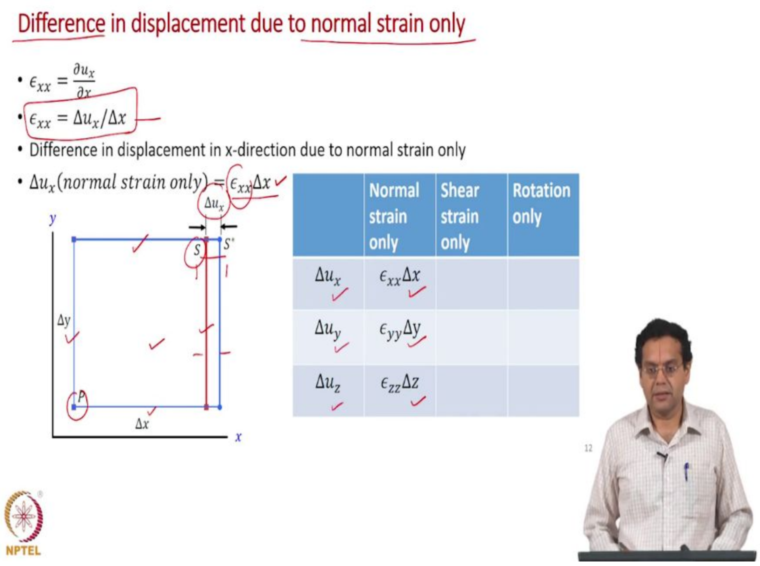

Now, earlier I said total difference in displacement, look at the title now, difference in displacement due to normal strain only, that is what we are going to see. So, of course, now, a larger figure is shown, we considering the plate, please not that we are considering a point P and a point S, the x distance is Δx, y distance is Δy. This is in line with our earlier consideration of points P and Q separated by Δx, Δy, Δz. Now, the plate undergoes normal strain only, what do I mean by that, there is a change in the length along the x-direction only. There is no change in angle.

Just want to mention this red is the original configuration or initial state. Remember this is blue line, but bellow that you have a red line. So, whatever I mark now is a initial state and then you subject this to normal strain only. Then what is a boundary, the blue line is the new state; part of the red is below the blue boundary. Now, subjected to the normal strain only and then along x axis only.

Now, we know that normal strain is related to the displacement gradient in terms of

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x}$$

I will write in terms of difference

$$\varepsilon_{xx} = \frac{\Delta u_x}{\Delta x}$$

As I told you we have working in terms of difference for understanding. What does this tell you in terms of physically if you want to read this statement, it relates normal strain to the difference in x displacement between two points with which are separated by Δx distance that is what we have here. We have two points which are separated by distance Δx.

And so this expression should relate to the difference in displacements between P and S. Now, the point P the way in which you have drawn there is no displacement for point P. So, the displacement of S becomes the difference in displacement. We are considering points P and S point, P there is no displacement so actually S moves to S* I should have noted here just displacement of point S only, but what is that I have marked their difference in displacement.

What is the reference point? My reference point is P, because there is no displacement of P displacement of S becomes difference in displacement. So, please keep this in mind this follows for all other discussion also. Though it may look like displacement, it is difference in displacement because we are always considering point P for which there is no displacement.

So, now we can what is that we are interested in, remember we are interested in the difference in displacement due to normal strain only. So, same expression, earlier we wrote as an expression for normal strain, but now we are going to write an expression for Δu_x which is for interested in. So,

$$\Delta u_x\,(\textit{normal strain only}) = \varepsilon_{xx}\Delta x$$

what it tells you is S to S* the different displacement or difference in displacement because of normal strain only can be obtained by $\varepsilon_{xx}\Delta x$.

So, once again this discussion looks like you have a plate and then of a finite length Δx, Δx. Please keep this in mind which is small region in solid two dimension region, where these are very very close to each other P and S are very close to each other, where visualization we are drawing a diagram and then saying a plate etcetera. Remember because epsilon x is a point definition as $\Delta x \rightarrow$, $\Delta x \rightarrow$.

Similarly, here also this points P and S are very very close to each other as we have as applicable to the definition of ε_{xx}. And so this gives you the difference in displacement, this expression gives you the difference in displacement along the x-direction because of normal strain only and that entry is made here. We have nine entries to make we have made one entry. Now, similarly if you consider the y-direction and z-direction, we can fill two more entries.

	Normal strain only	Shear strain only	Rotation only
Δu_x	$\varepsilon_{xx}\Delta x$		
Δu_y	$\varepsilon_{yy}\Delta y$		
Δu_z	$\varepsilon_{zz}\Delta z$		

What this table tells you is, what is the difference in x displacement because of normal strain along x-axis, y-axis and z-axis. And one component is now completed.

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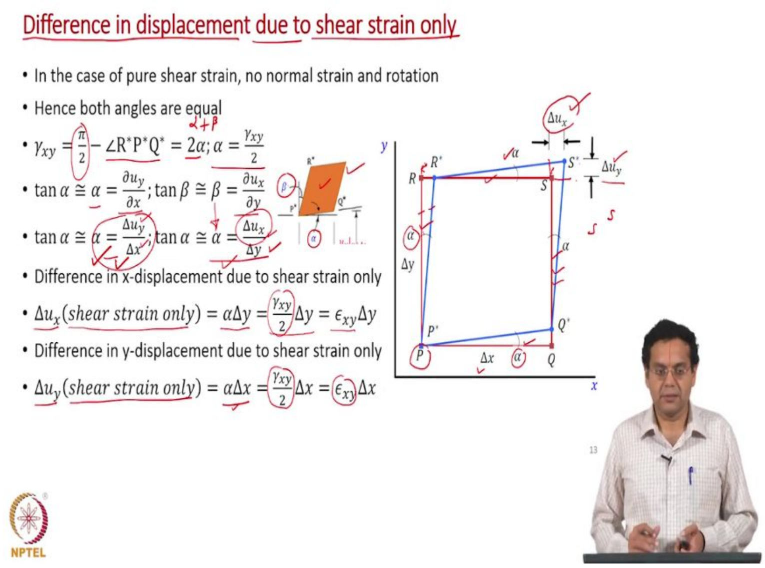

So, now let us move onto finding out difference in displacement due to shear strain only. Now, same plate is considered. Now, the initial status given by PQRS, the red boundary. And P*Q*R*S* is the final state, and the deformed boundary. And the way in which we have considered as you want to considered include only shear strain, so there is no change in length. P R is equal to P*R* in terms of length, similarly all other sites also. And also there is no rotation look at the diagonal PS and diagonal P*S*, there is no change. So, the only the change in angle has happened, no normal strain, no rotation.

Now, of course, the length Δx, Δy are shown let us proceed further to explain other nomenclature. We are considering the case of pure shear strain, where there is no normal strain and rotation. Now, what I shown here is this is a diagram, part of the diagram which are used to relate the strange the displacement gradient.

I have taken a small part of that I have shown here. We showed this angle is α, α made by P*Q* with the x-axis; angle made by P*R* y axis is β. Now, those angles where different in a general scenario where you had rotation, normal strain etcetera. But now we have only shear strain, we can easily imagine is going to be symmetric have only shear strain.

In this case, where we have only shear strain you just changes in angle, so α and β are equal. Now, we defined shear strain as, because α and β are same

$$\gamma_{xy} = \frac{\pi}{2} - < R^* P^* Q^* = 2\alpha; \quad \alpha = \frac{\gamma_{xy}}{2}$$

$\alpha = \frac{\gamma_{xy}}{2}$ is applicable when we have shear strain only that is the relationship. Now,

$$\tan tan\ (\alpha) \cong \alpha = \frac{\partial u_y}{\partial x}; \quad \tan tan\ (\beta) \cong \beta = \frac{\partial u_x}{\partial y}$$

We have already derived while driving discussing this case where we are relating strains to displacement gradient. We proved that $\alpha = \frac{\partial u_y}{\partial x}$, and $\beta = \frac{\partial u_x}{\partial y}$. We can you can refer those slides for this derivation. Now, for the present case, we will write in terms of difference, so write as

$$\tan tan\ (\alpha) \cong \alpha = \frac{\Delta u_y}{\Delta x}; \quad \tan tan\ (\alpha) \cong \alpha = \frac{\Delta u_x}{\Delta y}$$

Now, these expressions first expression for α, second expression for α correspond to this, this figure. Let me explain. Let us take this expression $\alpha = \frac{\Delta u_y}{\Delta x}$.

Now, if you consider the triangle you do not see a triangle there, why because I have shown a large value of alpha. Remember alpha is extremely small 10^{-6} as such a small value. What happens then this, R, R*, S, S* can be approximated to a triangle. Now, of course, it does not look like even a triangle. But remember this is going to R and R* is going to be very very close to each other and then you will S* somewhere here then R*, S*, S will be a triangle. And then you can apply this relationship for that triangle relating $\alpha = \frac{\Delta u_y}{\Delta x}$.

What does this relationship tell you, it relates α to the difference in y displacement for two particles which are separated Δx away that is what we have here also. Once again as like a last example, the difference in displacement is related to the point P and that is why S to S* is displacement, but denoted here as difference in y displacement.

So, this relationship is applicable for this case. And this relates α, the difference in displacement in the distance between them. What is that we are interested in we are interested in the in getting expression Δu_y from this equation. Now, let us take the second relationship $\alpha = \frac{\Delta u_x}{\Delta y}$ and that corresponds to this triangle. Once again in the limit of very very small α is becomes a triangle.

The second equation what does it tell you how α is related to difference in x displacement between two particles which are Δy away. Same thing is here we have P and then S, we are which are Δy away. So, this equation is applicable for this triangle. And we are going to use this we have written for α, but what is interest what is of interest was is the difference in the x displacement Δu_x and the we will write this expression for Δu_x.

So, now, we will write the difference in x displacement due to shear strain only. So,

$$\Delta u_x\,(shear\ strain\ only) = \alpha\Delta y = \frac{\gamma_{xy}}{2}\Delta y = \varepsilon_{xy}\Delta y$$

So, γ_{xy} shear strain half of that is denoted by another variable namely ε_{xy}. And what is the significance of this statement is that, you had a plate it is subjected to shear strain only, then what is the difference in displacement, Δu_x difference in displacement. What did we say that difference in displacement could be because of normal strain which I have already discussed could be because of shear strain.

So, we are saying that the difference in displacement that can happen because of shear strain in the x-direction is by given by Δu_x and that is what you have related to the shear strain. Now, similarly you can write the expression for the difference in y displacement due to shear strain only see when shear strain happens, there is not alone difference in x displacement, there is difference in y displacement also. We can very clearly see S moves to S*, so there is a displacement in x-direction S moves to S* so, there is displacement in the x-direction, displacement in the y-direction of course, we are calling it as difference in displacement related to P.

So, difference in y displacement

$$\Delta u_y\,(shear\ strain\ only) = \alpha\Delta x = \frac{\gamma_{xy}}{2}\Delta x = \varepsilon_{xy}\Delta x$$

What to conclude what we are done is if there is shear strain, what is the difference in displacement along x-direction y-direction.

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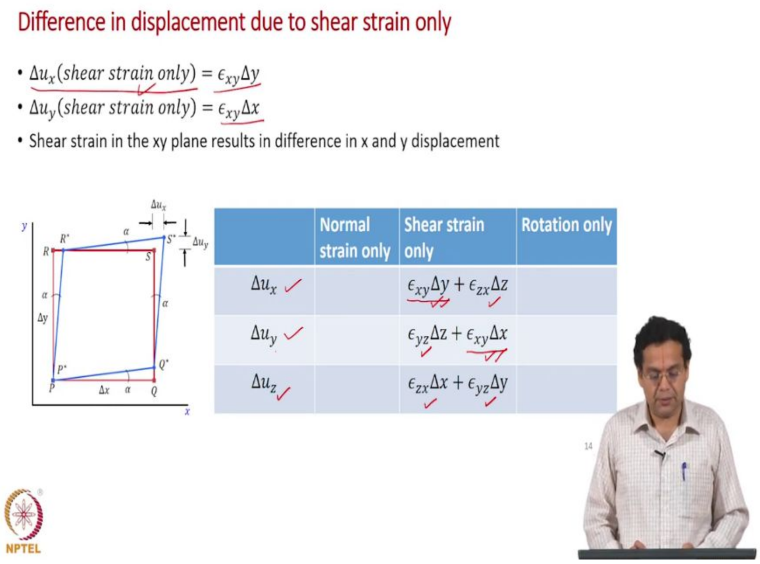

Just complete for other directions ok, this expression which we have written in the previous slide, difference in x displacement due to shear strain only, difference in y displacement due to shear strain only.

$$\Delta u_x\,(shear\ strain\ only) = \varepsilon_{xy}\Delta y$$

$$\Delta u_y\,(shear\ strain\ only) = \varepsilon_{xy}\Delta x$$

The nomenclature has been made very clear, difference in displacement in the x-directions so Δu_x . Now, we are considering nor the total difference in displacement, we are considering only that due to shear strain only that is way the within the bracket it is written as shear strain only. Now let us fill up the table.

	Normal strain only	Shear strain only	Rotation only
Δu_x		$\varepsilon_{xy}\Delta y + \varepsilon_{zx}\Delta z$	
Δu_y		$\varepsilon_{yz}\Delta z + \varepsilon_{xy}\Delta x$	
Δu_z		$\varepsilon_{zx}\Delta x + \varepsilon_{yz}\Delta y$	

Now, remember we said shear strain causes difference in displacement along x-direction and y-direction. So, you will have one entry for Δu_x you will have another entry for Δu_y that is what is shown here. We have one entry for Δu_x, which is $\varepsilon_{xy}\Delta y$; another entry for $\Delta u_y = \varepsilon_{xy}\Delta x$. So, these are the two terms which we have written based on the strain in the x, y plane. We are considered x, y plane, considered the strain which has resultant in displacement two directions; one in x-direction and y-direction. If you understood this very clearly extend to other pairs of directions.

So, strain in x, y plane for example, causes difference in displacement along x-direction and y-direction that should be kept in mind, that is of course, we just now discuss shear strain in the x, y plane results in difference in x and y displacement.