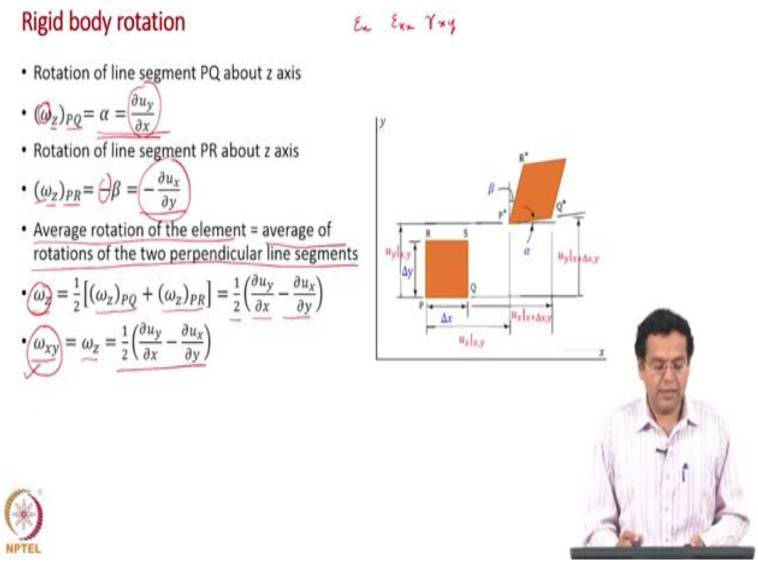
Continuum Mechanics And Transport Phenomena
Prof. T. Renganathan
Department of Chemical Engineering
Indian Institute of Technology, Madras

Lecture - 57
Strain Displacement Gradient Relation: Rotation and volumetric strain

(Refer Slide Time: 00:14)



Now, in the example which we discussed we introduce another term called rotation, more formally it is called rigid body rotation. Why is it rigid body rotation; the body as a whole rotates which is something to similar to your rigid body mechanics which are studied earlier. The; when the body rotates there is no internal strain stress nothing it just rotates that is all that is why it is called Rigid body rotation.

Because the entire body rotates there is no relative movement between any two points in the body, which means there is no normal strain there is no shear strain which mean other way of putting it is the body does not undergo any deformation. When we say deformation what did we say change in length change in angle, no deformation at all which means that you need not worry about deformable bodies it is just a rigid body and that is why more formally called as rigid body rotation.

Remember sometime back we said translation that is also rigid body translation, the entire body translates just like I am moving to someplace it is rigid body translation. If I rotate that is the rigid body rotation, so that is why it is called rigid body rotation. Now we will have to relate this rigid body rotation to displacement field and that is what we are going to do now. That is easy because already we have got all the expressions at hand a small sign has to be incorporated, let us see how do we do that.

Now, rotation of line segment PQ, moment you talk about rotation remember sometime back we talked about moment balance immediately we chosen axis about which you would take a moment. Similarly here you are talking about rotation. So, when I say rotation what is axis about which I talk about rotation, in this case the plate lies in the xy plane. So, which means my axis of rotation is along the z axis.

So, rotation of line segment PQ about the z axis. What is that? that is α, that is the rotation of line element PQ and how do you represent,

$$(\omega_z)_{PQ} = \alpha = \frac{\partial u_y}{\partial x}$$

The nomenclature used is ω and it is about z axis. So, subscript z for the line element PQ, so the nomenclature for rotation is that ω and about z axis.

Remember this rotation, the way in which you written is also a point from expression. We have related α to this as a point form because we made $\Delta x \to 0$, $\Delta y \to 0$, $Q \to P$, and $R \to P$. So, this rotation also how do you imagine at a point once again imagine a small line segment and how much it rotates is given by $\frac{\partial u_y}{\partial x}$.

Now, let us focus on the line element PR once again about z axis. Now if you look at the rotation PQ as rotated in the anticlockwise direction but PR has rotate in the clockwise direction. When earlier we are interested only in the change in angle, so we did not assign any sense of rotation we are interested in $\alpha + \beta$ angle alone. But now because we want to quantify rotation we will have to assign a direction as usual anticlockwise is positive. So, there is no extra sign here but now when we write the rotation of element PR which rotates clockwise direction because clockwise is negative I add negative sign.

$$(\omega_z)_{PR} = -\beta = -\frac{\partial u_x}{\partial y}$$

We assigned sense of direction now, earlier we are interested only in the difference between PRQ and P*R*Q*. So, we are interest only in $\alpha + \beta$ we need not worry about sign, but now we are talked about rotation how P Q rotates how P R rotates, P R rotates in clockwise and P Q rotates anticlockwise. So, we start assigning signs.

Now, when we defined a shear strain what did we say, we take two line elements and then what is the change in angle. Similarly now the definition of this rotation is that

Average rotation of the element = Average rotations of the two perpendicular line segments

Earlier for shear strain we consider two line segments looked at the change in angle between them from initial to final state. And now we are looking at the average rotation of these two lines.

So, average rotation of the element for this element is equal to average of the rotations of the two perpendicular line segments. Earlier also we considered two perpendicular line segments and looked at the difference in angle but now we are looking at the average of the rotation of these line segments from the initial state to the final state. Take one rotation and take the other rotation find the average that is how we define rotation of the element.

$$\omega_z = \frac{1}{2}\left[(\omega_z)_{PQ} + (\omega_z)_{PR}\right] = \frac{1}{2}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right)$$

This gives you the rotation for the element when I say element the two dimensional element.

So, always we imagine two ways physically you can imagine as a plate or it is a two dimensional region inside a solid object, because it can vary from point to point. Now in terms of a notation we had two subscripts for shear strain γ_{xy}.

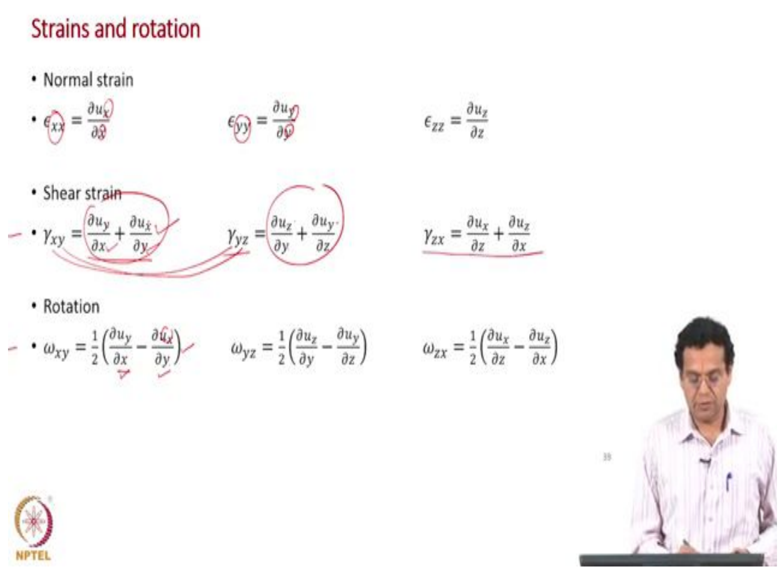
Now, to begin with we had ε_x and we want to have two subscript for normal strain also. So, we said ε_{xx}. Now similarly for rotation, the notation subscript is only one because it says rotation about the z axis what is the alternate way rotation in the xy plane. So, that every term we will have two subscripts. So, this tells you rotation in the xy plane.

$$\omega_{xy} = \omega_z = \frac{1}{2}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right)$$

The ω_z tells you rotation about z axis, equivalent way ω_{xy} is rotation in the x y plane. So, this is the nomenclature which we will use.

So, as we have done for normal strain sheer strain, we have related the rigid body rotation in terms of the displacement gradients once again these two gradients are very well known. They are known to us we already seen but subtraction and division by two use a physical significance namely the rotation.

(Refer Slide Time: 08:14)



So, let us summarize all of them and also write for other directions, that is the slide

- Normal strain: Ww discussed normal strain or derived normal strain for line element along x axis. Similarly, you can write for y axis and z axis; line elements along y axis z axis.

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x}; \qquad \varepsilon_{yy} = \frac{\partial u_y}{\partial y}; \qquad \varepsilon_{zz} = \frac{\partial u_z}{\partial z}$$

- Shear strain: we consider two elements which are along x and y axis, similarly we can consider along y and z axis, z and x axis. So,

$$\gamma_{xy} = \frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}; \qquad \gamma_{yz} = \frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z}; \qquad \gamma_{zx} = \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}$$

- Rotation:

$$\omega_{xy} = \frac{1}{2}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right); \qquad \omega_{yz} = \frac{1}{2}\left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right); \qquad \omega_{zx} = \frac{1}{2}\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right)$$

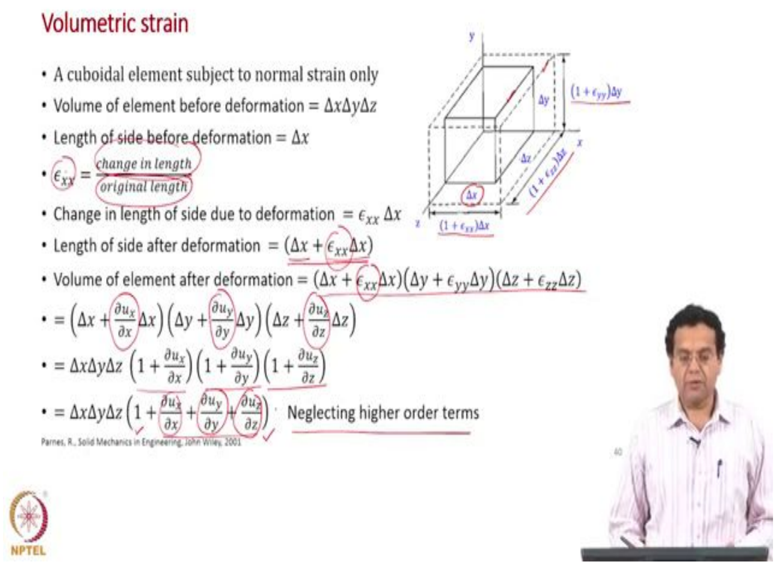
Just want to mention one thing in terms of remembering you always like to remember few expressions at least.

In terms of remembering shear strain and rotation, better to remember in terms of the derivative $\frac{\partial}{\partial x}$, we know they are cross derivatives. If you have x in the denominator you have u_y in the numerator. So, in terms of remembering, shear strain can be remembered anyway because it is plus sign, but remember rotation you need to remember the order the sequence is important.

So, one way to easily remember is start with $\frac{\partial}{\partial x}$, then next is $\frac{\partial}{\partial y}$. Moment you write $\frac{\partial}{\partial x}$, you know that numerator should be u_y. So, in that way you want make any mistake and of course, once you remember that shear strain is also followed in writing this way here also a only $\frac{\partial}{\partial x}$ is written first and $\frac{\partial}{\partial y}$ is written second.

So, both are analogous only sign changes different. So, that you can easily remember the expression for rotation and shear strain of course, you have a half here.

(Refer Slide Time: 11:07)



We have related normal strain, shear strain, rotation to the displacement gradients. Now we look at another strain which I have not defined we will define and also related to the displacement gradient that is called the volumetric strain. Just like the normal strain tells about change in length the shear strain tells about change in angle, rotation tells about rotation, volumetric strain tells about change in volume, in a 2D it will tell about change in area.

Now, to derive expression for that or to relate these two, we will consider cuboidal element the volumetric strain is change in volume original volume similar to our normal strain. Now; so, we will have to find expression for new volume and then old volume is cuboidal then we can find out expression of volumetric strain.

So, for deriving that we will consider cuboidal element as shown here subjected to normal strain only what does it mean; you have a cuboidal element and there is only change in length alone could be decrease or increase whatever can happen, but there is no change in angle. So, that is what is shown here, this is the cuboidal element. So, cuboidal element once again you can imagine a physical object or it is a 3D space inside a solid object.

So, Δx, Δy, Δz is the initial configuration or initial state and let us a represented by the continuous line, the dashed line represents the cuboid still it reminds a cuboid why is that; because there is only normal strain. So, it still remains a cuboid to begin with it was a cuboid and after let say subjected to normal stresses and so, on only normal strain happens, that is why in the final state represented by the dashed line it still remains a cuboid.

The lengths are different let see how do you find out the lengths. So, we are considering a cuboidal element of lengths Δx, Δy, Δz. The volume of element before deformation just a cuboidal volume. So, multiply all the lengths Δx, Δy, Δz. Now we have to find out the changed volume or the new volume or the volume in the final state how do you find out it remains a cuboid. So, if we find out the lengths we multiply all the lengths of the each of the side. So, length of the side before deformation is Δx.

Now, we have seen the normal strain has change in length by original length.

$$\varepsilon_{xx} = \frac{Change\ in\ length}{original\ length}$$

So, if you want to find out the change in length then you have to just multiply the ε_{xx} with the original length.

$$Change\ in\ length = \varepsilon_{xx}\,(original\ lenght)$$

$$Change\ in\ length = \varepsilon_{xx}\,\Delta x$$

So, that gives the change in length what is the new length and this change in length the old length. So,

$$length\ of\ side\ after\ deformation = \Delta x + \varepsilon_{xx}\,\Delta x$$

In this diagram it is shown that all the strains are positive because increase in length need not happen. Generally this ε_{xx} can be positive or negative automatically it will reflect a decrease in length or increase in length. In this case the diagram shows all lengths have increased. So, this is the new length the old length plus the change in length gives a new length.

Similarly along the y direction

$$length\ of\ side\ after\ deformation = \Delta y + \varepsilon_{yy}\ \Delta y$$

Similarly along the z direction,

$$length\ of\ side\ after\ deformation = \Delta z + \varepsilon_{zz}\ \Delta z$$

Now volume of element after deformation is just a product of all the lengths in the final state.

$$Volume\ of\ element\ after\ deformation = (\Delta x + \varepsilon_{xx}\ \Delta x)(\Delta y + \varepsilon_{yy}\ \Delta y)(\Delta z + \varepsilon_{zz}\ \Delta z)$$

Now, let us do some simplification, remember we are to relate with the displacement gradient because of physical significance we introduced ε_{xx}, now we will replace this ε_{xx} with $\frac{\partial u_x}{\partial x}$ similarly ε_{yy} and ε_{zz}.

$$= (\Delta x + \tfrac{\partial u_x}{\partial x}\ \Delta x)(\Delta y + \tfrac{\partial u_y}{\partial y}\ \Delta y)(\Delta z + \tfrac{\partial u_z}{\partial z}\ \Delta z)$$

Now let us take out Δx, Δy, Δz they are common

$$= \Delta x \Delta y \Delta z \left(1 + \tfrac{\partial u_x}{\partial x}\right)\left(1 + \tfrac{\partial u_y}{\partial y}\right)\left(1 + \tfrac{\partial u_z}{\partial z}\right)$$

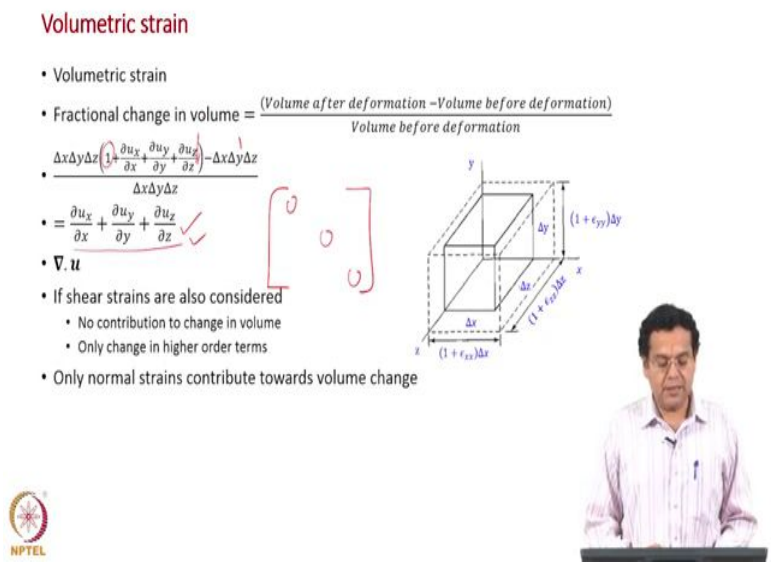
Now, we will have to multiply all this. Now when we multiply what we do is, we neglect all the terms which are squared, second order third order etcetera why do we do that? Remember all our discussions are under the infinitesimal strain theory under that assumption the normal strains are very small let say 10^{-6} compare to that when your square becomes 10^{-12} So, which is negligible.

$$= \Delta x \Delta y \Delta z \left(1 + \tfrac{\partial u_x}{\partial x} + \tfrac{\partial u_y}{\partial y} + \tfrac{\partial u_z}{\partial z}\right)$$

So, we neglect all the higher ordered terms and only retain the first order terms You will have terms which are products of these derivatives. In fact, product of all the derivatives all those are neglected second order terms onwards we are neglected only the first order terms are retained.

So, in what is that we have found out? The volume of the element after deformation has been found out.

(Refer Slide Time: 17:39)



Now we can find out the expression for volumetric strain; volumetric strain how do you define?

$$Fractional\ change\ in\ volume = \frac{Volume\ after\ deformation - Volume\ before\ deformation}{Volume\ before\ deformation}$$

Analogous to our normal strain definition new length minus old length divided by old length similarly here volume after deformation minus volume before deformation divided by volume before deformation.

Now, let us express them in terms of our variables, you just now found out what is the volume after deformation

$$= \frac{\Delta x \Delta y \Delta z \left(1 + \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) - \Delta x \Delta y \Delta z}{\Delta x \Delta y \Delta z}$$

So, now, $\Delta x \Delta y \Delta z$ cancels out and finally we have

$$= \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}$$

This can be represented in terms of our gradient vector as divergence of the displacement field.

$$\nabla .u = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}$$

So, once again we have come across these terms of just gradients earlier and in terms of our displacement gradient tensor, these are the diagonal elements. When you sum up you get a nice physical significance tells you the fractional change in volume. So, $\nabla .u$, the divergence of the displacement field, one way is look at a divergence of displacement field other ways look at it as tells you the fractional change in volume.

Now, when we started the derivation, we said there are only normal strains what happens if you include shear strain ok? If shear strains are also considered no contribution to change in volume still you will get the same expression the reason is they will only contribute to the higher order terms we are not discussing that.

If you include shear strains meaning change in angle the final result will still be same, there is a fractional change in volume will still be same reason is you will have terms here additional terms because of the change in angle, but all of them will be higher ordered terms only that is why it will not contribute towards the final expression.

Conclusion is that, only normal strains contribute towards volume change within our infinitesimal strain theory. Within our assumption if I have an object and there is a change in angle volume still remain same but obviously, there is change in length there is going to be change in volume. Conclusion is that only normal strains contribute towards volume change.

So, what we have done here is, introduced another strain namely volumetric strain and which is fractional change in volume and derive an expression relating volumetric strain to the displacement gradients, they are nothing but this is sum of the diagonal elements of the displacement gradient tensor.