Continuum Mechanics And Transport Phenomena
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Lecture - 56
Strain Displacement Gradient Relation : Normal and shear strain

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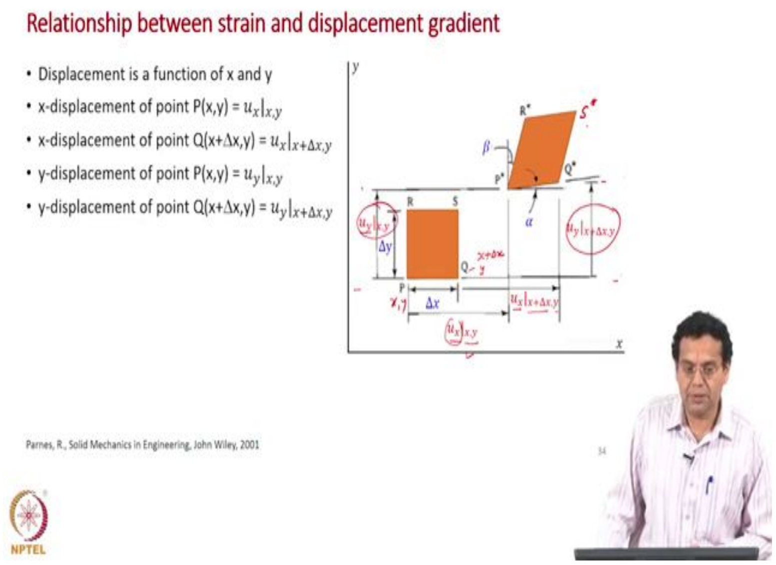

Moment you, let say type in your Google Strain Displacement Gradient this figure is the most popular figure you will come across, same figure has been taken from Ryman Parnes and let us understand this gradually. The x axis and y axis are shown and then you have plate PQRS is shown, in the initial state undeformed configuration. Now two ways of visualizing this; one is as I told you a plate, easy you to understand, other way should understand is it is a square region inside a solid object.

Because you will see, as we go along we will make the plate to a point as usual, so then of course, you can not say it is the plate. So, two ways of imagining, it is a plate to visual observation or a 2 dimensional region in a solid object and that is denoted by P Q R S, the length of the sides are Δx, Δy. Now this plate is subjected to a force and it has taken a final state denoted by P*R*Q*, S* is not denoted because it is not important for us. So, PQRS plate has become P*R*Q*S*. If you compare the final state and initial state easily can understand that, there is change in length, PR is not same as P*R* that is normal strain.

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Similarly P*Q* and versus PQ, look at the angle there is change in angle, look at the rotation there is rotation, what about translation yes there is translation. So, you consider the most generic configuration; the initial plate has undergone translation, is undergone rotation also, there is change in length also, change in angle also, that is what we want to represent.

Now coming back to the description, let us focus on P and P*; P has move to P* or more formerly P has got displace to P*. How do you represent? P and P* are relating x displacement. So, if you look at this one, it says the distance between P and P* is x displacement; and then that depends on remember u_x, u_y depends on x and y, so it is evaluate that x comma y.

This nomenclature we have seen earlier for our derivations. The vertical bar tells you that we are evaluating at a particular location we not come across two subscripts; but now u_x is evaluate at x, y. Repeat again P has got displaced to P* that displacement is u_x; but that depends on where P is present, the coordinates of P are (x, y), so u_x evaluated at (x, y). Now what about Q? Because the length of PQ is Δx, the x coordinate of Q is $x + \Delta x$.

So, this coordinate if you mention it is $x + \Delta x$, of course same y, y coordinate is y. So, Q has got displaced to Q* and x displacement is u_x, evaluated at ($x + \Delta x$, y); same like as we have done for a control volume, one phase is x other phase is $x + \Delta x$. Similarly here looking at the two points, so one tells you displacement at (x, y), other tells you displacement at ($x + \Delta x$, y).

Now P* and Q* have undergone vertical displacement also, now we are looking focus only on the horizontal displacement. Now let us focus on the vertical displacement, P has move to P* and that is what is denoted here as vertical displacement u_y, once again evaluated at (x, y). So, u_y vertical displacement, that is the vertical distance between p and p star and evaluated at (x, y).

Now Q has also shifted to Q*, or moved to Q* vertically displaced. The vertical displacement is u_y; but where is it evaluated now, ($x + \Delta x$, y), , so u_y evaluated at ($x + \Delta x$, y). So, we have completely described all the nomenclature shown, one thing is left I will describe that as well. What else you shown? The angle P*Q* makes horizontal is α; which means that the element P Q has undergone a small rotation of α, similarly PR is undergone a small rotation of β to P*R*.

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So, all the nomenclature has been discussed just to summarize plate PQRS in an initial state goes to P*R*Q*S* in the final state. And we have discussed x displacement of P, x displacement of Q, and then similarly y displacement of P and Q and also we discussed what does alpha and beta represent. With this nomenclature let us proceed and that is what is shown here,

- Displacement as the function of x and y,
- x displacement of point P(x, y) = $u_x|_{x,y}$,
- x displacement of point Q ($x + \Delta x$, y) = $u_x|_{x+\Delta x,y}$.
- Now talking about y displacement of point P (x,y) = $u_y|_{x,y}$
- y displacement of point Q ($x + \Delta x$, y) = $u_y|_{x+\Delta x,y}$

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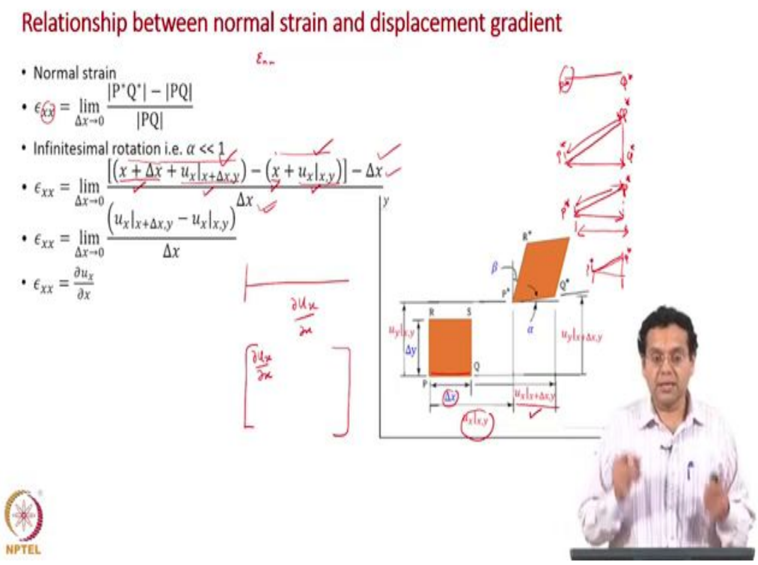

Now we are proceeding towards deriving a relationship between normal strain and displacement gradient. Now let us start with the definition of normal strain.

$$\varepsilon_{xx} = \frac{|P^*Q^*| - |PQ|}{|PQ|}$$

The definition of normal strain is change in length by original length. We define introducing ε_{nn} , the way to begin with we said ε_{nn} . Now our direction is along x axis, because we are going to consider the line element PQ. So, that is why this is ε_{xx} , normal strain for nine element oriented along x axis.

Now, what is the final length? Final length is the length of P* and then Q*. So, length of P*Q*, and the initial length is the length of P Q, divide by initial length PQ, there is something known to us. But now this change in length by original length in the limit of $\Delta x \to 0$. That is why I told you two ways of imagining; one is the imagining as a plate easy to visualize, other one is a 2 D region inside a solid object. And as usual we want a local point form of a relationship between normal strain displacement gradient any relationship should be valid every point, that is why the relationship which I going to derive is where every point inside a solid object.

Now, so we have to evaluate this change in length over original length in the limit of $\Delta x \to 0$. Now I will make an assumption, infinitesimal rotation what do I say PQ slightly rotates very small, extremely small to P*Q*. What happens, in that case instead of taking this length P*Q*, I will take the projected length of P*Q*. So, I will repeat again, actually I supposed to take the actual length of P*and then Q*; because the rotation is very very small I am not going to take this actual length, but I will project this Q* and I will consider only this length as same as the original length, which is very much valid and the $\alpha \ll 1$, is very small, that is the assumption.

Infinitesimal rotation that is $\alpha \ll 1$. Now what is the length of projected P*Q*, please keep that in mind, we are going to write this that is my final length to some approximation. So, let us write down

$$\varepsilon_{xx} = \frac{\left[\left(x+\Delta x+u_x|_{x+\Delta x,y}\right)-\left(x+u_x|_{x,y}\right)\right]-\Delta x}{\Delta x}$$

Now, let me now draw the project P*Q*, what is this length of this line, difference x coordinate that is all; x coordinate of Q* minus x coordinate of P*. What is x coordinate of Q*, it is a original x coordinate plus the displacement; remember that is what we did in the earliest slides to find out x coordinate of C, what did you do one plus displacement of that point. Similarly here it is original x coordinate, what is that of Q* the original x coordinate is $x + \Delta x$ to that you add the displacement of the point Q.

What is displacement of point Q; $u_x|_{x+\Delta x,y}$. So, if you add these two you will get the final x coordinate. Now for point P, it has moved from P to P*; now we have to find out the x coordinate of P*. How do you do that, the original x coordinate plus the displacement of point P; what is the displacement $u_x|_{x,y}$.

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So, if you add these two you will get the new x coordinate of P*. And of course, so when you subtract these two you will get the length of approximated P*Q*, the projection of P*Q*. So, let us simplify this,

$$\varepsilon_{xx} = \frac{\left[u_x|_{x+\Delta x,y} - u_x|_{x,y}\right]}{\Delta x}$$

Now we will take limit $\Delta x \to 0$, what happens this becomes a partial derivative

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x}$$

We have found a relationship between strain and displacement gradient. Remember, when we took the one dimensional example we had a wall, we attach the wire, stretched it etcetera; the one dimensional displacement was $\frac{\partial u_x}{\partial x}$.

What is the alternate way of interpreting because one of the components of the displacement gradient tensor.

But now look at the physical significance, because it has got a main physical significance saying that normal strain. What does it mean, remember all are point definitions, at a point take a small length and based on this application of force ε_{xx} *or* $\frac{\partial u_x}{\partial x}$ tells you the fractional change in length of that line segment at that particular point.

Once again as I told you, little difficult to imagine, at a point imagine a small very very small line segment; what does it is final length minus original length divide by original length is represented by $\frac{\partial u_x}{\partial x}$ which we call as normal strain. So, we are going to a physical significance were a simple derivative, partial derivative $\frac{\partial u_x}{\partial x}$; this is clear, then we can go at other things quickly. So, now whenever you look at $\frac{\partial u_x}{\partial x}$ immediately what should come to your mind is normal strain, different representation is there different view point.

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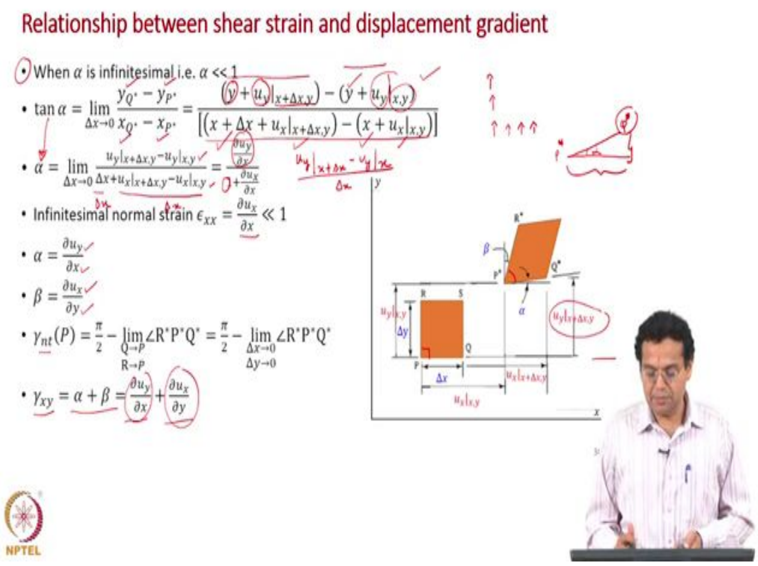

Now, let us there are relationship between the shear strain and displacement gradient. Now, once again same assumption we make α is infinitesimal very small $\alpha \ll 1$; how do you write, tan(α), opposite side by adjacent side. So, let me draw that P*Q* so, opposite side by adjacent side;

$$\tan tan\ \alpha = \frac{y_{Q^*} - y_{P^*}}{x_{Q^*} - x_{P^*}} = \frac{\left(y + u_y|_{x+\Delta x,y}\right) - \left(y + u_y|_{x,y}\right)}{\left[\left(x + \Delta x + u_y|_{x+\Delta x,y}\right) - \left(x + u_y|_{x,y}\right)\right]}$$

So, this denominator is not new to us, $x_{Q^*} - x_{P^*}$ is nothing, but the length of the projected in our earlier case; projected P*Q* which I have already evaluated just for revision Q* is a new position. So, the old x coordinate $x + \Delta x + u_y|_{x+\Delta x,y}$, that gives the x coordinate of Q*, x coordinate of P* is $x + u_y|_{x,y}$.

Now let us proceed further, because we said α is infinitesimal, I am approximating

$$\tan tan\ \alpha = \alpha$$

So,

$$== \frac{\frac{\partial u_y}{\partial x}}{1 + \frac{\partial u_x}{\partial x}}$$

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Now, second assumption infinitesimal normal strain, you have already seen that $\frac{\partial u_x}{\partial x}$ represents normal strain. I will assume that the normal strain is much smaller than 1, $\frac{\partial u_x}{\partial x} \ll 1$. So, the denominator you can neglect $\frac{\partial u_x}{\partial x}$, denominator becomes just unity. So,

$$=$$

Change of y displacement along x direction. Y displacement is vertical; but now what we are $\frac{\partial u_y}{\partial x}$ tells you, how this vertical displacement changes in the x direction ok, that is the meaning of $\frac{\partial u_y}{\partial x}$. So, the way in which the vertical displacement changes along the x direction, we going to got a physical meaning for that which tells you the rotation of the element P Q.

Now, repeat the same thing you will get for β as

$$\beta = \frac{\partial u_x}{\partial y}$$

The displacement is along y direction; the derivative is along with respect to x direction; here β the displacement along x direction, the variation is along the y direction they are the other way round unlike the normal strain. Normal strain see here, you are saying what is the change of x displacement in x direction; but α and β you are looking at y displacement along x direction, x displacement along y direction.

Now, we are now in the process of deriving expression for shear strain. So, let us write the definition for shear strain

$$\gamma_{nt}(P) = \frac{\pi}{2} - \angle R^*P^*Q^* = \frac{\pi}{2} - \angle R^*P^*Q^*$$

So, this angle 90 degrees minus this angle $R^*P^*Q^*$ not for a finite value of Δx Δy, but the limit of $\Delta x \to 0$, $\Delta y \to 0$ and that angle is $\alpha + \beta$. So,

$$\gamma_{xy} = \alpha + \beta = \frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}$$

We have got an expression for shear strain which is the change in angle. So, once again just like we discuss for normal strain, these gradients are familiar to us. In the 2 dimensional case though of course, we did not sum up and discuss, separately we saw this gradient. But now you see, when you sum up you got a good physical significance; what is the physical significance at a point imagine too small lines and what is the change in angle from initial

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state to final state, that is what you obtain by summing these two derivatives. Look at them as derivatives more mathematical, meaning change of u_y with x, u_x with y; but when you sum up you get the shear strain a more physically meaningful quantity.

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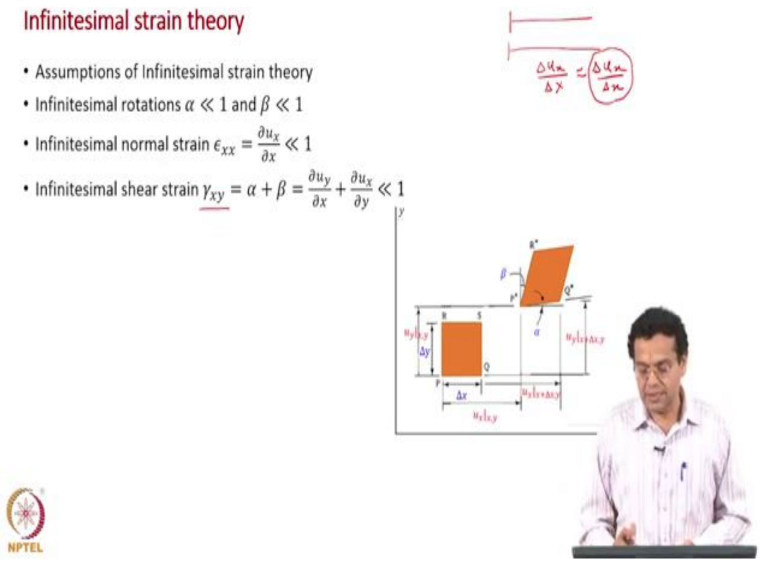

Now, we made a few assumptions, let us summarize all of them and this summary as a formal terminology called infinitesimal strain theory. So, we are discussing solid mechanics strain and solid mechanics. We will discuss under this assumption, remember sometime that we discussed, we had a wire or thread and then we said it undergoes a small displacement or a small strain, under that condition we said

$$\frac{\Delta u_x}{\Delta X} = \frac{\Delta u_x}{\Delta x}$$

And then we said we are going to work with $\frac{\Delta u_x}{\Delta x}$ and it is valid under the assumption of a small displacement or small strain; this also is a line with that all these are called infinitesimal strain theory.

Now, it is an assumption in the case of solid mechanics, why when you subject a solid to a force it can undergo a finite strain; we said most of the time it is very very small; but it can be a large value also. We assumed it a that they are very very small, that is why it is an assumption in solid mechanics. Later on you will see that when you go to fluid mechanism when you carry over all that it is not an assumption in the case of fluid mechanics; it is an assumption because of solid mechanics.

And remember we are discussing solid mechanics under that assumption, because we have come to solid mechanics we take some concepts to fluid mechanics. So, we are not discussing in general finite strain theory, we are only discussing the part of it which is infinitesimal strain theory; so that I can take over these concepts to fluid mechanics and that is not assumption there, you will understand that later.

So, that is the scope of this discussion. If you look at very rigorous solid mechanics book, they will discuss both finites strain theory and infinitesimal strain theory. We are not discussing finite strain theory, that will not give us linear expressions, that will results in non-linear expressions; that is beyond the scope of our discussion, not require also, just to learn whatever is required whatever we can take over to fluid mechanics that is. So, what are assumptions of infinitesimal strain theory?

- First, we said infinitesimal rotations, $\alpha \ll 1$; similarly $\beta \ll 1$
- Then we said, when we derived expression for shear strain, what did we say infinitesimal normal strain; which means that $\varepsilon_{xx} = \frac{\partial u_x}{\partial x} \ll 1$
- And then because $\alpha \ll 1$; $\beta \ll 1$; obviously, the sum $\alpha + \beta \ll 1$, which means that the shear strain $\gamma_{xy} = \alpha + \beta = \frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y} << 1$.

So, this complete set of assumptions is called infinitesimal strain theory. So, all our discussions are within these assumptions, it is not a limitation at all for us. We can straight away apply all these for fluid mechanics; let that we will see much later.

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