Continuum Mechanics And Transport Phenomena
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Lecture - 54
Displacement Field and Displacement Gradient – Part 2

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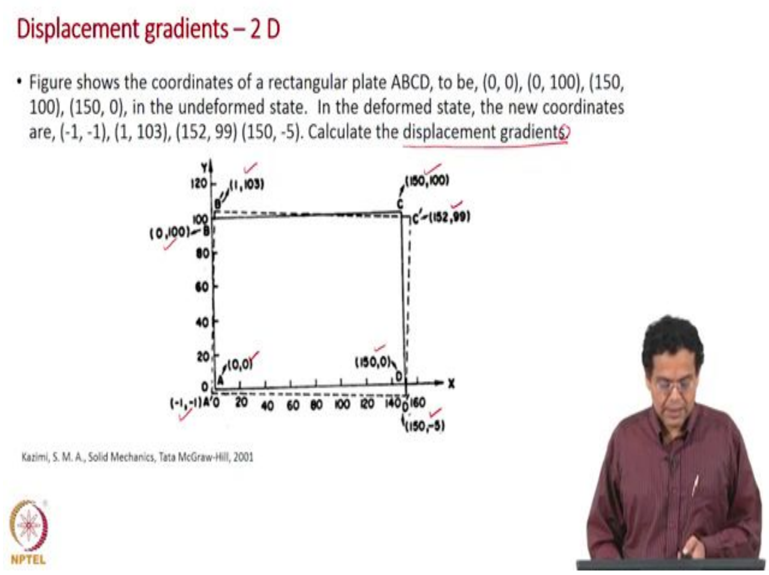

Now we will take an example and then evaluate the Displacement Gradient, we are seeing Displacement Field. Now we will evaluate displacement gradient for the two dimensional example. This example is from the solid mechanics book by Kazimi.

So, you have a plate and then the coordinates are shown before deformation and after deformation, before applying the force after applying the force or initial state final state. Figure shows the coordinates of rectangular plate A, B, C, D to be (0, 0) and then (0, 100) and then (150, 100) and then (150, 0) in the undeformed state. In the deformed state the new coordinates are shown $(-1\ -1)$ and then (1, 103) and then (152, 99) and then $(150,\ -5)$. We are asked to calculate that displacement gradients. Let us do Let us see how do we calculate.

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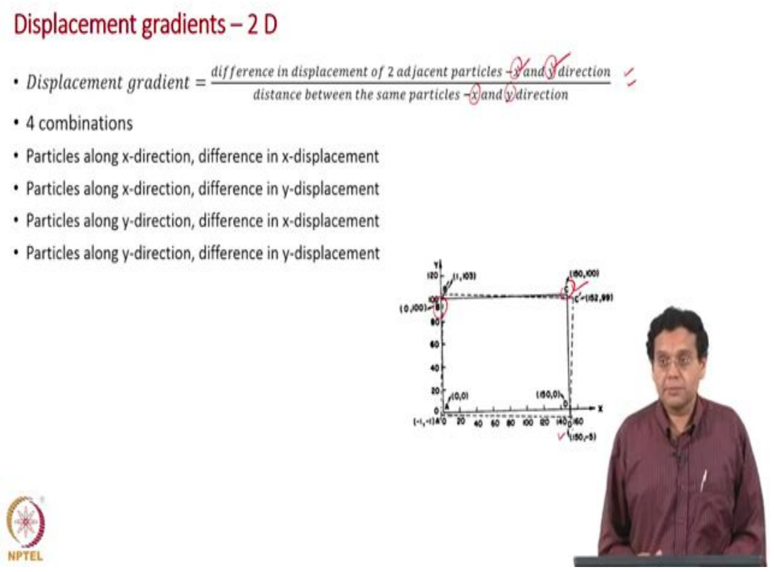

Now, let us start with the definition of the displacement gradient which I have seen in few slides ago, few slides back we are seeing that definition of displacement gradient. We said

$$Displacement\ gradient = \frac{difference\ in\ displacement\ of\ two\ adjacent\ particles}{distance\ between\ the\ same\ particles}$$

Now, earlier we considered only one direction, so there was no mention of direction at all it understood we are working in x direction only, but now the two particles which you consider can be along x direction can be along y direction. And for particle considered along x and y direction you can find difference in displacement along x direction and y direction. That is what is shown here.

So, the denominator tells the direction along which are going to consider two particles. For example, let us say x direction let us say we consider the particles B and C, if you consider B and C for these two particles I can find out difference in the x displacement and y displacement. That gives two possibilities for the numerator.

Now, we can consider two particles along y direction, for example C and then D. And considering these two particles I can calculate that difference and displacement along x and y direction once again you get another two set values. So, finally, we have four combinations of displacement gradient: two for that direction along which are considering the particles, two for the direction of displacement.

So, let us write here

Particles along x direction for them difference in x displacement.

Particles along x direction once again, but now difference in y displacement.

Similarly particles along y direction, difference in x displacement.

Particles along the y direction once again, but different y displacement.

So, numerator x and y direction denominator x and y direction. So, you can calculate four displacement gradients. Let us do that now.

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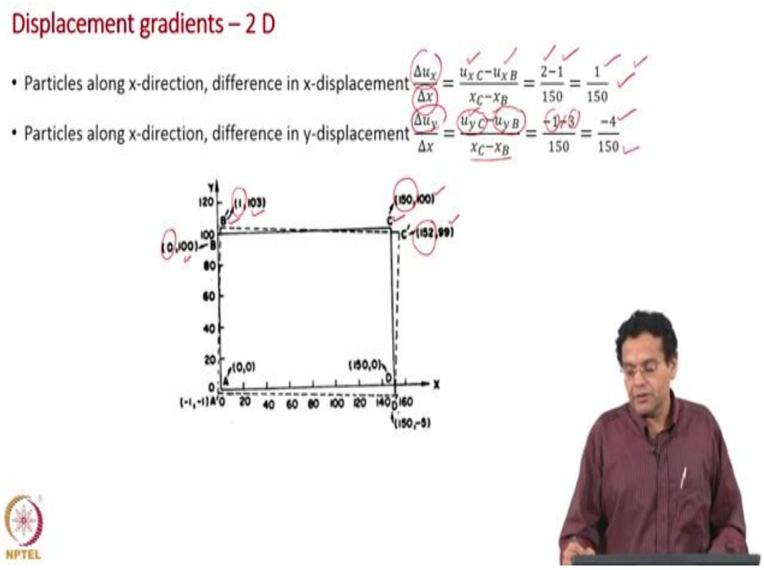

Now, we will first consider particles along the x axis. In the present case you are going to consider particles B and C. So, particles along x direction. Now, if you consider and then first we are going to consider difference in x displacement. How do you write formally,

$$Particles\ along\ x\ direction,\ difference\ in\ x\ displacement \frac{\Delta u_x}{\Delta x} = \frac{u_{xC} - u_{xB}}{x_C - x_B}$$

Remember u x tells displacement of a particle, we are going to consider two particles and look at the difference between them. That is why it says $\frac{\Delta u_x}{\Delta x}$.

Let us consider particles C and then B which are along the x axis. Now, it says difference in displacement which means that x displacement of particle C minus x displacement of particle B, and of course difference in their x coordinates. Now, let us take particle C, x displacement is nothing, but a difference in x coordinate, the final x coordinate is 152, initial x coordinate is 150. So, the displacement of particle C is 2 units.

$$u_{xC} = 152 - 150 = 2$$

Now, let us come to particle B. The final x coordinate is 1, initial x coordinate is 0, so that this x displacement of particle B is 1 unit.

$$u_{xB} = 2 - 1 = 1$$

And, the difference in x displacement

$$\Delta x = 150$$

So,

$$\frac{\Delta u_x}{\Delta x} = \frac{u_{xC}-u_{xB}}{x_C-x_B} = \frac{2-1}{150} = \frac{1}{150}$$

Now, we are consider the same particles along x direction, but now look at the difference in y displacement, and I say y displacement it is change in y coordinate. So, how do I write

$$\frac{\Delta u_y}{\Delta x} = \frac{u_{yC}-u_{yB}}{x_C-x_B}$$

Now, let us look at particle C. Now, the y coordinate has moved from 100 to 99. So, the y displacement of particle C is

$$u_{yC} = 99 - 100 =- 1$$

Look at particle B, 100 has become 103. So, the y displacement is

$$u_{yB} = 103 - 100 = 3$$

So,

$$\frac{\Delta u_y}{\Delta x} = \frac{u_{yC}-u_{yB}}{x_C-x_B} = \frac{-1-3}{150} =- \frac{4}{150}$$

The denominator is once again 150. Same particles are considered along the x axis the distance between them is 150.

So, we are considered two particles along x axis B and C, looked at their difference in displacement along x axis, and difference in their y displacement.

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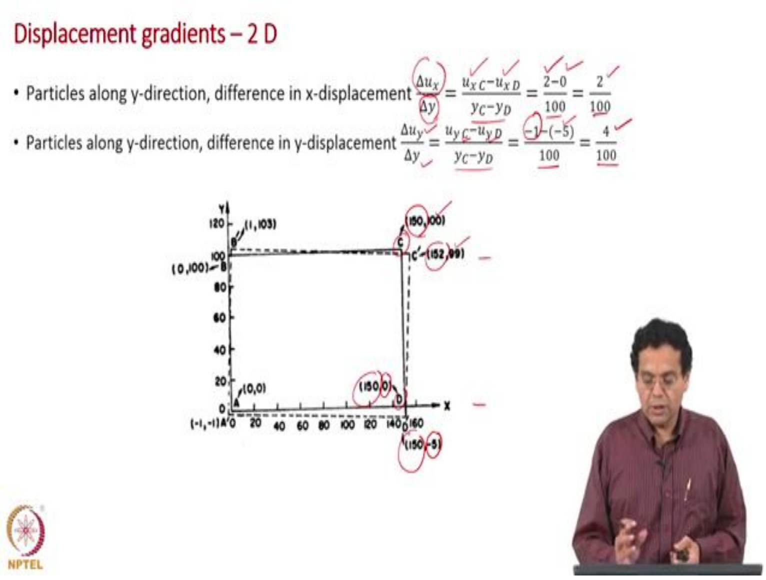

Now, just we will repeat this: by taking two particles along the y axis we are going to consider particle C and particle D. So, particles along the y direction: first we will consider difference in their x displacement. How do you represent formerly?

$$\frac{\Delta u_x}{\Delta y} = \frac{u_{xC} - u_{xD}}{y_C - y_D}$$

Now, let us look at point C. Now, for this case we focus on the x displacement. So,

$$u_{xC} = 152 - 150 = 2$$

Now, coming to point D look at the x displacement

$$u_{xD} = 150 - 150 = 0$$

Now, the vertical distance between the two particles, $\Delta y = 100$. So,

$$\frac{\Delta u_x}{\Delta y} = \frac{u_{xC} - u_{xD}}{y_C - y_D} = \frac{2-0}{100} = \frac{2}{100}$$

Now, the last combination considering the same particles along the y direction and looking at the difference in the y displacement y displacement. So, how do you represent formerly?

$$\frac{\Delta u_y}{\Delta y} = \frac{u_{yC} - u_{yD}}{y_C - y_D}$$

Now, let us consider particle C what is the y displacement of C

$$u_{yC} = 99 - 100 =- 1$$

Now, let us come to point D

$$u_{yD} =- 5 - 0 =- 5$$

So,

$$\frac{\Delta u_y}{\Delta y} = \frac{u_{yC} - u_{yD}}{y_C - y_D} = \frac{-1-(-5)}{100} = \frac{4}{100}$$

So, we got now four values of displacement gradient let us arrange them.

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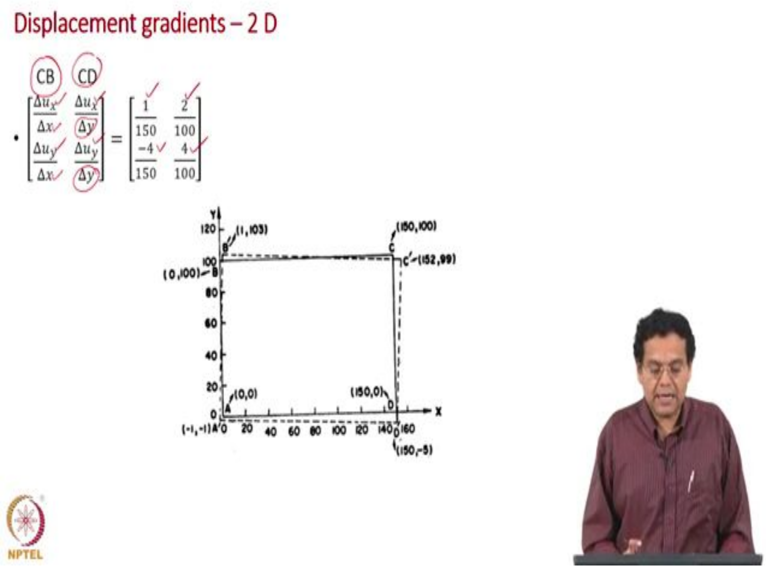

The way in which we have arranged is; first we consider particles along the x axis which were C and B the that is denoted by CB here, we calculated two gradients for them. Then we consider particles along the y axis C and D.

$$\begin{bmatrix} \frac{\Delta u_x}{\Delta x} & \frac{\Delta u_x}{\Delta y} & \frac{\Delta u_y}{\Delta x} & \frac{\Delta u_y}{\Delta y} \end{bmatrix} = \begin{bmatrix} \frac{1}{150} & \frac{2}{100} & -\frac{4}{150} & \frac{4}{100} \end{bmatrix}$$

So now, I have arranged taken those values arranged column wise. Why specifically column wise we will understand later, right now I have taken all the numerical values from the previous slides and then filled up here. So now, what is that we have seen? We have seen a displacement field in 2 D and then also discussed displacement gradients for 2 D. Because of two directions along which I can consider particles two directions along which I can consider displacement four displacement gradients can be found out. And that is what we have done now.

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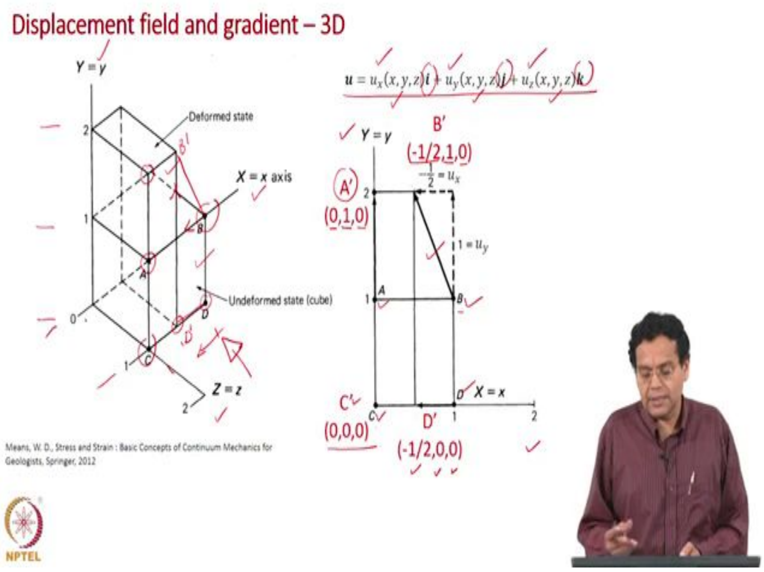

Now, let us move to three dimensional case. A very good representation, a very good example from stress and strain. Now the initial configuration or the undeformed state is a cube. You applied a force and it has become a cuboid like this ok. So, along the x direction the length has decreased; along in the x direction original length has decreased.

Now, along the y direction the original length of this much, the final length is become twice of that. Along the z direction no change at all. So, the cube has become cuboid with decrease in length along x axis increase in length along y axis, that is what has happened.

Now, what is shown on the right hand side is the side view that is why you have x axis here and then on the y axis here. Now, what are the things shown here in this diagram let us go one by one. The initial position of A, B, C, D are shown the initial configuration. Now, the same A, B, C, D are shown in the final configuration also as A'.

Now, let us come to B; So, it has moved in the B' direction and this arrow mark represent the direction. B is the original position and B' is the final position. Now, C has not changed at all, C and C' are same. Now let us go to D; D has moved along the direction and this is D' that is what is shown in the figure, this arrow mark shows that D has moved along the x axis; of course along the negative x axis by half a unit. So, this is D dash.

Now, what else is shown? Below the new position three values are shown. What are they? They represent displacement of these A, B, C the particles along x, y, z direction earlier we are gradually moved from x to x and y now to x, y, z. So, the numbers below the new coordinates represent the displacement of the particles A, B, C, D, because it can happen in three directions three values are given.

So, let us take the case of A; there is only vertical displacement along the y direction by 1 unit. So, you have value 1 here, no displacement along x direction, no displacement along z direction. So, it indicated as (0, 1, 0).

Now, let us take B; B has moved, but in terms of coordinate along coordinate direction it has moved by half a unit along negative and the negative direction along x axis and 1 unit along positive y axis. So, it indicated as (-1/2, 1, 0).

Now, coming to C to C' no movement at all. So, displacement 0 along all the three directions.

Coming to D to D' it has moved by half a unit in the negative x direction and there is no movement along y axis z axis, and that is why it indicated as (-1/2, 0, 0).

So, we are shown a three dimensional geometry initial configuration final configuration, you are shown one view of that the particle showing A, B, C, D; initial position, final position, arrow marks going from initial to final position and the displacements of the particles as well. Now, with all this information we will find out the displacement gradient.

Before that, how do we represent displacement field in three dimensional case;

$$u = u_x\,(x, y, z)\,i + u_y\,(x, y, z)\,j + u_z(x, y, z)k$$

The displacement in the x, y, z directions can be function of x, y, z. And that is what we have seen here as well. The displacements whatever numbers shown can be functions of x, y z, and it may be a constant, but in general they are functions of x, y, z.

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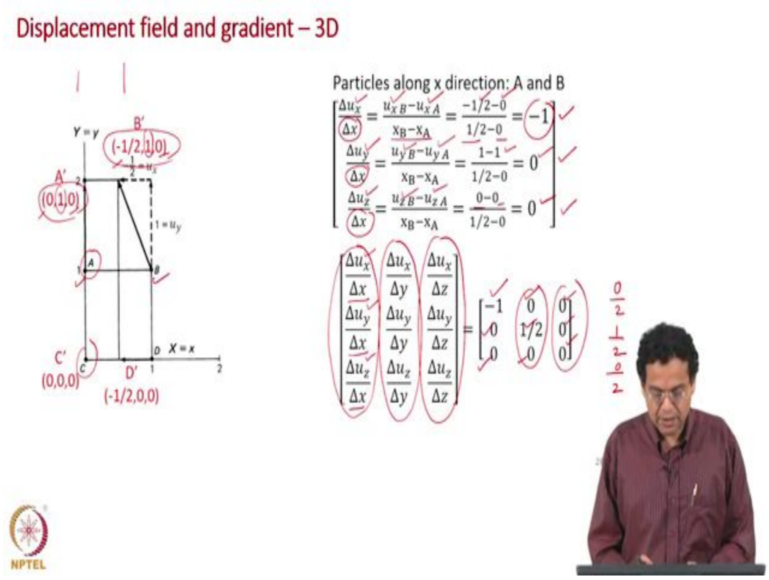

Now, we will have to find out the displacement gradient. So, how many entries do we expect? 9 entries we expect. Three directions along which you can consider the particles and for every direction I can consider their x displacement, y displacement, z displacement, difference in their this x, y, z displacements.

So, let us take first particles along x direction which means I am considering particles A and B. Now, having considered particles along A and B I am going to consider their difference in x displacement, y displacement, and z displacement. So, if you look at the denominator it is always Δx, because we are considering particles along x axis. Numerator difference in that x displacement, y displacement, z displacement.

$$\frac{\Delta u_x}{\Delta x} = \frac{u_{xB}-u_{xA}}{x_B-x_A} = \frac{(-\frac{1}{2}-0)}{\frac{1}{2}-0} = -1$$

$$\frac{\Delta u_y}{\Delta x} = \frac{u_{yB}-u_{yA}}{x_B-x_A} = \frac{1-1}{\frac{1}{2}-0} = 0$$

$$\frac{\Delta u_z}{\Delta x} = \frac{u_{zB}-u_{zA}}{x_B-x_A} = \frac{0-0}{\frac{1}{2}-0} = 0$$

Now, remember we said these numbers represent that displacement of the particles along the respective directions x, y, z direction. So, to evaluate the numerator it is enough if you look at the difference in the displacement along with respective directions.

Just let me repeat: we are considering particles A and B for the first entry we are looking at the difference in x displacement. And the three numbers here below the final coordinate tells you that displacement of the particles. We will have to look at the difference in displacement that is all is required. For the first entry we will have to look at the difference in x displacement.

So, let us complete the matrix. Now, there will be 9 entries each column corresponds to the direction in which we are considering particles and each row corresponds to the direction of displacement.

$$\left[\frac{\Delta u_x}{\Delta x}\ \frac{\Delta u_x}{\Delta y}\ \frac{\Delta u_x}{\Delta z}\ \frac{\Delta u_y}{\Delta x}\ \frac{\Delta u_y}{\Delta y}\ \frac{\Delta u_y}{\Delta z}\ \frac{\Delta u_z}{\Delta x}\ \frac{\Delta u_z}{\Delta y}\ \frac{\Delta u_z}{\Delta z}\right] = \left[-1\ 0\ 0\ 0\ \tfrac{1}{2}\ 0\ 0\ 0\ 0\right]$$

So now, we have filled up all the 9 entries, it was very obvious that because of 3 D we will have 9 combinations and we have found out all the 9 displacement gradients.

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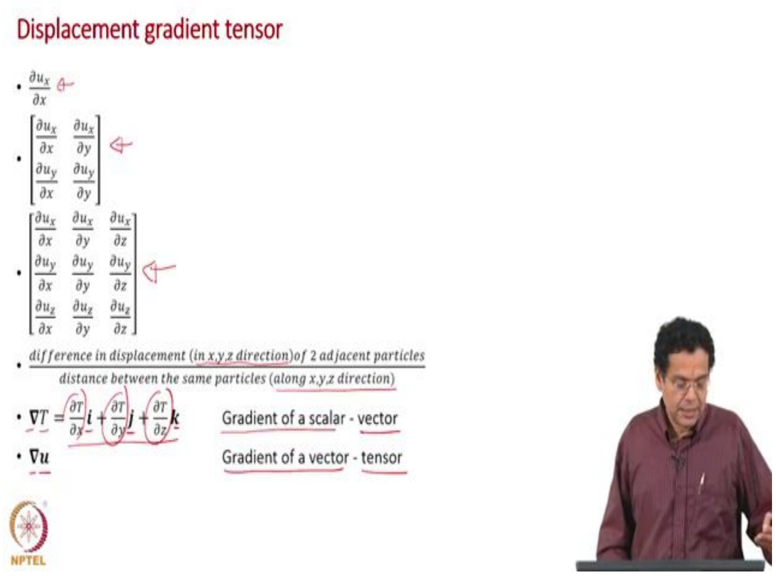

Now, you would have easily sensed that you have been indicating two directions all the time. Any physical quantity which requires two direction to describe is a tensor. So, we are introduced in fact very gradually a displacement gradient tensor. So, initially we got it was 1D it looked like a partial derivative for you as.

- $\frac{\partial u_x}{\partial x}$

We slowly move on to 2 D and then we wrote

- $\left[\frac{\partial u_x}{\partial x}\ \frac{\partial u_x}{\partial y}\ \frac{\partial u_y}{\partial x}\ \frac{\partial u_y}{\partial y}\right]$

Now, the just previous slide we have moved to the three dimensional case and we have got 9 entries for the displacement gradient tensor.

- $\left[\frac{\partial u_x}{\partial x}\ \frac{\partial u_x}{\partial y}\ \frac{\partial u_x}{\partial z}\ \frac{\partial u_y}{\partial x}\ \frac{\partial u_y}{\partial y}\ \frac{\partial u_y}{\partial z}\ \frac{\partial u_z}{\partial x}\ \frac{\partial u_z}{\partial y}\ \frac{\partial u_z}{\partial z}\right]$

What are the two directions? As I told you one direction is for the particles along which direction we are considering and one direction for the displacement. Now, just understand this of course, we know that in using the definition which introduced earlier for displacement gradient we said difference in displacement of two adjacent particles divided by the distance between the same particles. We have considered particles along x, y, z direction that is one direction, we have considered displacement in x, y, z direction for those particles that gives another direction. So, we have two directions to represent each entity in this matrix. And hence it is a displacement gradient tensor.

Now, this displacement gradient tensor is similar to a gradient of a temperature, in terms of physical significance. What does gradient of a temperature tell you, how do you represent first of all? Temperature of the scalar, you have a gradient of a temperature moment to take a gradient of a scalar it becomes a vector because temperature field is there moment you tell gradient if you attach a direction temperature varies along x axis in this way, temperature varies along y axis in this way with some magnitude or some value minus or positive.

$$\nabla T = \frac{\partial T}{\partial x} i + \frac{\partial T}{\partial y} j + \frac{\partial T}{\partial z} k$$

So, moment you take a gradient you attach a direction to temperature. So, temperature field is scalar but now I say rate of change of temperature along x axis, rate of change of temperature along y axis so you attach a direction.

So, gradient of a scalar is a vector physically why, scalar has no direction moment you take a gradient attach a direction moment you attach a direction becomes a vector. Now if you extend analogously to this displacement what you have got is a gradient of a displacement (∇u), displacement itself is a vector.

Now, we are going to say how displacement varies along x axis, y axis, z axis instead of saying this how displacement varies along x axis, y axis, z axis we said two particles two points etcetera without saying this. Now, because we know formerly we can use very formal statement.

Let me repeat: displacement itself is a vector we are considering gradient of displacement. So, we are attaching a direction on a on top of a direction which makes gradient of a vector as a tensor. So, first direction of whichever is first: one direction is the direction of the displacement x, y, z direction, second is the direction of the gradient direction in which you are interested in the variation, along which direction interested in the variation that is second direction.

First direction of the displacement, second is the direction in which interest in the variation of the displacement. So, those two directions result in a tensor. So, if you understand scalar temperature, gradient of temperature then easy to extend to gradient of a displacement field and that is a displacement gradient tensor.

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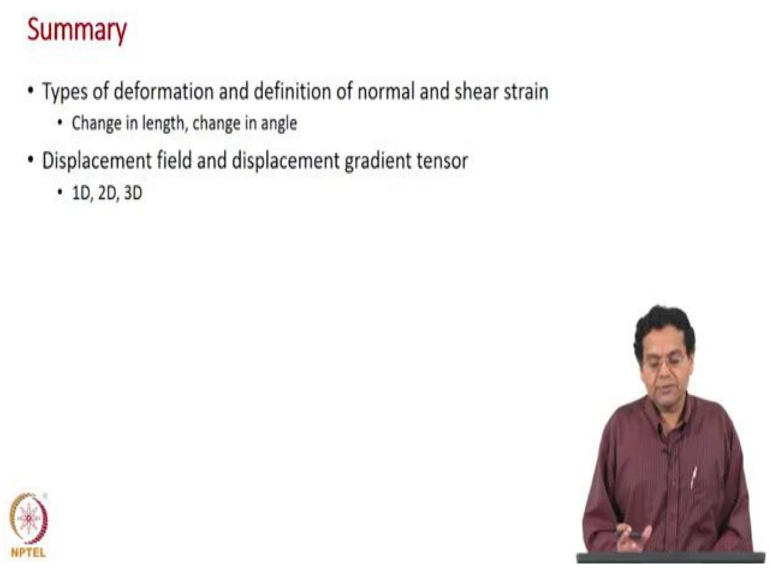

So to summarize, we started with defining types of deformation, it should be it could be change in length or change in angle, it could be and as well. And we quantified it by definition, we express change in length change in angle in terms of normal and shear strain. And then we said this normal strain, shear strain are somewhere depending on the displacement of the particles or the points. So, we are discussed displacement field, displacement gradient tensor slowly from 1D to 2D to 3D. The next we will have to relate these two and that is what we will discuss in the next lecture.