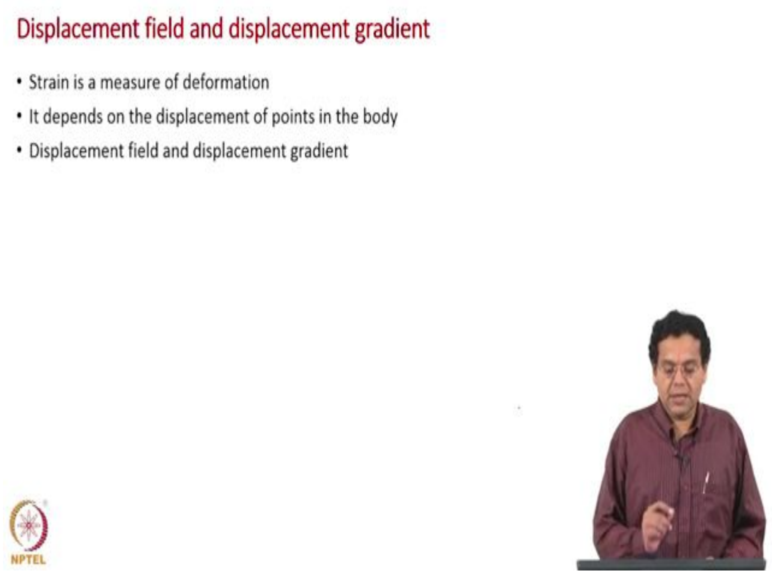
**Continuum Mechanics And Transport Phenomena**
**Prof. T. Renganathan**
**Department of Chemical Engineering**
**Indian Institute of Technology, Madras**

**Lecture - 53**
**Displacement Field and Displacement Gradient – Part 1**

(Refer Slide Time: 00:14)

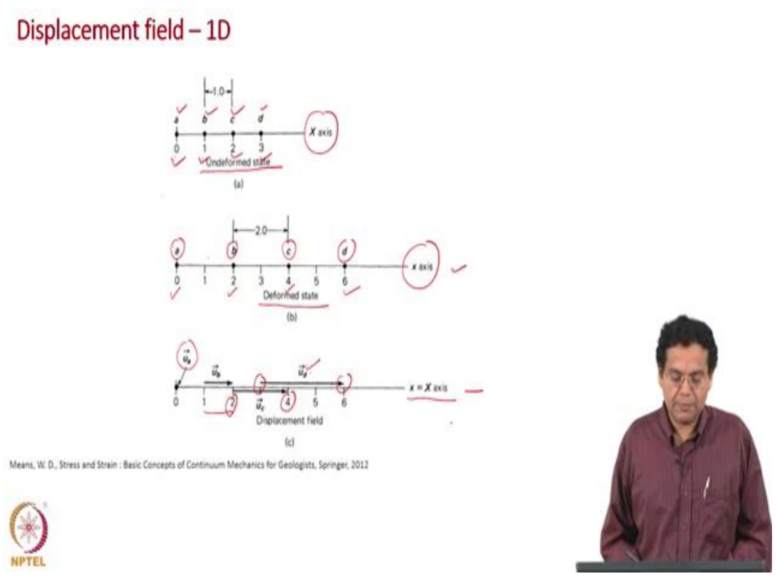



We are into the second concept under strain and deformation which says displacement field and displacement gradient. Now, we saw that the strain is a measure of deformation; we saw deformation could be change in length change in angle we quantified using strain that is why the sentence says strain is a measure of deformation. Now, the way in which we discussed strain either normal strain, shear strain we said we mark some points and then remember we discuss the how the points move we had a symbol called variable called u which represent displacement.

So, we mark some points p q or r p q and they got displaced to some other positions. So, which means that this strain depends on the displacement of points in the body. Now, after this discussion on displacement field and displacement gradient we are going to relate both of them. So, before relating we will discuss what is the displacement field, what is displacement gradient. So, to summarize this slide we first introduced deformation quantified it using strain, but these depend on the displacement of the points or solid particles and so, we are discussing now what displacement field and displacement gradient.

The word field is known to us right from several classes earlier field represents, the variation of any property in special location example we discussed temperature field velocity field etcetera. So, just like temperature field velocity field pressure field here we have displacement field. So, moment you look at displacement field that is the idea that should come to your mind that it is going to tell about displacement as a function of x, y, z just like temperature, pressure, velocity etcetera. We have come across temperature gradient similarly we are going to discuss about displacement gradient here of course, conceptually they are more involved.

(Refer Slide Time: 02:36)




The way in which we are going to discuss displacement field and gradient is parallelly, we are going to discuss displacement field displacement gradient, we are going to discuss hierarchically also what is hierarchy? Slowly introduce for 1 dimensional case, then go to 2 dimensional case, then go to 3 dimensional case that is a hierarchy. So, first take 1D discuss displacement field and gradient go to 2D discuss both of them similarly 3D that is the plan.

Now, let us start with the 1D case, we are going to discuss about displacement field, what is shown here (in the above slide image) of course, a very good discussion from our book by W D means on stress and strain. We can imagine a plate or a wall and then a wire attached to to that. And you mark some points on the wire a, b, c, d and they are at locations 0, 1, 2 and 3. Now, you stretch the wire; and that is what is shown in the second diagram. Now the points or particles have move to some other location, the a, b, c, d have moved to some other location. They point a is not moved in fact, just saying it is something like attached to the wall let us say b has moved to 2, c has moved to 4 and d has moved to 6.

Now, in terms of nomenclature which I have used few times earlier, the first figure represents the underperformed state or the initial state. The second figure represents the deformed state because you applied a force and continue to apply the force. Now, in terms of coordinate axis I am using 2 coordinate axis X axis which represents the old coordinates and x represents the new coordinate axis. The points a, b, c, d in terms of values X = 0, 1, 2, 3 for the undeformed state and x = 0, 2, 4, 6 are the position in the deformed state.

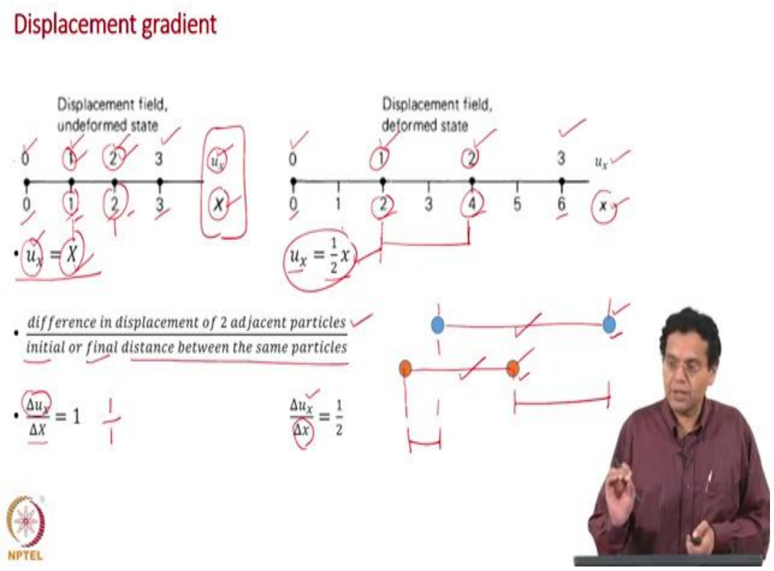
Now, how do we formally say point a has not got displaced, but b has displaced from 1 to 2 and then c has displaced from 2 to 4 and d has displaced from 3 to 6. Remember the scale is same only the nomenclature of axis is different what do I mean by that you take the same scale; scale cannot change; only the axis we are used to represent the position or the coordinate is different in the old configuration and the new configuration for old I use X for the new I use x.

Now, what does the third diagram represent? It represents the displacement of all the four points or the four solid particles; let us see how does it represent point. Now, $u$ represents displacement it's a vector right now it looks like a scalar, but actually it's a vector. So, $u$ vector for point a, point b, points c and point d.

Now, point a has not moved, so null vector $u_a$ is a null vector b has moved from 1 to 2. So, the arrow mark shows a displacement from 1 to 2, c has moved from 2 to 4, so a displacement of 2 units of course, all along the x axis d which was a position 3 has moved to 6. So, this represents displacement of 3 units for point d. So, the third diagram represents because we are discussing the displacement field look at the nomenclature it says x axis and X axis are superimposed that also shows we are at the same scale.

So, the third diagram represents the displacement for the different particles a, b, c, d, it shows an arrow from the initial position to the final position. Let us move on to the next slide and what is shown here formally is called the displacement field. Why is a displacement field? it shows displacement as a function of position what happens to displacement field for different particles this what we are going to discuss the next slide in a more detailed way.

(Refer Slide Time: 07:32)




Now, the displacement field, $u$ a vector, I have displacement only in the x direction. So, I have not denoted that as $u_x$ displacement in the x direction just like our velocity in the x direction here we are displacing x direction. Now, the displacement field $u_x$ can be represented in term in 2 ways what is that?

The displacement can be represent in terms of the old axis that is X axis or the displacement can be represent in terms of the new position axis or x axis let us see how do we represent. Remember these are the displacements remember we had 0 1 2 and 3 the displacements are same what is shown above the liner displacements. Similarly, the displacements are same 0, 1, 2, 3, but the axis with which you relate that defers.

The left hand side figure you are related this displacement to the old coordinate. So, you are showing the displacement 0, 1, 2, 3 against the old position 0, 1, 2 and 3, the right hand side we are showing the displacement against the new position 0, 2, 4 and 6. So, the independent variable is different in the 2 figures, but the dependent variable the displacement is same in both the cases the displacement the same that cannot be changed certainly, but with which you are trying to relate the independent variable the position that defers it could be the old position 0, 1, 2, 3 or to be the new position 0, 2, 4, 6.

Now, I will use let us say grammatically correct English statement, if I want to explain the first figure I should say that 0, 1, 2, 3 are the displacements that will happen for the particles at 0, 1, 2, 3 because that is the initial state before you applied the force that is the undeformed state. Now, we applied the force it has come to another equilibrium state we are in the final state. Now, if I want to explain you see grammatically correct statement for the right side figure I should say these are the displacements that has occurred for the particles which were at 0, 1, 2, 3 or which are currently at 0, 2, 4, 6. So, in one case displacement that will happen in other case displacement that has happened.

Now, what does this two figures tell you the displacement field in this case just $u_x$ can be related as a function of X or can be related as a function of x.

$$u_x = X; \qquad u_x = \frac{1}{2}x$$

So, at 0 you have 0 displacement, at 1 you have displacement 1. So, just $u_x = X$. On the right hand side the displacement is half the new coordinate position. So, $u_x = \frac{1}{2}x$, for 0 you have 0 for 2 you have 1 for 4 you have 2. So, relationship its connects the displacement and the new coordinate is $u_x = \frac{1}{2}x$.

So, as I told you the independent variable a either is X or x the dependent variable is the displacement that is same the functionality which you express other function which we used to express or the axis which you use to express that can change either the old coordinate or the new coordinate.

Now, the title of this slide says displacement gradient we are going to define displacement gradient let us define that.

$$Displacement\ gradient = \frac{Difference\ in\ displacement\ of\ 2\ adjacent\ particles}{Initial\ or\ final\ distance\ between\ the\ same\ particles}$$

Let us understand this clearly, it says 2 adjacent particles. As usual why 2 adjacent particles they are very close to each other we get a point definition of displacement gradient everywhere that is a common theme throughout, all our equations all our variables should be well at a point definition should be well at a point. So, we consider 2 adjacent particles of course, here as per the diagram it may be slightly away, but theoretically considering 2 adjacent particles.

And let us say for example, you are considering particles which were at 1 and 2 they are displacements are 1 unit and 2 units. Now, difference in displacement of 2 adjacent particles the displacement of particle at 2 is 2 which was at 1 is 1, so difference in displacement is 1 unit. Now, the denominator is the initial or final distance between the particles. So, now, there is a possibility of 2 choices, the distance between the 2 particle in the old configuration is 1 unit, but in the new configuration is 2 units. So, now, the same definition

$$\frac{\Delta u_x}{\Delta X} = 1; \qquad \frac{\Delta u_x}{\Delta x} = \frac{1}{2}$$

Let me repeat we are considering 2 particles which were at 1 and 2; their displacements are 1 and 2. So, the difference $\Delta u_x$ tells difference in displacement that is 1; denominator the first case we are taking the initial distance between the same particles initial distance between them is 1. So, giving us value of displacement gradient of 1, the denominator we are using $\Delta X$ denote distance between the initial distance between the particles.

Second alternative is numerator there is no ambiguity or present difference in displacement of 2 adjacent particles, but denominator I take the distance between the final distance between the particles I take the final distance between the particles which is 2 units. So, I have $\frac{1}{2}$. The two values of two different ways in which you can express the displacement gradient.

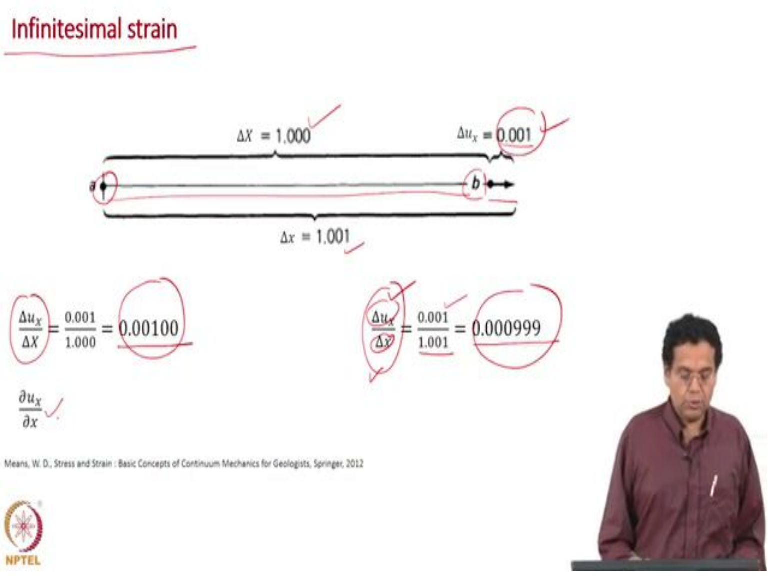
Now, the small very small animation helps you understand better let us imagine like on the same road we have 2 persons standing and they are initially at these locations one represented by blue circle other represented with the orange circle. So, these are their initial position and this is the distance between them in the initial configuration let us say and both of them are the same line just for understanding I have shown separately. Now, let us say they start walking and then after the same time let us say one walks much faster the other walks slower and these are the final position of the 2 persons.

Now, let us say I want to evaluate the displacement gradient for these 2 persons. Now, this is a displacement of the first person and this is a displacement for the second person, the numerator is difference between these two lengths or distances. Now, denominator either you can take the initial distance between them or the final distance between them that is the analogy between here we are discussing particles little bit easier to understand if you take 2 persons there are two different locations they start walking depending on the rate at which they walk there are two different positions and the first person displacement is so much second person displacement or the distance walked by the person is so much that difference is the numerator. Now, divide by the distance between them you take the initial distance or the final distance.

So, what we have to introduced is a displacement field, why is the displacement field in this case its only 1 dimension. So, $u_x$ only is considered here is a function of X or x we said displacement any field is a expressing a property or a variable in terms of x, y, z coordinates in this case only 1 component $u_x$ as a function of only 1 variable either X or x.

Now, you have been telling X and x etcetera that that of course, is difficult to understand and take forward also, but we are going to discuss is a condition under which we need not differentiate both the definitions of the displacement gradient. So, that we need not worry about either we consider X or x.

(Refer Slide Time: 16:26)




Formally what I going to discuss is called infinitesimal strain let us see what it is. We take the same case of let us say a threat attached to a point a now we apply a force this point b now gets displaced point a is at the same position, but the displacement of point b is very very small it just gets displaced by 0.001. Now, if you look at the difference in displacement, the displacement of point a is 0 but the point b got displaced by a small value 0.001. So, difference in displacement is 0.001.

$$\Delta u_x = u_b - u_a = 0.001 - 0 = 0.001$$

Now, what is the distance between them? In the initial state we took a let us say wire of threat of length 1 unit that is a distance between a and b. Now, in the final configuration the distance between them is 1.001

$$\Delta X = 1.000; \quad and \quad \Delta x = 1.001$$

So, initial distance between them as 1 unit the final configurations distance between them is 1.001, displacement does not depend on whether you take initial distance or final distance that is anyway 0.001.

Now, we will calculate the displacement gradient based on the 2 options we have; the first case

$$\frac{\Delta u_x}{\Delta X} = \frac{0.001}{1.000} = 0.00100$$

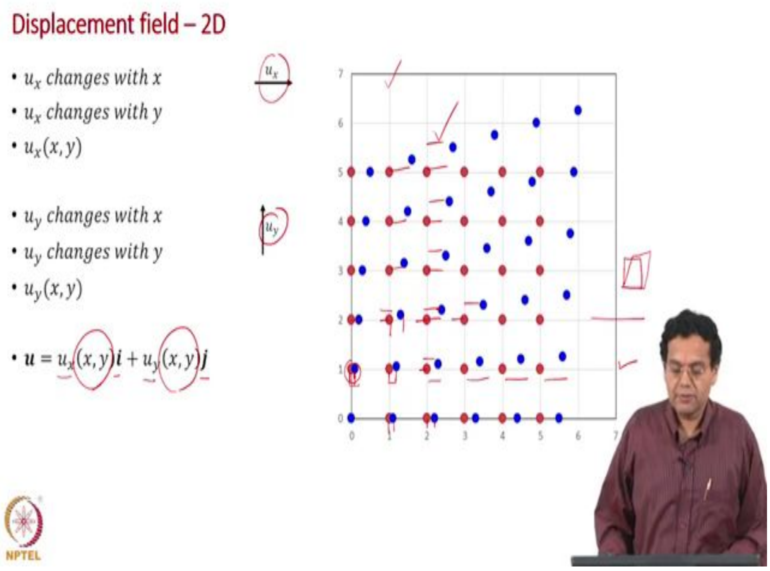
Coming to option 2

$$\frac{\Delta u_x}{\Delta x} = \frac{0.001}{1.001} = 0.000999$$

So, what do we conclude if the displacement is very very small. If the strain is very very small, then both the derivatives are same and we will use our very familiar derivative $\frac{\Delta u_x}{\Delta x}$ or in terms of partial derivative $\frac{\partial u_x}{\partial x}$ , eventually $u_x$ is going to vary with the x, y, z etcetera. So, we have to represent using partial derivative and that is why we call this as infinitesimal strain this is an assumption under which are going to work through of solid mechanics.

We will discuss more about this later right now the conclusion from this slide is that if the displacement is very very small if you want to take some normalized way the strain is very small if it is infinitesimal, then either you can work in terms of this definition of the displacement gradient or this we are very familiar with the small x and we are going to work with $\frac{\partial u_x}{\partial x}$ .

(Refer Slide Time: 20:23)




Now, let us move on to same discussion extended to 2 dimensional case, we will discuss displacement field displacement gradient. Now, what is shown here (in the above figure) is a plate the red dots show the some points identified particles identified in the plate in the undeformed configuration the initial state. We apply a force it gets deformed it reaches another equilibrium state, now whatever points we are identified earlier have now got displaced and those are shown as the blue dots.

Now, if you look at this, the case of a 2 dimensional displacement we are taken this example to illustrate this illustration to discuss 2 dimensional displacement field. Now, because it is 2D we will have to consider $u_x$ and $u_y$. The $u_x$ is the displacement in x direction. Now, let us take an example let us take these particles around in this horizontal line, if you see these two particles the same particle in the initial state and final state it has got horizontal displays there is a $u_x$. If you take these two the $u_x$ there is a $u_x$ the displacement in the x direction is different from these two these two points.

Now, as you go along the x axis the horizontal displacement or $u_x$ keeps increasing as a function of x. So, $u_x$ changes with the x, remember $u_x$ is displacement in the horizontal direction around the x axis. So, the x coordinate distance between these two dots represent the $u_x$ value that $u_x$ value keeps increasing along the x axis. Now, let us take particles around this vertical line, now once again focus on the x displacement there is a small x displacement and then slightly more x displacement slightly more. So, the x displacement is increasing as you go along the y direction.

So, $u_x$ changes with y as well $u_x$ represent displacement the x direction this example shown it is varying along the x axis it is varying along the y axis specifically in this example they are increasing along x axis increasing along y axis.

So, $u_x$ is a function of x comma y is analogous to the x component of velocity varying along x direction y direction; x velocity is easy to understand because it is displacement because they are not very familiar with that slightly difficult to get along with it, but just tells you the x direction moment of a particle which we are focusing and just difference in x coordinate gives you x displacement. Now, let us talk about the y displacement; displacement in the y direction which is $u_y$ .

Now, what should we consider? Let us consider the particles along this horizontal axis. Now, I want to focus on the vertical displacement which means that I should consider the difference in the y coordinate. Here, there is hardly any vertical displacement, but now if you see there is a vertical displacement between the red and blue dot that vertical displacement is more here. So, the vertical displacement increases along the x axis that is what you are going to say next $u_y$ changes with x. So, $u_y$ increases with x. So, vertical displacement is a function of x coordinate.

Now, if you focus and let us say particles around this region, then we are considering these two particles. And there is no displacement in the y direction, if you consider these two remember they are the same particles in the initial state and final state. So, if you consider the red and blue dots there is y displacement, if you consider these two particles there is once again y displacement it is more than the earlier set and this y displacement keeps increasing as you go along the y axis. So, $u_y$ changes with y as well.

So, the vertical displacement changes along the x axis; vertical displacement changes along the y axis also $u_y$ as a function of is a function of x comma y. Combining these two the $u$ vector can be expressed as

$$u = u_x\,(x, y)\, i + u_y\,(x, y)\, j$$

This is similar to velocity field $v = v_x(x,y)\,i + v_y(x,y)\,j$ . So, we have seen is an example for a displacement field in 2 dimensional case, where we have both displace x direction, y direction and both of them vary in x and y direction.