Continuum Mechanics And Transport Phenomena
Prof. T. Renganathan
Department of Chemical Engineering
Indian Institute of Technology, Madras

Lecture - 50
Differential linear momentum balance: Closure problem

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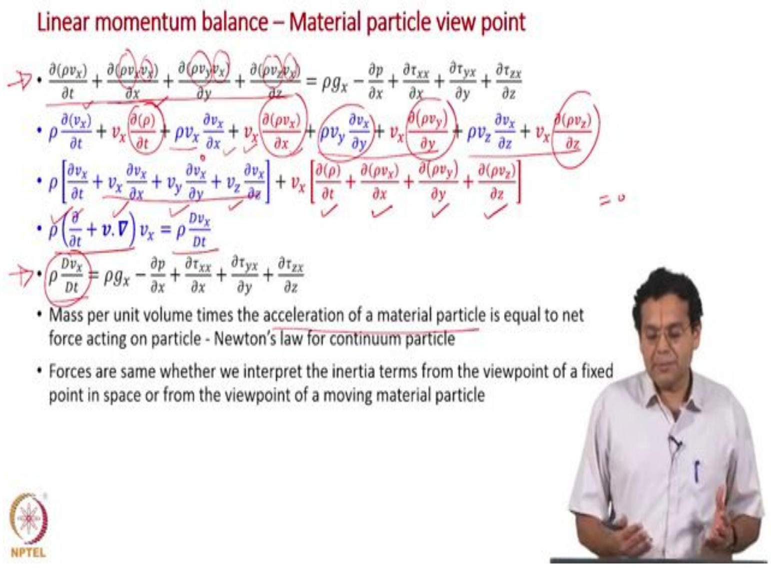

So far the linear momentum balance was viewed from a control volume point of view, what does that mean; you derive the balance equation taking a small control volume then we shrink it to a point and then we were discussing in terms of rate of change of momentum of course, all per unit volume, rate of change of momentum in that small volume and whatever is whatever momentum flowing in flowing out and then right hand side we have sum up all the forces.

Now, we are going to take a different look at the left hand side right hand side is going to still remain same and that is why the title says linear momentum balance from a material particle view point. Let us go through that and then you will understand the title clearly. One other importance of this slide is that we are going to do this series of step later on also few other place in the course. So, we will repeat this later on as well.

So, the idea of this discussion is that we have derived the linear momentum balance and interpreted the left hand side in terms of what is happening over a small region, rate of change of momentum in that region and whatever momentum flow enter and leaving in that region. We are going to have a different view point for the left hand side let us see how do we do that.

$$\frac{\partial(\rho v_x)}{\partial t} + \frac{\partial(\rho v_x v_x)}{\partial x} + \frac{\partial(\rho v_y v_x)}{\partial y} + \frac{\partial(\rho v_z v_x)}{\partial z} = \rho g_x - \frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}$$

Now, I will use the product rule and expand the derivatives on the left hand side. Now,

$$\rho\frac{\partial(v_x)}{\partial t} + v_x\frac{\partial(\rho)}{\partial t} + \rho v_x\frac{\partial(v_x)}{\partial x} + v_x\frac{\partial(\rho v_x)}{\partial x} + \rho v_y\frac{\partial(v_x)}{\partial y} + v_x\frac{\partial(\rho v_y)}{\partial y} + \rho v_z\frac{\partial(v_x)}{\partial z} + v_x\frac{\partial(\rho v_z)}{\partial z}$$

What is that we have done take in the left hand side, applied product rule for all the terms first term is very straight forward. The second third and fourth term we have clubbed ρv_x together and v_x separately. The significance of ρv_x is the mass flux and we will see why do we do that in next step as well.

So now I will combine all the terms

$$\rho\left[\frac{\partial(v_x)}{\partial t} + v_x\frac{\partial(v_x)}{\partial x} + v_y\frac{\partial(v_x)}{\partial y} + v_z\frac{\partial(v_x)}{\partial z}\right] + v_x\left[\frac{\partial(\rho)}{\partial t} + \frac{\partial(\rho v_x)}{\partial x} + \frac{\partial(\rho v_y)}{\partial y} + \frac{\partial(\rho v_z)}{\partial z}\right]$$

Now, terms in the second bracket is a equation of continuity we know that goes to 0; both for compressible flow and incompressible flow

$$\frac{\partial(\rho)}{\partial t} + \frac{\partial(\rho v_x)}{\partial x} + \frac{\partial(\rho v_y)}{\partial y} + \frac{\partial(\rho v_z)}{\partial z} = 0$$

So, and second term just vanishes this terms inside the square bracket is nothing, but the continuity equation or the differential mass balance is equal to 0. Now, left hand side if you look at that

$$\rho\left(\frac{\partial}{\partial t} + v.\nabla\right)v_x = \rho\frac{Dv_x}{Dt}$$

Of course, we discussed the substantial derivative many classes earlier and we also mentioned that we will come across later in the course this is the point where we are coming across the substantial derivative.

If you look at the terms in the within the first square brackets they are nothing, but $\left(\frac{\partial}{\partial t} + v.\nabla\right)v_x$. So, the left hand side can be written as $\rho\frac{Dv_x}{Dt}$; ρ is the density and $\frac{Dv_x}{Dt}$ is the substantial derivative of velocity of course, in the x direction.

So, what is that we have done, we have taken the left hand side and did some simple mathematical steps used the equation of continuity and then we have proved that the left hand side can be written as $\rho\frac{Dv_x}{Dt}$, as I told you we will come across similar steps later on energy balance also.

Now, let us rewrite the linear momentum balance expressing the left hand side in terms of this substantial derivative.

$$\rho\frac{Dv_x}{Dt} = \rho g_x - \frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}$$

So, now left hand side, I would say easier to remember also compared to the earlier linear momentum balance this equation is easier to remember and starting point for many problems as well. So, many application start this start with this form of the linear momentum balance equation and easy to remember.

Now, how does the significance change how or how do we interpret this equation? The left hand side if you read it, first you have density which is mass per unit volume and then that is multiplied by $\frac{Dv_x}{Dt}$ of course, we know acceleration of a material particle. Remember, what all significance of substance derivative the derivative following a fluid particle fluid particle.

So, $\frac{Dv_x}{Dt}$ represents acceleration of a material particle of course, material fluid particle what is there in the right hand side forces acting on the fluid particle. What is this nothing, but Newton's second law, but for a continuum fluid particle of course. So, the left hand side interpretation when we wrote the first time was, we have a small volume there is some rate of change of momentum or momentum entering and leaving etcetera. Now, the left hand side interpretation is different you are following a fluid particle and it represents mass into acceleration of that fluid particle of course, divided by volume density into acceleration of a fluid particle.

So, left hand side has changed from a Eulerian view point to a Lagrangian view point earlier it was for a fixed volume what is happening in the small region. Now, if you follow a fluid particle how does it is density into acceleration change and that is nothing, but governed by the forces acting on the fluid particle. Now, remember all this started from our Newton second law of motion for a system what happens when you make the size of a system go to 0; you get a fluid particle. Why? We said a system is made up of several fluid particles.

So, if you take a system of finite size and shrink it and shrink it you will get a fluid particle that is what you have got now because remember the derivation we shrank to a particular point. So, we started with Newton second law of motion for a system, now this equation is also is for a fluid particle how do you relate when you shrink a size of a system it becomes a fluid particle as you goes to zero size and that is why the left hand side now represents the density into acceleration of a fluid particle.

And, one more observation is that the left hand side are all called as the inertial terms. Now, independent of how do you view the inertial terms? How did we view the inertial terms for the first instance? It is based on a fixed control volume. Second time, we interpreted the inertial terms in terms of a moving fluid particle or a material particle. Independent of that the right hand side is same there was no change in the right hand side.

So, whether you use a inertial terms in terms of it is control volume as in terms of change of momentum, momentum entering and leaving either you take that view point or you take the view point of a moving fluid particle, what is that? Density into its acceleration, the right hand side is independent of that; independent of these two viewpoints.

Now, you look at the title linear momentum balance from a material particle view point. So, two different ways of writing the linear momentum balance equation. We have discussed at length the first viewpoint; second view point just requires a rearrangement of the left hand side including the continuity equation and you get this viewpoint. See, just rearrangement gives a different completely different viewpoint for the left hand side of course, right hand side is some of forces acting.

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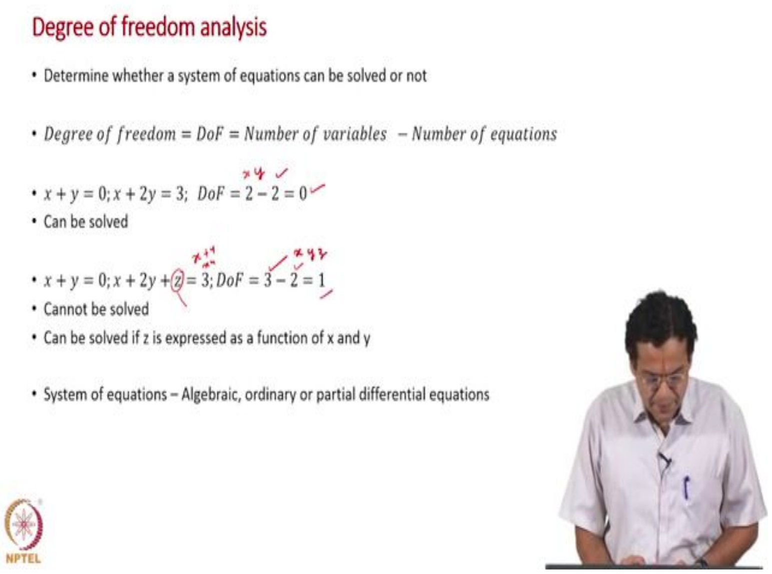

We will proceed to the next topic we said closure problem. To understand closure problem we should understand the terminology degree of freedom analysis. It is a very common terminology which we come across frequently in mass energy balance calculation in a process calculation course.

Just as a quick revision we will have a simple example which explain what degree of freedom analysis means ok. First of all, the degree of freedom analysis helps to determine whether a system of equations can be solved or not. In terms of a mass energy balance calculation suppose if you are given a problem should your degree of freedom analysis you can find out whether enough information is given to you so that you can solve for the rest of the unknown variable.

Usually you will your flow sheet will have some known variables unknowns variables, you do a degree of freedom analysis and find out whether enough information is available to you so that you can solve for rest of the unknown variables.

In fact, the Gibbs' phase rule which all of us are familiar with is $f = c - p + 2$, number of components number of phases that is also an application of degree of freedom analysis and of course, that is in terms of intensive variables.

Degree of freedom is defined as,

$$DOF = Number\ of\ variables - \ Number\ of\ equations$$

More formally, number of unknown variables and number of independent equations. Equations which account for should be independent of each other which means you should not be able to get one equation by a linear combination of other equations.

For our purpose any all of the equations are independent and number of variables are unknown variables. Let us take a simple example let us say I have a system of algebraic equations

$$x + y = 0$$

$$x + zy = 3$$

Now, I want to find out whether this system of equations can be solved. So, I can do a degree of freedom analysis.

The number of variables = 2, x and y

The number of equations =2

So, DOF = $2-2=0$

Zero degrees of freedom tells you that we can solve this set of equation.

Now, let us take another example. This can be solved now let us take another example

$$x+y=0$$

$$x+2y+z=3$$

Now, if you do a degree of freedom analysis

The number of variables = 3, x, y and z

The number of equations =2

So, DOF = $3-2=1$

Which means that I cannot solve the system of equations unless you specify z. Now, if you specify $z = f(xy)$. If you replace x as some function of let us say $x+y$ or xy, some equation then you have 2 equations and 2 unknowns then you can solve this system of equations.

And, we have illustrated for the case of algebraic equations need not be algebraic. It could be equations, partial differential equation. We are going to see an application for partial differential equations because our conservation equations are partial differential equation ok. So, degree of freedom analysis tells you whether system of equations can be solved or not, it is defined as number of variables minus number of equations.

We have seen two examples where one diff degree of freedom is 0, the second one this degree of freedom is 1 but if you express $z = f(xy)$ it can be solved and that is the point you should note we will carry out this understand this, it becomes very easy to understand what we are going to discuss now next.

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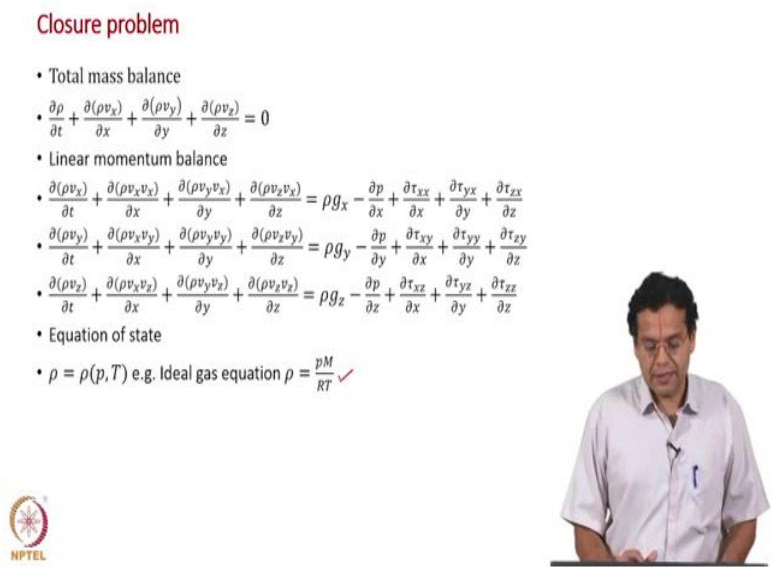

The topic which I am going to discuss now is formally called a closure problem, we will understand why is it so, as we go along. So, let us write down all the equations which I have derived so far.

The total mass balance of course, a differential form

$$\frac{\partial(\rho)}{\partial t}+\frac{\partial(\rho v_x)}{\partial x}+\frac{\partial(\rho v_y)}{\partial y}+\frac{\partial(\rho v_z)}{\partial z}=0$$

Then we will write the linear momentum balance, we have to write all the three components, the x component, the y component and the z component

$$\frac{\partial(\rho v_x)}{\partial t}+\frac{\partial(\rho v_x v_x)}{\partial x}+\frac{\partial(\rho v_y v_x)}{\partial y}+\frac{\partial(\rho v_z v_x)}{\partial z}=\rho g_x-\frac{\partial p}{\partial x}+\frac{\partial \tau_{xx}}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z}$$

$$\frac{\partial(\rho v_y)}{\partial t}+\frac{\partial(\rho v_x v_y)}{\partial x}+\frac{\partial(\rho v_y v_y)}{\partial y}+\frac{\partial(\rho v_z v_y)}{\partial z}=\rho g_y-\frac{\partial p}{\partial y}+\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \tau_{yy}}{\partial y}+\frac{\partial \tau_{zy}}{\partial z}$$

$$\frac{\partial(\rho v_z)}{\partial t}+\frac{\partial(\rho v_x v_z)}{\partial x}+\frac{\partial(\rho v_y v_z)}{\partial y}+\frac{\partial(\rho v_z v_z)}{\partial z}=\rho g_z-\frac{\partial p}{\partial z}+\frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}+\frac{\partial \tau_{zz}}{\partial z}$$

Then remember this rho is the density so, we need a equation of state we will have to include that as well which comes from a thermodynamic course that equation of state could be an ideal gas equation or non-ideal gas equation for simplicity for example, we take a ideal gas equation of state.

$$\rho=\rho(p,T) \quad e.g\ ideal\ gas\ of\ equation\ \rho=\frac{pM}{RT}$$

Now, let us do a degree of freedom analysis for these set of equations. We have the total mass balance, the linear momentum balance and the equation of state. Just want to repeat the equation of state is required because we have density and we said that then the density is related to pressure and temperature and that relationship should come from a equation of state.

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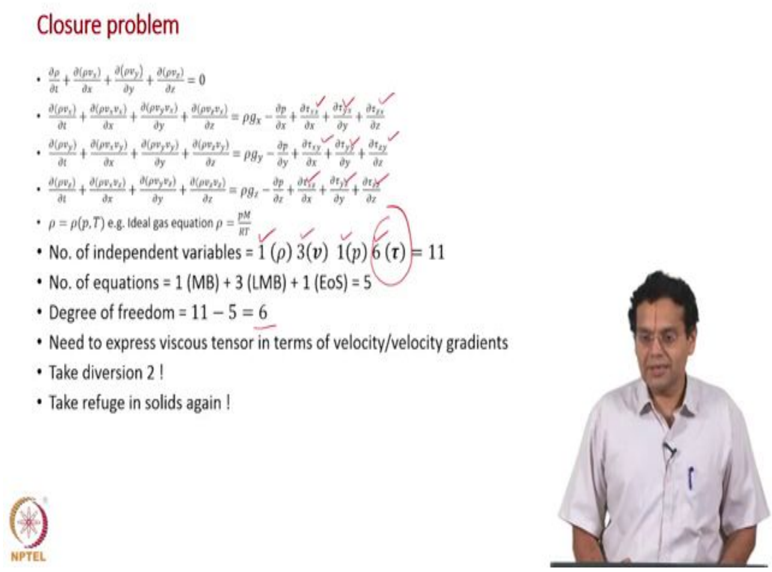

The equations are repeated here, we will do a degree of freedom analysis now for these set of partial differential equations. So, now, let us account for the number of variables number of equations; when I say variables I mean the dependent variables not the independent variables like time, x, y, z coordinates.

The number of independent variables we have are

- ρ (1 count),
- Velocity (Velocity is a vector so 3 components)
- Pressure (1 variable) and then
- 9 components of the viscous stress tensor, but we proved that the cross shear components are same leaving as only with 6 independent viscous stress tensor components and that is what is written here

So, totally independent variables = 1 + 3 + 1 + 6 = 11 , Number of equations = 1(mass balance) + 3 (linear momentum balance) + 1 (equation of state) =5

So, find the degree of freedom as

DOF = Number of independent variables – Number of equations = 11 – 5 = 6

We have seen the previous example that the degree of freedom should be 0 for systemic equation to be solved. We have 6 DOF so which means that you cannot solve.

And, what are the 6 unknowns? They are very clear because the 6 independent components of viscous stress tensor are the unknowns. If you express that in terms of the usually existing variables, if you express that in terms of the already existing variables just like we did in the last example what did we do z in terms of x and y.

Similarly, here if you express this τ the components of tau let us say τ_{xx}, τ_{yx} etcetera in terms of velocity then this degree of freedom will become 0. Not exactly velocity, you will see later on that we are going to express in terms of velocity gradients.

And, we take a diversion 2 to solid mechanics to understand what velocity gradients are understand velocity gradients and understand what displacement gradients are and that is why we take a diversion 2. So, as usual any difficulty we run to solids. So, we will take a refuge in solids again.

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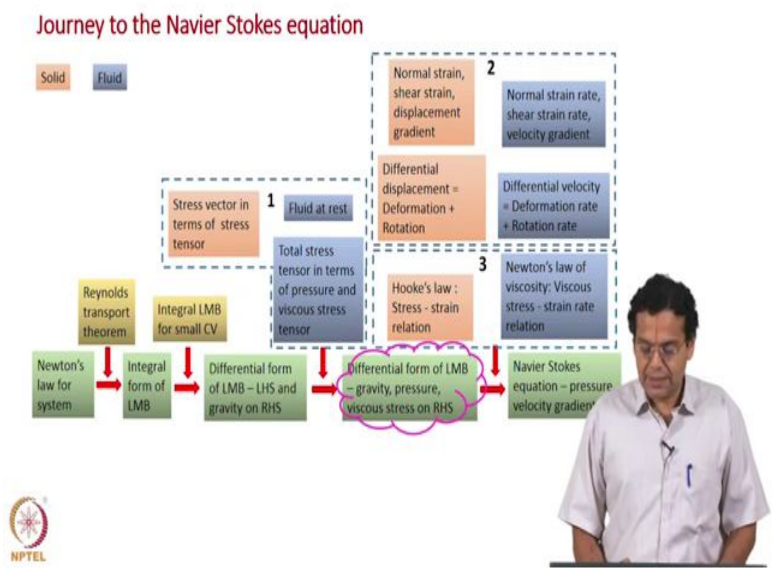

And, let me explain that with this journey to the Navier Stokes and where are we now and where are we proceeding. To begin with we started with Newton's second law of motion for system, applied the Reynolds transport theorem, derived the integral form of linear momentum balance, applied this for a small control volume, derived the differential form in the left hand side and the gravity on the right hand side.

To understand surface forces we went to solid mechanics, understood stress factor and stress tensor, applied that to fluids, discussed fluids under rest to discuss hydrostatic stress, discussed about total stress tensor in terms of hydrostatic stress and the viscous stress tensor for fluids. With all this the previous lecture we applied the we completed the right hand side of the linear momentum balance, now we are ready to understand this the differential form of linear momentum balance, what does it include? Of course, the left hand side is there and the right hand side we have gravity, we have pressure and viscous stress. So, anyway going to solve for pressure, the viscous stresses are unknown.

You want to relate this viscous stress to velocity actually velocity gradients, understand velocity gradients which understand what displacement gradients are and that is why we take a second diversion here, go to solid mechanics understand displacement gradient and then we can understand velocity gradients and that is that is where we are looks like almost we are there but we need to understand furthermore ok and this problem is called the closure problem, that is, closure meanings meaning that expressing the components of the viscous stress tensor in terms of velocity gradients you close the system of equations.

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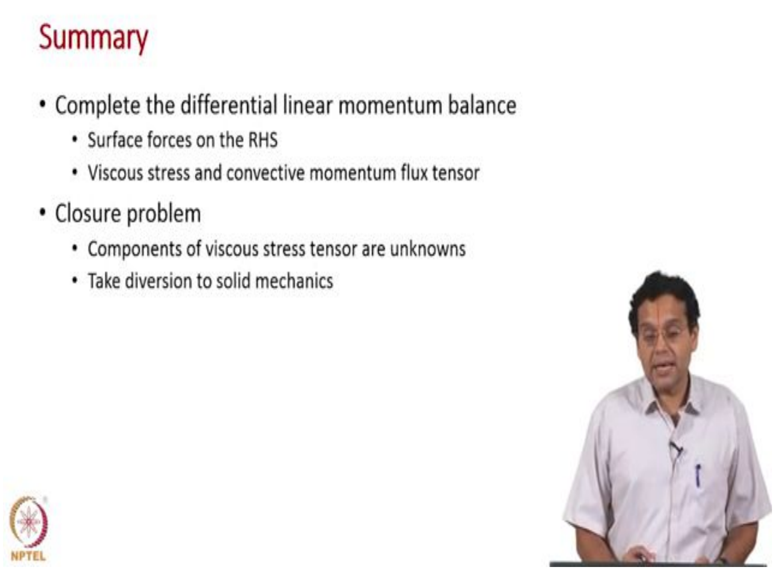

To summarize this part we completed the differential linear momentum balance in the right hand side, including the surface force in the right hand side, and we included the surface force due to pressure and due to viscous stresses and we also discussed the viscous stress tensor and the convective momentum flux tensor as well. Viscous stress tensor is kind of known to us because analogous to tau for solids, the conductive momentum flux tensor was is new to us.

We also discussed the closure problem which means that the components of viscous stress tensor are unknowns, they need to be expressed in terms of velocity or more precisely velocity gradients and to understand displacement gradients we take a diversion to solid mechanics.