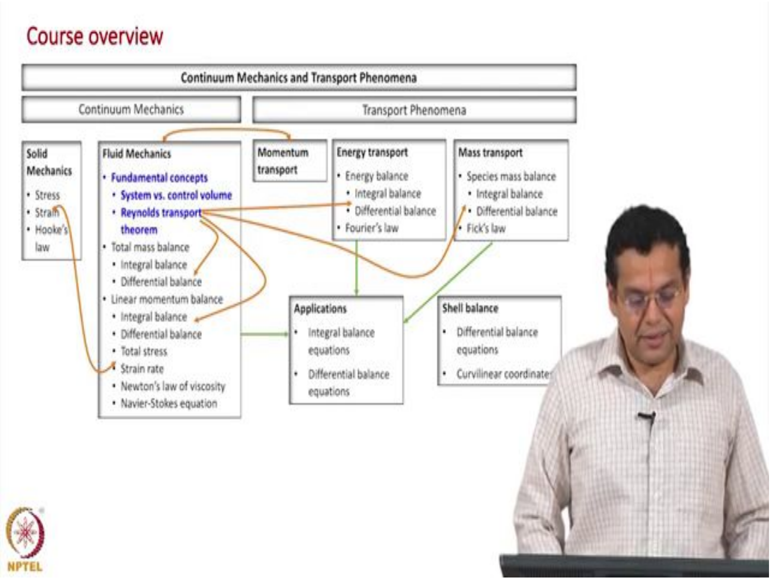
# Continuum Mechanics And Transport Phenomena
**Prof. T. Renganathan**
**Department of Chemical Engineering**
**Indian Institute of Technology, Madras**

## Lecture - 05
## Continuum Hypothesis

(Refer Slide Time: 00:13)



Course overview

Continuum Mechanics and Transport Phenomena

Continuum Mechanics

Transport Phenomena

Solid Mechanics
- Stress
- Strain
- Hooke's law

Fluid Mechanics
- Fundamental concepts
  - System vs. control volume
  - Reynolds transport theorem
- Total mass balance
  - Integral balance
  - Differential balance
- Linear momentum balance
  - Integral balance
  - Differential balance
  - Total stress
  - Strain rate
  - Newton's law of viscosity
  - Navier-Stokes equation

Momentum transport

Energy transport
- Energy balance
  - Integral balance
  - Differential balance
- Fourier's law

Mass transport
- Species mass balance
  - Integral balance
  - Differential balance
- Fick's law

Applications
- Integral balance equations
- Differential balance equations

Shell balance
- Differential balance equations
- Curvilinear coordinate

We will start with course overview which was there the title of this course Continuum Mechanics and Transport Phenomena. Of course, that splits into continuum mechanics part and transfer phenomena part. In that we are going to start with the Fluid Mechanics part and within that Fluid Mechanics, we are going to start with the fundamental concepts. That is highlighted. Fundamental concepts has several topics in it.

(Refer Slide Time: 00:50)

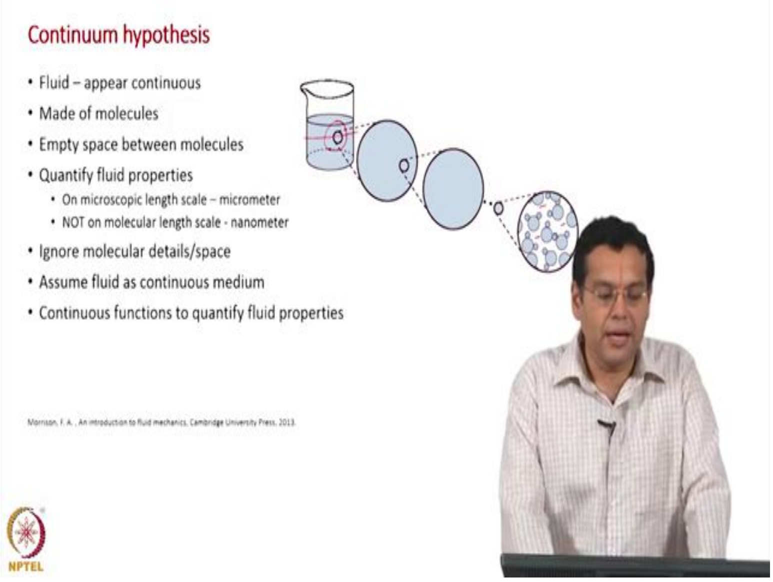
Fundamental concepts - Outline

- Continuum hypothesis
- Two approaches for describing flow
- Substantial derivative
- Visualization of flow patterns
- System and control volume
- Reynolds transport theorem

First we start with the Continuum Hypothesis where we discuss what we mean by the term continuum, then we discuss two approaches for describing flow. One is following the fluid flow and one is at a particular point and then, we discuss substantial derivative, a derivative which is something special to continuum mechanics and transfer phenomena, a general transfer phenomena.

Then we discuss how to visualize flow patterns and then, more importantly discuss what do we mean by a system which is same as the system we come across in thermodynamics course and then, control volume which is representing maybe a heat exchanger reactor etcetera. Then the laws of physics are stated for the system and we need conservation equations for the control volume. So, we need a theorem which takes us from system to the control volume and that is the Reynolds Transport Theorem. So, we will discuss that at the end. So, these topics will be covered over a few lectures.

(Refer Slide Time: 02:07)



Continuum hypothesis

- Fluid – appear continuous
- Made of molecules
- Empty space between molecules
- Quantify fluid properties
  - On microscopic length scale – micrometer
  - NOT on molecular length scale - nanometer
- Ignore molecular details/space
- Assume fluid as continuous medium
- Continuous functions to quantify fluid properties

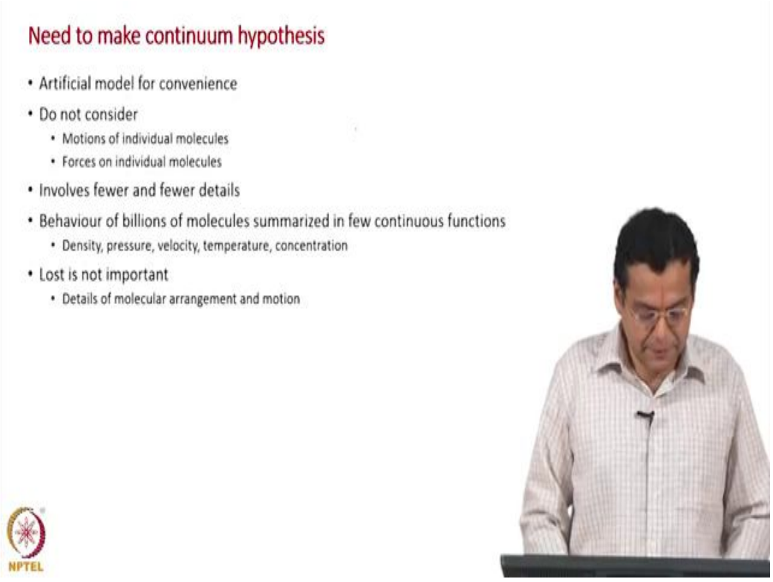
Morrison, F. A. , An introduction to fluid mechanics, Cambridge University Press, 2013.

So, let us start with term Continuum Hypothesis. What you see here is beaker of water and if you see it with the naked eye, you feel it is completely filled with water, but now if you keep zooming in, zooming and magnifying it and then if you magnify to large extent, you will see water molecules there and then there are void spaces. So, fluid appears as continuous to the naked eye, but at the molecular level it is made of molecules.

Now there are empty spaces between the molecules as we have seen. Now we want to quantify fluid properties on the microscopic length scale which is at the scale of the water in the beaker and let us say a few millimeters or few micrometers in the volume in the beaker and we are not going to quantify on the molecular length scale which is now scale of nanometers, and which means that you are going to ignore all the molecular details and assume fluid as a continuous medium. That is what we mean continuum hypothesis.

Though at the molecular level, it is made up of molecules, we are just going to ignore that molecular picture and we are going to say everywhere without leaving any space, the space occupied by liquid or gas whatever matter it is. That is what we mean continuum hypothesis. Now what is the implication of this? All the fluid properties can be taken as continuous functions of space. If you take the molecular picture, there will be discontinuities at these locations. If you assume a continuum hypothesis, you can assume continuous variation of all the fluid properties. That is the implication of continuum hypothesis.

(Refer Slide Time: 04:24)



### Need to make continuum hypothesis

- Artificial model for convenience
- Do not consider
  - Motions of individual molecules
  - Forces on individual molecules
- Involves fewer and fewer details
- Behaviour of billions of molecules summarized in few continuous functions
  - Density, pressure, velocity, temperature, concentration
- Lost is not important
  - Details of molecular arrangement and motion

Now, what is a need to make continuum hypothesis? First of all we should know it is an artificial model. Why it is an artificial model? The reality is actually consists of a lot of molecules of water with void spaces.

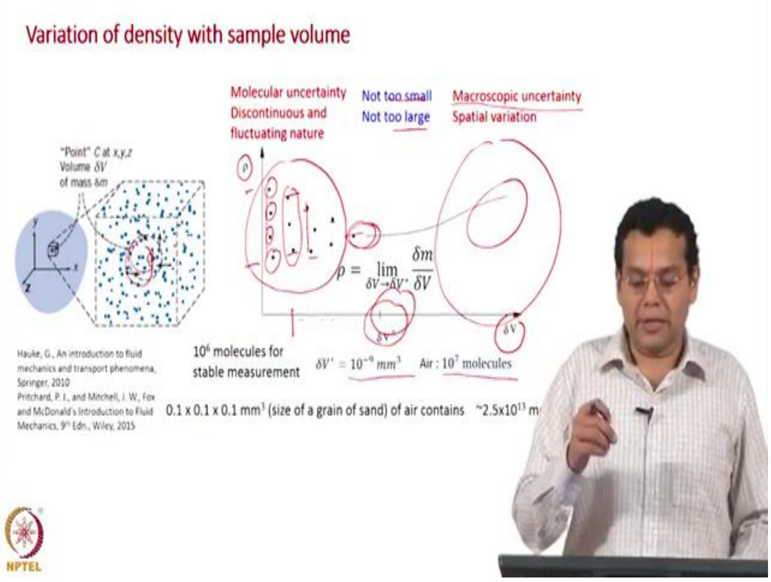
We are making an approximation to that artificially as for convenience. So, what is the convenience we are talking about? As we have seen we are not going to consider the motions of individual molecules, forces acting on them, the velocity distributions etcetera. All those are not going to be considered because we do not consider the molecular details the way in which we discuss involves only fewer and fewer details, and we are going to summarize represent behavior of billions of molecules in terms of few continuous functions. What are those continuous functions? Density, pressure, velocity, temperature, concentration all these are measurable by instruments, can measure pressure; can measure velocity, density, temperature, concentration.

So, the molecular behavior of billions of molecules is represented by few continuous functions. As we have seen continuous functions have values throughout the locations which ignore molecular behavior and these continuous functions are measurable quantities namely density, pressure, velocity, temperature and concentration. So, these continuous functions indirectly represent the molecular behavior which to be exact we should represent in terms of their velocities, momenta acting on them, forces acting on them etcetera,. Now, what is lost?

What is that we have lost? We have lost the molecular behavior and that is not important for engineering purposes.

We do not talk in terms of forces acting on individual molecules; we do not talk about the molecular velocities. We just talk in terms of variables which we measure in our industry or day to day activity, in terms of density, pressure, velocity, temperature, concentration. So, whatever last is not important from our level of requirement or for our level of accuracy. So, details of molecule arrangement and motion is lost, but not important for us.

(Refer Slide Time: 06:49)



Variation of density with sample volume

Molecular uncertainty
Discontinuous and
fluctuating nature

Not too small
Not too large

Macroscopic uncertainty
Spatial variation

$$\rho = \lim_{\delta V \to \delta V'} \frac{\delta m}{\delta V}$$

$10^6$ molecules for
stable measurement

$\delta V' = 10^{-9}\ mm^3$  Air : $10^7$ molecules

0.1 x 0.1 x 0.1 $mm^3$ (size of a grain of sand) of air contains ~$2.5\times10^{13}$ m

Hauke, G., An introduction to fluid mechanics and transport phenomena, Springer, 2010
Pritchard, P. J., and Mitchell, J. W., Fox and McDonald's Introduction to Fluid Mechanics, 9th Edn., Wiley, 2015

To explain further about this continuum hypothesis what I have shown here is a small volume, very small volume and where there are molecules inside the volume which I have considered and now I am going to vary this volume from very small volume to very large volume and find out the density within that small volume.

So, I plotted density on the y axis, and the volume which are considering on the x axis. Now if I take very small volume, the number of molecules within that volume is going to be very less let us say a few molecules, 5 molecules, 10 molecules. So now every time you make a measurement, there is going to be huge fluctuation in density. That is what this region shows sometimes mega density, here sometime as a density here. Taking a small volume there are a number of molecules are going to be present in that region are going to fluctuate, a lot and number of molecules is going to determine the mass and then has a density and then this is going to fluctuate a lot. Now let us say the measurement volume increase. Now more number of molecules are present in that volume let us say a few hundreds. Now the fluctuation, the density is slightly smaller compared to the earlier smaller volume, number of molecules on average maybe varying let us say from 90 to 110. Instead of varying from 1 to 10, it may vary from 90 to 100. The variation is going to be less, that is why the density which you measure or calculate is going to vary over smaller range compared to this larger range.

Now let us say if you keep increasing the volume, the fluctuation, the density decreases almost become without any fluctuation. So, this volume let us say something kind of optimal volume. So, what we have seen is as you increase the volume which are going to sample, the fluctuation in the density reduces and becomes almost without any fluctuation. Now let us see what happens if you keep increasing the volume further.

Now, let us say if you increase the volume what happens, the volume occupied by this region is going to span over a larger region and then you are going to lose out on this spatial variation. What does it mean? let us say in this particular room you have some temperature variation, velocity variation. Our ultimate objective is to capture the variation spatially also.

So, let us say near the air condition the temperature is low. Somewhere near the door it is slightly warmer and there is temperature distribution and velocity distribution as well. Suppose if your volume keeps increasing and increasing, you lose out on this spatial variation. So, you cannot capture the spatial variation. So, at one end you have discontinuous and fluctuating nature which we call as molecular uncertainty. So, that is also not preferred.

On the other side if the volume is extremely large, you lose out on the spatial variation which was a larger volume, of course there won't be any fluctuation, but you won't be able to describe the density across different spatial points or velocity or temperature. So, we do not want very low sampling volume and then we do not want very large volume also, both are disadvantageous to us. One is termed as molecular uncertainty; other is called the macroscopic uncertainty. Why is it? Because the volume over which was sampling is large and we will not be able to describe the spatial variation. That is why I said some optimal volume which we do not hear as $\delta v^*$ which is not too small, which is not too large as well.

Now, what is this volume? how do you get an idea of this volume? Based on measurements and simulation, it has been found that. If you have $10^6$ molecules, remember we said few molecules, 10s of molecules, 100s of molecules, 1000s of molecules if in your space that are $10^6$ molecules, then you get a stable measurement without any fluctuation.

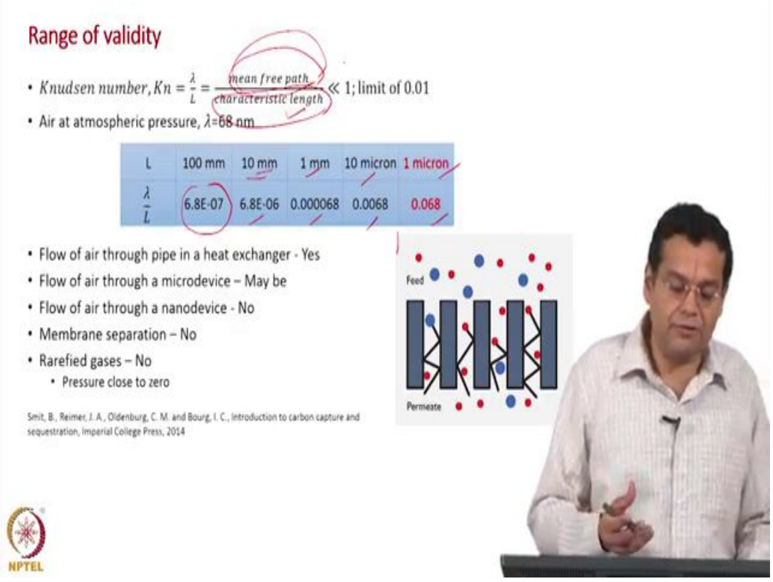
Now let us see what is the volume that would contain roughly $10^6$ molecules if you use ideal gas law and then at standard conditions of temperature and pressure and then use Avogadro's number of molecules. For the case of air if you take an extremely small volume of $10^{-9}$ millimeter cube extremely small volume, you have $10^6$, $10^7$ molecules of air. So, very small volume contains our desired number of molecules. How what is the other way of looking at it?

Suppose if you take a cube of 0.1x0.1x0.1, what does it mean? You let us say a centimeter and then one-tenth of that there is a millimeter, one-tenth of that and you imagine a cube of point each of point one side. So, let us see our usual measurement instruments that we easily occupied that volume. You know it is a very small instrument. It will occupy certainly even more than that. Even the most very sharp instrument will occupy by 0.1x0.1x0.1.

What is a number of molecules present in very small region? $2.5 \times 10^{13}$ molecules for air which are far away from the required $10^6$ molecule. So, even the measurements which are making practically occupies a volume where it is representative of much more than $10^6$ molecules.

So, what I discussed here is the variation of density with sample volume and we do not want molecular uncertainty, we do not want macroscopic uncertainty, we want a small volume which is not very small, which is not very large, has enough number of molecules, so that there are no fluctuations. At the same time we will be able to capture the spatial variation also.

(Refer Slide Time: 13:25)



## Range of validity

- Knudsen number, $Kn = \frac{\lambda}{L} = \frac{\text{mean free path}}{\text{characteristic length}} \ll 1$; limit of 0.01
- Air at atmospheric pressure, $\lambda$=68 nm

| L | 100 mm | 10 mm | 1 mm | 10 micron | 1 micron |
|---|---|---|---|---|---|
| $\frac{\lambda}{L}$ | 6.8E-07 | 6.8E-06 | 0.000068 | 0.0068 | 0.068 |

- Flow of air through pipe in a heat exchanger - Yes
- Flow of air through a microdevice – May be
- Flow of air through a nanodevice - No
- Membrane separation – No
- Rarefied gases – No
  - Pressure close to zero

Smit, B., Reimer, J. A., Oldenburg, C. M. and Bourg, I. C., Introduction to carbon capture and sequestration, Imperial College Press, 2014

How do you quantify this range of validity of the continuum hypothesis? So far we have been discussing qualitatively, now we want to put some numbers and quantify this is a region over which continuum hypothesis is valid, over which I can apply my equations and this region where I cannot apply the equations. So, this whole course we are going to assume continuum hypothesis. So, you should know this region over which the equations which we are going to make in this course are going to be valid. Now such a number is called the Knudsen number which is defined as the mean free path divided by the characteristic length.

$$Knudsen\ number = \frac{mean\ free\ path}{characteristic\ length}$$

What is this? Mean free path is the average distance travelled by the molecules between successive collisions. The molecules are in constant collision and what is the average distance which they travel between two successive collisions,. That is a mean free path. What is a characteristic length? Characteristic length is just length roughly the length of the dimension of the pipe or region over which we have our flow is being considered. For example, if you have a flow through a pipe, the diameter of the pipe can be a characteristic length.

So, Knudsen number is defined as the mean free path divided by the characteristic length. The Knudsen number should be much smaller than one and the limit given is 0.01. So, all of our equations are the scope of this course is for Knudsen number less than 0.01. The equation, which we derive in this course may not be valid when Knudsen number crosses the limit order of 0.01.

Now, let us take an example to get an idea of this. If you consider air atmospheric pressure, the mean free path 68 nanometers that can be calculated. Now let us take this air flowing through let us say pipe of characteristic length of different dimensions shown in the slide. Let us say if it goes through a pipe of 100 millimeter diameter, then Knudsen number is extremely small, much smaller than 0.01. Now, keep reducing the size of the pipe, let us say air goes through a 10 millimeter pipe. Still your Knudsen number is small, then 1 millimeter pipe the Knudsen number of small 10 microns, it is still smaller than 0.01, but now if this air flows through a pipe of 1 micron diameter, the Knudsen number crosses the limit of 0.01. Now the continuum hypothesis may not be applicable over this range or beyond this. If you go smaller and smaller sizes than this, then continuum hypothesis breaks down. So, our scope of the whole course plays let us say lies within this region quantified by Knudsen number less than 0.01.

Now, let us take some examples to get an idea where continuum hypothesis is valid and where it is not valid, ok. Flow of air through a pipe and heat exchanger maybe let us say a few centimeters to few inches. So, our Knudsen number is going to be much smaller. So, it is going to be certainly valid. Now let us say flow of air through a microscopic device maybe because we are almost on the verge so may or may not be valid to be safer side.

It may not be valid if you want approximate, you can take continuum hypothesis to be valid. Suppose this air flows through a nanometric device, then continuum hypothesis will not be valid. Your Knudsen number will cross the limit of 0.01 and you cannot apply the equations which are going to derive in this particular course.

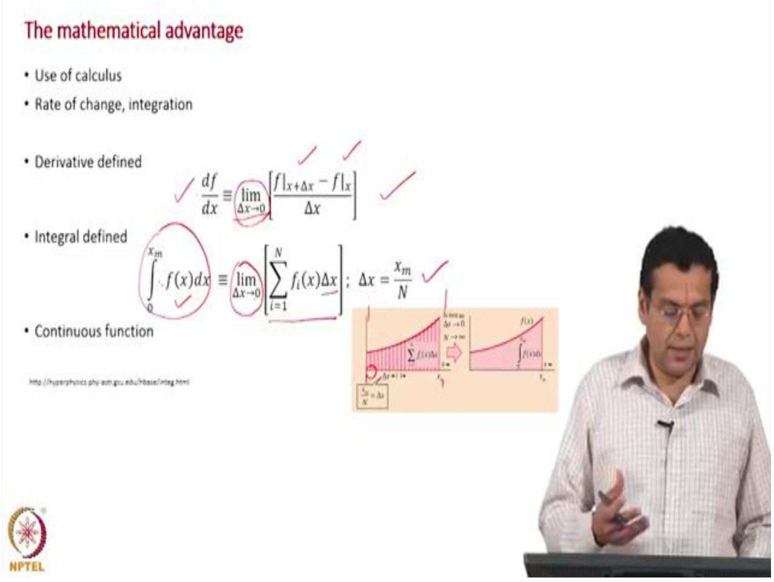
Let us take another example of membrane separation. The membrane separation, the size of the membranes are going to be let us say the order of nanometer range once again you cannot apply the continuum hypothesis for this case. All these cases what we have discussed so far the flow through nano device, the membrane separation etcetera. the Knudsen number was large because of the smaller characteristic length. The nano device has characteristic length of order of nanometers. Similarly for the membrane process, the pores are the order of nanometers.

So, the Knudsen number became larger because of the very small value of the characteristic length. Let us take another case where the Knudsen number is large because of the large mean free path. That is what happens if you consider rarefied gases. What do we mean by rarefied gases, gases at extremely low pressure. So, you have very few molecules per unit volume.

So, they travel a large distance between successive collisions. So, even though your space may be larger, but your mean free path is extremely large making the Knudsen number having a larger value, then thus causing breakdown of continuum hypothesis. So, the continuum hypothesis can break down either because of a very small characteristic length which is usual case. Certain examples exist where the continuum hypothesis breakdown because of the larger mean free path and that tells us the scope of the equations we are going to discuss in the whole of transfer phenomena course. Remember in the introduction, we said the three length scales are involved in a transfer phenomena discussion. One is the macroscopic which was are of equipment size. One is macroscopic which is inside the equipment; one is the molecular length scale. So, these all revolve around the molecular length scale and that is not within the scope of this particular course.

So, we have quantified this continuum hypothesis in terms of a Knudsen number and that is what we have done in this particular slide.

(Refer Slide Time: 19:55)



The mathematical advantage

- Use of calculus
- Rate of change, integration
- Derivative defined

$$\frac{df}{dx} \equiv \lim_{\Delta x \to 0} \left[ \frac{f|_{x+\Delta x} - f|_x}{\Delta x} \right]$$

- Integral defined

$$\int_0^{x_m} f(x)dx \equiv \lim_{\Delta x \to 0} \left[ \sum_{i=1}^{N} f_i(x)\Delta x \right]; \ \Delta x = \frac{x_m}{N}$$

- Continuous function

http://hyperphysics.phy-astr.gsu.edu/hbase/integ.html

What is the mathematical advantage or mathematical implication of continuum hypothesis, ok? The calculus which you have learnt in your mathematics course, itself can be used only if you assume continuum hypothesis. You cannot even use calculus if continuum hypothesis is not valid. So, even for your calculus to be used you require continuum hypothesis. What does it mean?

The rate of change, $\frac{d}{dx}$ , integration all those are defined only if you assume continuum hypothesis. Let us see how is it? I think all of us are familiar with this definition of the derivative

$$\frac{df}{dx} = \left[\frac{f|_{x+\Delta x} - f|_x}{\Delta x}\right]$$

df/dx is equal to, you take two spatial locations which are separated by a small distance x and $x + \Delta x$. Take that function value at $x + \Delta x$, take the function value at x and then divide $\Delta x$, take the limit $\Delta x \rightarrow 0$. That is how we define the differential of a function.

Now what is to be paid attention is the limit $\Delta x \rightarrow 0$. We are saying $\Delta x \rightarrow 0$ which means that this function should be describable at every spatial location, that is what we defined as the continuum hypothesis. If you are not assuming the continuum hypothesis, there will be some spaces and you cannot take $\Delta x \rightarrow 0$. The fact that we are taking $\Delta x \rightarrow 0$ means that that function is should be definable at every spatial location. That means, for example, let us say f is density. I should be able to describe density at every spatial location. Only then you can even define a differential. We are going to come across differential equations throughout this course and the moment you see a differential that is meaningful only if you assume continuum hypothesis that is the mathematical significance of assuming high.

So far we have seen the physical significance of the continuum hypothesis. What we have seen is the mathematical significance, mathematical implication. We may take this limit very frequently in this course, but you should remember that this limit assumes the continuum hypothesis.

Now similarly the definition of integral let us take this example. Let us say I have a function and then I want to find the integral of this function between two limits. Now this integral can be represented by a summation. How do you do that?

$$\int_0^{x_m} f(x)\,dx = \left[\sum_{i=1}^{N} f_i(x)\,\Delta x\right] \quad ; \quad \Delta x = \frac{x_m}{N}$$

You have to take the values of the function at different x and then multiply by the $\Delta x$, sum over the number of increments and this integral is represented by the summation. When you have this $\Delta x$, the width of each segment tends to 0 or the number of elements which you consider is very large which means that here also you say that this is valid when $\Delta x \rightarrow 0$ which means it continuum hypothesis to be valid.

So, both for defining integral we will come across integral expressions, when we discuss integral balance equations and differential, when we discuss about differential balance equation, when we write a macroscopic balance we will come across integrals, when we write a microscopic balance we will come across differentials and we should know that these differentials and the integrals are valid only in the limit of $\Delta x \rightarrow 0$ which requires the assumption of the continuum hypothesis. And of course, all these implies that whatever property when I say property density, velocity, etcetera, temperature all those should be continuous functions.