Continuum Mechanics And Transport Phenomena
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Lecture – 46
Fluids at rest

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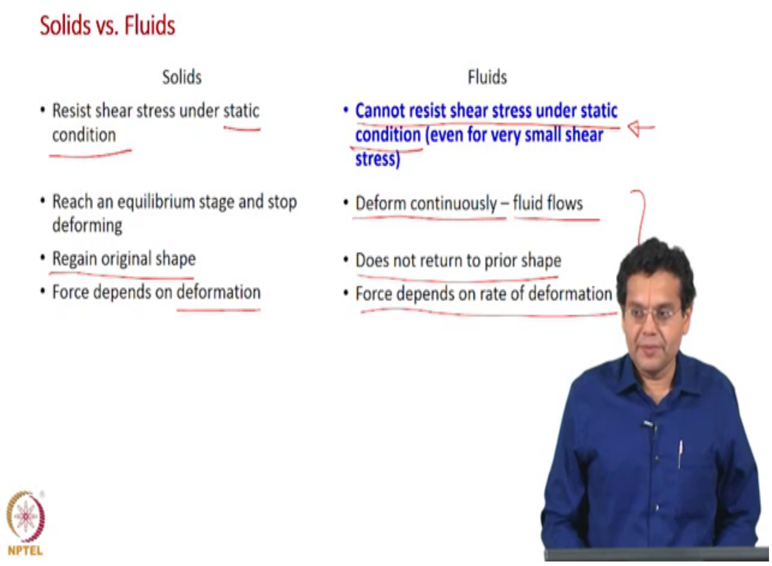

We will take the lead from this characteristic feature of fluids. Fluids cannot resist shear stress under static condition. So, in a static fluid there are no shear stresses.

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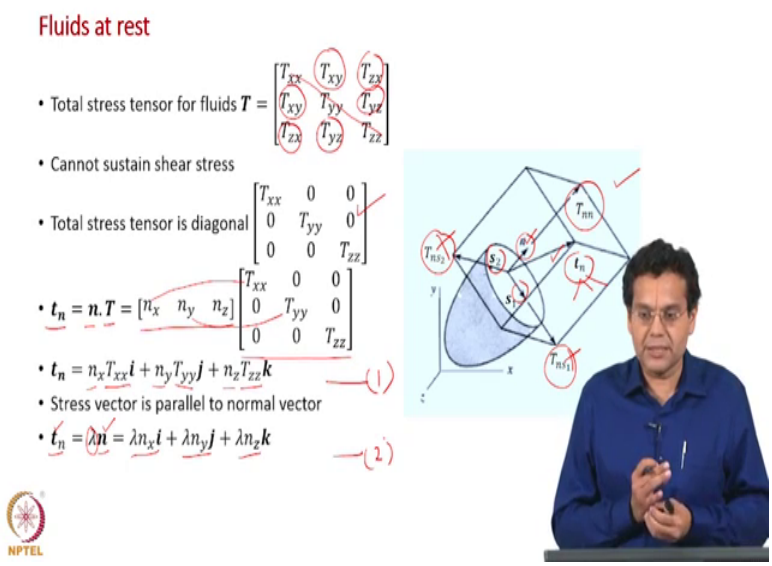

So, now we shortly understand what the total stress tensor means. So, let us consider Fluids at rest; the total stress tensor for fluids was introduced as

$$T = \left[T_{xx}\, T_{xy}\, T_{zx}\, T_{xy}\, T_{yy}\, T_{yz}\, T_{zx}\, T_{yz}\, T_{zz}\right]$$

The total stress tensor is symmetric I have shown a symmetric total stress tensor here ok. As we are discussed all the discussions for solids are applicable for fluids as well for total stress tensor. So, what I shown here is a, symmetric total stress tensor or we can say, it cannot sustain shear stress. What does it mean? This stress tensor, if you apply for a fluid at rest under static condition, then you will not have all this shear stress components at all.

The diagonal elements are the normal stress components, the off diagonal elements are the shear stress components. Now we said, if you have a fluid in a rest; we are considering fluid under rest condition and because there are no shear stresses all the off diagonal elements become 0. And so, the total stress tensor is diagonal for a fluid under static condition, under a rest condition.

$$T = \left[T_{xx}\, 0\; 0\; 0\; T_{yy}\, 0\; 0\; 0\; T_{zz}\right]$$

Now, we will find out the stress vector for a fluid under rest condition.

$$t_n = n.T = \left[n_x\, n_y\, n_z\right]\left[T_{xx}\, 0\; 0\; 0\; T_{yy}\, 0\; 0\; 0\; T_{zz}\right]$$

We apply the Cauchy's formula, we also seen that the Cauchy's formula is applicable for fluids as well in terms of total stress tensor. The total stress vector is equal to; for our understanding matrix multiplication with total stress tensor. So, this is the components of the unit normal vector and then these are the components of the stress tensor, it is a matrix only the diagonal components are there.

So, simple matrix multiplication will give us the stress vector as,

$$t_n = n_x T_{xx} i + n_y T_{yy} j + n_z T_{zz} k$$

So, the stress vector has been expressed in terms of the components of the total stress tensor and the components of the unit normal vector. So, we have got expression for total stress vector in one way. We will now write another expression for total stress vector in another way.

For that look at the diagram shown (in the above referred slide), which is same as what seen for solids, you should imagine this is a fluid body and then a surface in the fluid body, just to give a feel I made it colored as blue. But now look at the components the normal stress is T_{nn} and the shear stresses are T_{nS_1}, T_{nS_2} and then directions n, S_1, S_2 are shown. The stress vector we resolved along the normal and the tangential directions.

Now, this applicable for a fluid under moving condition; now imagine this to be a surface in a fluid under static condition, there are no shear stresses at all. So T_{nS_1}, T_{nS_2} are not there. So, this normal direction and the stress vector or at angle to each other because of the presence of shear stress components; because this stress vector has three components 2 shear stress and 1 normal stress component, it is at an angle to the normal vector.

Now, suppose if this shear stresses are absent which is the present case; then what will happen, the stress vector will be parallel to the normal vector. There are no shear stress components at all, only one component and that will be parallel to the n vector. That is what we are going to state now, stress vector is parallel to the normal vector. And then when a vector is parallel to another vector how do we write, the stress vector in this case is parallel to the normal vector just a scalar multiple of the other vector.

$$t_n = \lambda n = \lambda n_x i + \lambda n_y j + \lambda n_z k$$

So, if a vector is parallel to another vector, we express that as scalar multiple. So, this is our expression for stress vector using the second method. Like to summarize this slide again, we made a physical observation based on experimental observation; experimental observation tells you that a fluid under static condition there are no shear stresses. Experimental observation is same that we are used in two different ways; the first implication of the absence of shear stress is that, the stress tensor is diagonal, because of that we got the first expression for stress vector.

Once again going back to the same experimental observation, that there are no shear stresses in a static fluid; we infer that the stress vector normal should be parallel to each other and hence write the second expression for stress vector; same physical observation, experimental observation resulting in two different expressions, two equivalent expressions for stress vector, both should be same we will equate it.

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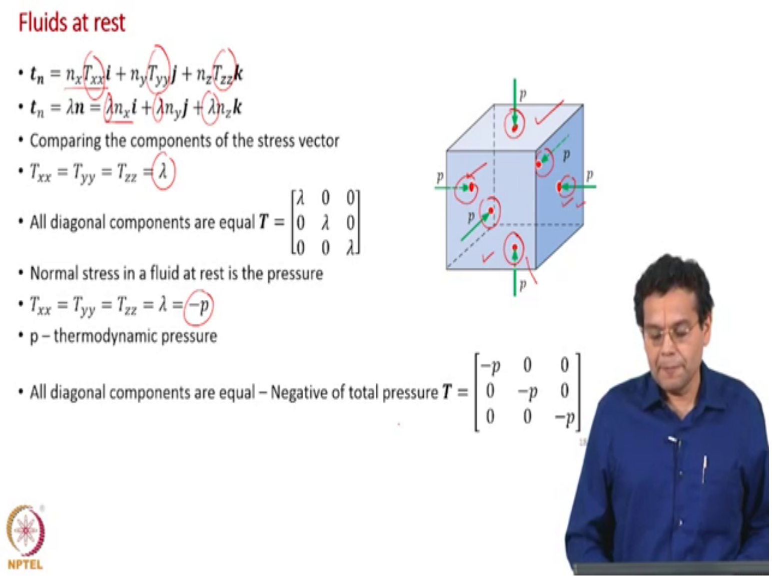

That is what we going to see here, the first expression is,

$$t_n = n_x T_{xx} i + n_y T_{yy} j + n_z T_{zz} k$$

$$t_n = \lambda n = \lambda n_x i + \lambda n_y j + \lambda n_z k$$

The first equation is based on the fact that the stress tensor becomes diagonal; second is based on the fact that the stress vector becomes parallel to the normal vector. Now component wise they should be equal to each other, what does it mean,

$$\lambda n_x = n_x T_{xx} \rightarrow \lambda = T_{xx}$$

Similarly,

$$\lambda n_y = n_y T_{yy} \rightarrow \lambda = T_{yy}$$

$$\lambda n_z = n_z T_{zz} \rightarrow \lambda = T_{zz}$$

Now, comparing the components of the stress vector, stress vector is same two different ways of expressing.

$$T_{xx} = T_{yy} = T_{zz} = \lambda$$

What is a physical equivalence of this statement? If you have a static fluid; first observation is that there are only normal stresses. If there are only normal stresses, all the normal stresses should be equal to each other that is a conclusion and that whatever number or scalar we calling it as λ. I will repeat in a fluid under static condition there are only normal stresses, if there are only normal stresses all of them should be same we are calling it as λ.

Now, of course, other way of putting it is all diagonal components are equal, so now, our stress tensor becomes diagonal.

$$T = [\lambda\ 0\ 0\ 0\ \lambda\ 0\ 0\ 0\ \lambda]$$

So, if you have only normal stresses, only if all of them are equal then it will be under equilibrium and force balance will be valid, that is other way of interpreting this.

Now, the normal stress in a fluid at rest is the pressure and so,

$$T_{xx} = T_{yy} = T_{zz} = \lambda = -p$$

Then this p is the thermodynamic pressure, exactly what you have come across in thermodynamic course, no difference at all. We have seen pressure is compressive and so, it is a negative p.

So now, we said that all the normal stresses are equal; in a fluid under static condition, normal stress is pressure, because pressure is compressive it is a negative p. p is the thermodynamics pressure, which we come across in thermodynamic course. All the diagonal components are same and they are negative of pressure. So, if you write the total stress tensor in terms of pressure, it becomes

$$T = [-\ p\ 0\ 0\ 0\ \ -\ p\ 0\ 0\ 0\ \ -\ p\ ]$$

Now let us look at the stress element; how do you represent stress element? We have seen a more general representation of stress element, here the components are only the normal stresses. So, if you want to represent this as stress element, if you take the right hand side face remember it is negative p; which means that on a positive face your force should be along the negative x axis. And then on a negative face because it is minus p your force should be along positive x axis; similarly along y axis and then similarly along z axis.

On all the positive faces, the force should be along the negative axis direction. So, the summary of this slide is, the stress tensor is diagonal for fluids at rest and diagonal elements are negative of pressure.

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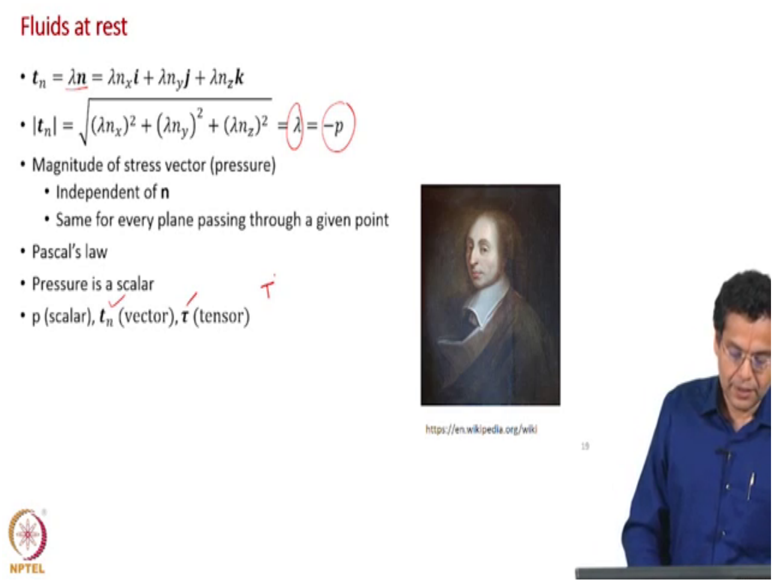

Now, let us proceed further, look at one more characteristic; the stress vector was along the normal. So, we represented as

$$t_n = \lambda n = \lambda n_x i + \lambda n_y j + \lambda n_z k$$

This expression we have seen already now, let us find out, the magnitude of the stress vector

$$\left|t_n\right| = \sqrt{\left(\lambda n_x\right)^2 + \left(\lambda n_y\right)^2 + \left(\lambda n_z\right)^2} = \lambda =- p$$

So, the magnitude of the stress vector is the pressure of course, with the negative sign; but what is observation? we are saying that imagine this to be a fluid object have a surface in the fluid, what we are saying, the stress under rest condition. We are saying that, there were stress vector acting and then the magnitude of the stress vector is the pressure of course, the negative sign.

But what is more important observation is that, right hand side does not depend on n, does not depend on the components n_x, n_y, n_z. What is it mean? whatever be the angle of the surface, their magnitude of the stress vector is the pressure which is same and is independent of the orientation of this surface. So, the pressure acting is same for every plane passing through a given point.

That is what we said, the n represents the direction of the plane. So, every plane the or if you take a plane like this whatever be the orientation of the plane at the same point; of course, imagine a very small plane, the pressure is same.

Now, that statement is the Pascal's law. At the point pressure is same in all directions, that is what we have stated there; the magnitude of stress vector is a pressure and that is independent of the plane,. So, pressure is independent of direction and that is the Pascal's law. And that is why, pressure is a scalar. of course we know that any quantity is independent of direction is a scalar; like density, temperature etcetera. Only difficulty is, little bit difficult to understand pressure is a scalar; reason is very simple why, that I will explain now.

Pressure is force per area, stress vector is force per area, stress tensor is force per area; so if you go by units, looks little confusing. But now pressure is a scalar; why?, as we have seen now, we have proved that pressure at a point is independent of direction. Any quantity, so independent of direction only which is can be characterized by magnitude alone it is a scalar, that is why pressure scalar though the units are force per area.

Let us go to the stress vector, the force acting on a given plane, because of plane it is specified and it is a force, it is a vector. There is a direction, because the direction of the area is given, you need only the direction of the force which is direction of the stress vector and because you require only one direction of course, magnitude is there, it is a vector.

Now, go to stress tensor; what does it say, at the same point you can have any number of planes, stress tensor will give you information about stress vector acting on all the planes. So, we are not specifying any plane at all there, that is why it is a stress tensor.

So, we are not specifying any direction at all for the plane that is why you have direction for force, direction for the plane as well resulting in a stress tensor. This could be τ or in our case it could be the T total stress tensor also; just want to illustrate that though the units are same it could be a scalar as pressure, vector as stress vector could be a tensor as τ or T.

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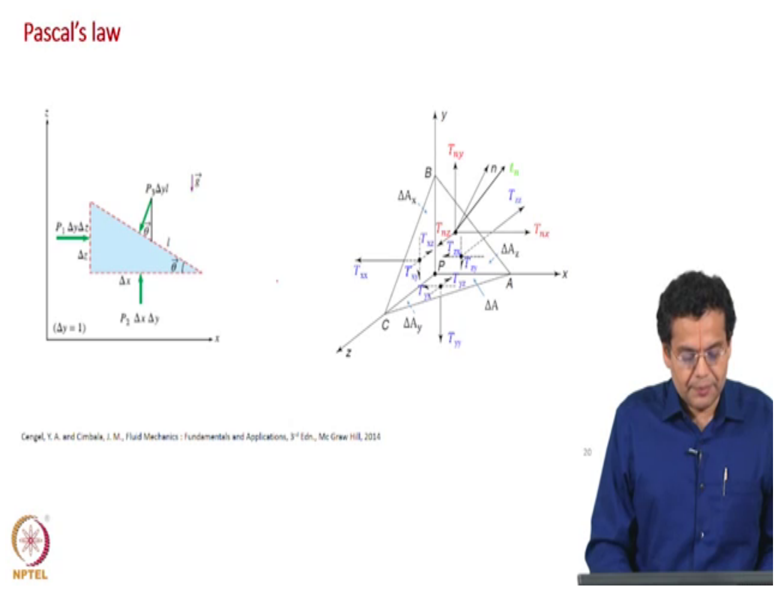

Just want to compare the way in which Pascal's law is discussed in fluid mechanics book and then what we have discussed. It may so appear that, what we have discussed is different from what in the books, just want to show that both are same. We have taken this picture from this book Cengel, what they show is a wedge shaped element and they show pressure acting on the phases and then look at the any surface incline at an angle and they show the pressure acting on that phase. To begin with nomenclature P_1, P_2, P_3 etcetera has given.

This is what is this wedge shaped element is nothing, but our 2D version of our tetrahedron; because we discuss under 3D condition we took tetrahedron. Look at one projection of that and then another plane attached to that, this is a plane at an angle θ with the horizontal. We also said, we have a plane at any angle, so both are equivalent that is done for 2D, 2 dimensional case we have done 3 dimensional case. What are the other differences? Pascal's laws derived in your let us say second chapter fluid mechanics book and a fluid mechanics book slowly gradually develops from fluid under rest condition, then without viscous forces, with viscous forces etcetera.

So, because it is considering fluid under static condition, they do not talk about a general stress; they talk about only pressure as the only normal stress. Because we talked about a general stress, we said more physically, fluid under static condition has only pressure as the normal stress. So, in more specific and then discuss about only pressure to, we are more general saying as T_{xx}, T_{yy}, T_{zz}; but finally, of course, we have also came down to pressure only. Then what they do here is write a force balance in the x direction, here in this case x direction x direction, etcetera; that is what we also did, what is our Cauchy's law, a force balance in the respective direction.

Of course, so both are analogous, I was little more general than what is given there. That is the scope of that chapter at that level of entry into fluid mechanics book, it is an simpler way. But because we are discussing and then taking concepts from solid mechanics we are more general discussion, conceptually both are same. They also finally prove that

$$p_1 = p_2$$

Similarly, we also proved

$$T_x = T_y$$

Finally, they prove that p_3 is same, we also proved that the magnitude of stress vector is pressure. So, eventually it is all equivalent to each other.

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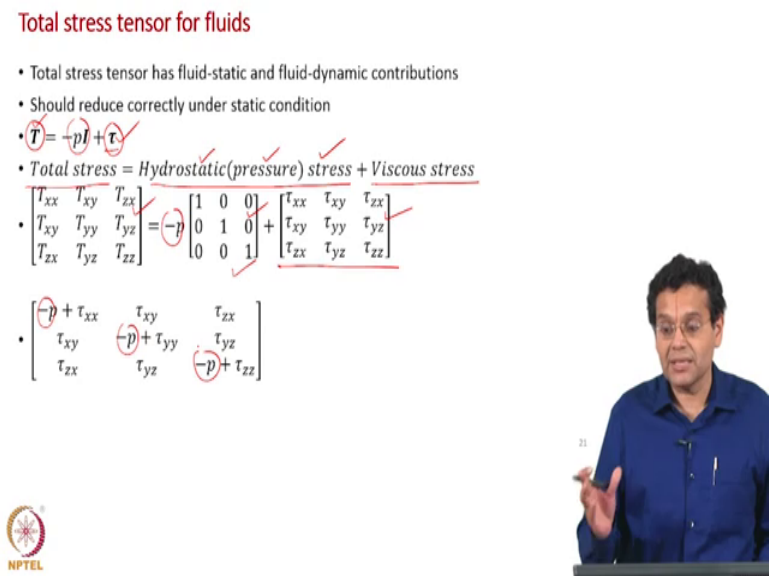

Then now, we are in a position to express total stress tensor for fluids. We have first introduced total stress tensor and I just made a statement that; a fluid under static condition has some stress, when it is flowing it has some additional stress. Fluid under static condition whatever stress it has what we have discussed now, we have to just add the additional stress.

Total stress tensor has fluid static and fluid dynamic contributions. Now, you are in a better position to understand this statement and then should reduce correctly under static condition. We have two components; so one should vanish when the fluid is not moving and only the static component should remain, that should be taken into consideration.

So, the how do we represent the total stress tensor

$$T = -pI + \tau$$

Total stress = Hydrostatic (pressure) stress + Viscous stress

The total stress in a moving fluid has two components; one the component because of the static condition which we call as hydrostatic stress, other additional stresses developed because of fluid in motion which are called as viscous stresses; why viscous stress, you will understand later; right now we will coin a name called viscous stress.

So, the total stress in a fluid has two components, just under static condition it has some stress; we have seen that at is purely due to pressure. That is why I have written as hydrostatic stress in bracket I have written pressure. Now, when it starts moving additional stresses develop which are the viscous stresses; the τ for fluids and the τ in solids are exactly analogous.

Only the T was different, now we understand why we introduced T when we came to fluids; T has two components, hydrostatic and viscous. The τ in fluids and the τ in solids analogues I mean how is it, you will understand as we go along; but that is why same nomenclature is used, physical significance everything is same. But T has one extra additional component; that is why, moment we enter fluids first line I said let us introduced total stress in fluids.

$$T = -pI + \tau$$

$$\left[T_{xx}\ T_{xy}\ T_{zx}\ T_{xy}\ T_{yy}\ T_{yz}\ T_{zx}\ T_{yz}\ T_{zz}\right] = -p[1\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 1] + \left[\tau_{xx}\ \tau_{xy}\ \tau_{zx}\ \tau_{xy}\ \tau_{yy}\ \tau_{yz}\ \tau_{zx}\ \tau_{yz}\ \tau_{zz}\right]$$

And if you sum up term by term.

$$\left[T_{xx}\ T_{xy}\ T_{zx}\ T_{xy}\ T_{yy}\ T_{yz}\ T_{zx}\ T_{yz}\ T_{zz}\right] = \left[-p + \tau_{xx}\ \tau_{xy}\ \tau_{zx}\ \tau_{xy}\ \ -p + \tau_{yy}\ \tau_{yz}\ \tau_{zx}\ \tau_{yz}\ \ -p + \tau_{zz}\right]$$

So, the off diagonal elements there is no change, only along the diagonal elements; we have seen that under static condition the stress tensor becomes diagonal. So, to the diagonal elements minus p gets added. So to, just in one line if you want to summarize whatever we discussed now, this stress tensor is analogous to solids number 1; number 2 to the diagonal elements minus p gets added, if you want to just mathematically say; we all discuss the physics very in detail. Bottom line take one point from this is that slide is that, 2 components, some stress acts when the fluid is under rest, when it flows some additional stresses are there.

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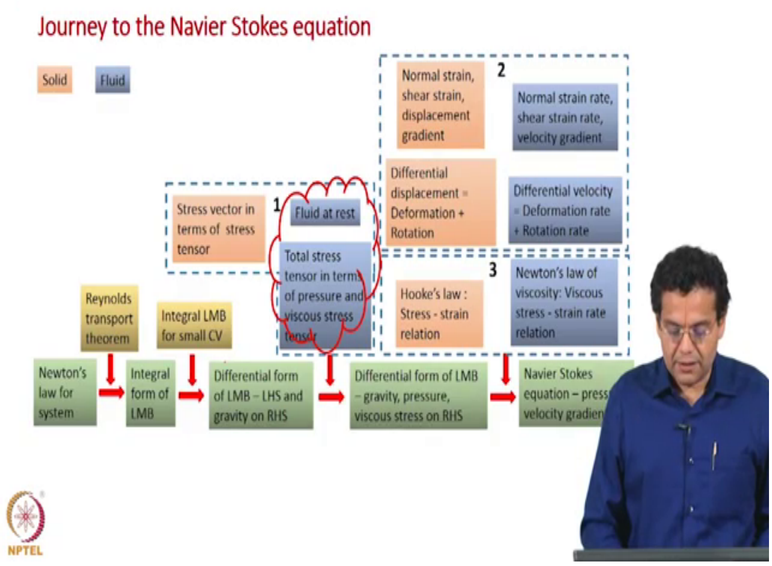

So, now, we can understand that highlighted portion. We were in third box, having derived the left hand side of differential linear momentum balance, body force in the right hand side. And then for surface forces on the right hand side we took a diversion to solid mechanics; discussed about stress vector, stress tensor.

Now, whatever highlight becomes very clear to you now, based on the discussion we are so far. We came to back to fluids, first discussed fluid at rest. And then now look at the term here it says, total stress tensor in terms of pressure and then stress tensor that is the summary of what I discussed under fluids. Now we have to go back to our right hand side of linear momentum balance and complete it.

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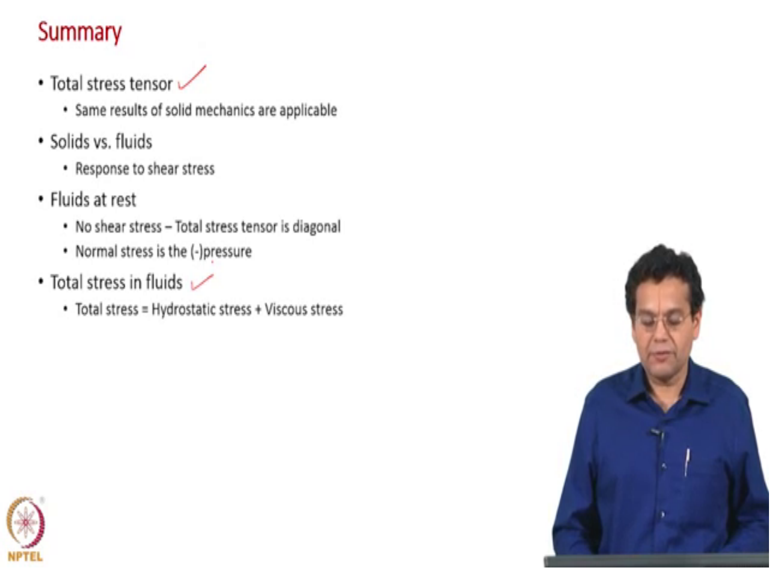

So, just before that let us summarize what I discussed so far; total stress tensor was introduced for case of fluids, same results of solid mechanics are applicable; we discuss the difference between solids versus fluids, specifically in terms of response to shear stress. And the most important difference is that, no shear stress in a fluid under rest condition which leads to that total stress tensor is diagonal, also leads to that the stress vector is parallel to the normal vector and only normal stresses are present and they are nothing but the thermodynamic pressure, negative because they are compressive, pressure is compressive. So, based on that, we are able to write expression for total stress and fluids. Here we just introduce, here only we are able to express that. Total stress has two components, hydrostatic stress plus viscous stress and one stress acts when the fluid is in rest condition; additional stress acts when the fluid is flowing.