Continuum Mechanics And Transport Phenomena
Prof. T. Renganathan
Department of Chemical Engineering
Indian Institute of Technology, Madras

Lecture - 43
Properties of stress tensor – Part 2

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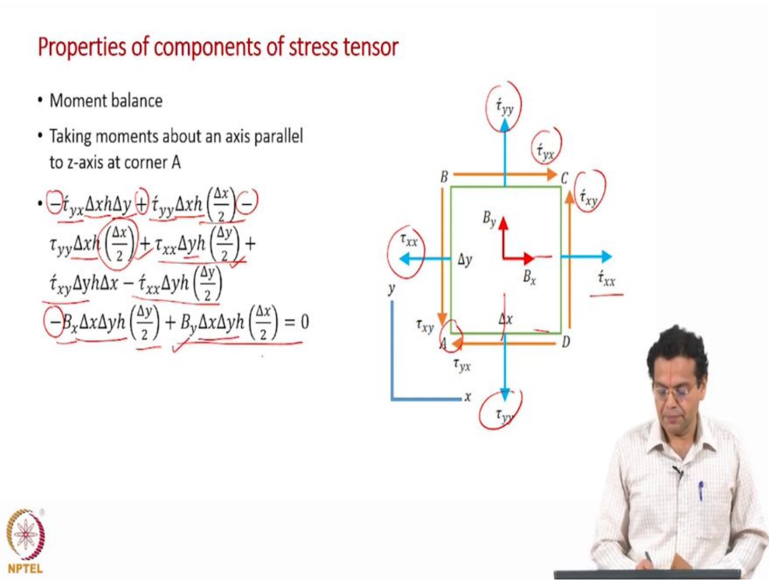

We will now derive another property for the components of stress tensor which will help us to simplify the stress tensor. In this case, we do a moment balance for the same geometry. And for the case of moment balance we need a axis of rotation, so that we can identify the lever arm for each of the force. So, let us do that.

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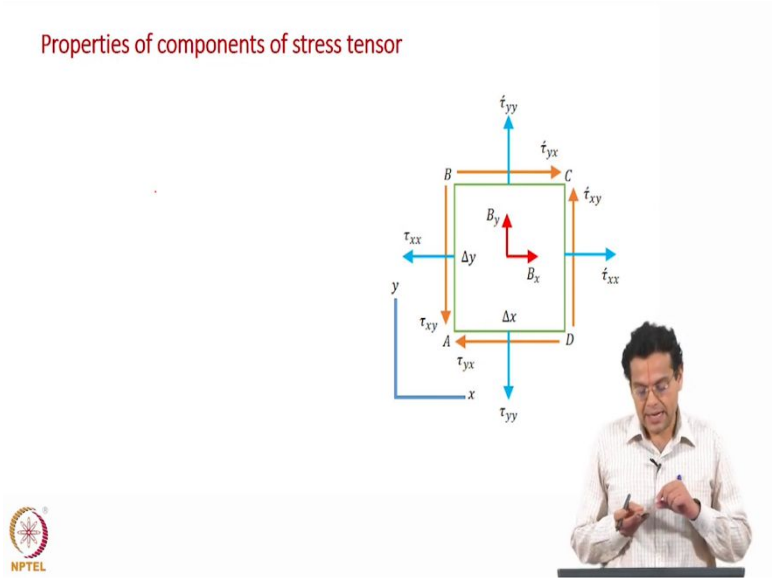

So, moment balance we are going to write. We take moments about an axis parallel to z axis at corner A. We have plane in the xyz axis and I had taken a z axis at corner A. Now, let us write down the moment balance. We will have to account for the sign, the force and the perpendicular distance, 3 are to be accounted for each of them.

$$-\dot{\tau}_{yx}\Delta x h\Delta y + \dot{\tau}_{yy}\Delta x h\left(\frac{\Delta x}{2}\right) - \tau_{yy}\Delta x h\left(\frac{\Delta x}{2}\right) + \tau_{xx}\Delta y h\left(\frac{\Delta y}{2}\right) + \dot{\tau}_{xy}\Delta y h\Delta x - \dot{\tau}_{xx}\Delta y h\left(\frac{\Delta y}{2}\right) - B_x\Delta x\Delta y h\left(\frac{\Delta y}{2}\right) + B_y\Delta x$$

And we consider clockwise as negative anti-clockwise is positive. How do you assign clockwise, anti-clockwise? Let us say you have pivoted your plate here and if you are focusing on one force let us say that force deflects your plate in this direction that is anti-clockwise and then you pivot it let us say that force deflects your plate in the clockwise that is clockwise sense. That is how we identify the sign. Let us keep this in mind and let us look at the sign of each. So, $\dot{\tau}_{yx}$, we pivot it and then allow that force alone it will rotate in the clockwise direction. That is why it is negative, anti-clockwise is positive. So, we multiplied the stress with the area to get the force and multiply with the perpendicular distance and taken the sense of rotation as well.

Now, let us go to $\dot{\tau}_{yy}$. Now, this will rotate in the anti-clockwise direction. If you pivot it here and allow this force this will rotate in the anti-clockwise direction and that is why this is positive.

And let us go to the next term τ_{yy} . So, now, this will rotate in the clockwise direction. This is acting in this direction, if you pivot it here this will rotate in the clockwise direction and that is why you have negative sign here. So, similarly you can explain other terms for $\dot{\tau}_{xy}$, $\dot{\tau}_{xx}$, and τ_{xx} .

Now, coming to the body force. We have B_x per unit volume, so we multiply by volume you get the force per unit volume multiplied by volume you get the force and if you leave the B x it rotates in the clockwise direction that is why this negative. Analogously for the that due to the y direction body force B_y multiplied by volume perpendicular distance. That is a moment balance.

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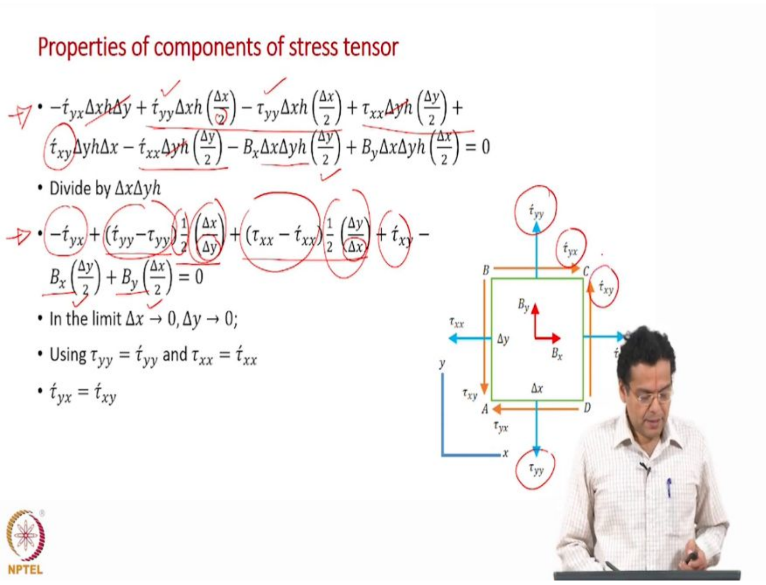

Now, let us write that quickly the moment balance here.

$$-\dot{\tau}_{yx}\Delta xh\Delta y+\dot{\tau}_{yy}\Delta xh\left(\frac{\Delta x}{2}\right)-\tau_{yy}\Delta xh\left(\frac{\Delta x}{2}\right)+\tau_{xx}\Delta yh\left(\frac{\Delta y}{2}\right)+\dot{\tau}_{xy}\Delta yh\Delta x-\dot{\tau}_{xx}\Delta yh\left(\frac{\Delta y}{2}\right)-B_x\Delta x\Delta yh\left(\frac{\Delta y}{2}\right)+B_y\Delta x$$

Now, we will divide by $\Delta x\Delta yh$ that is the next step,

$$-\dot{\tau}_{yx}+\dot{\tau}_{yy}\left(\frac{\Delta x}{2\Delta y}\right)-\tau_{yy}\left(\frac{\Delta x}{2\Delta y}\right)+\tau_{xx}\left(\frac{\Delta y}{2\Delta x}\right)+\dot{\tau}_{xy}-\dot{\tau}_{xx}\left(\frac{\Delta y}{2\Delta x}\right)-B_x\left(\frac{\Delta y}{2}\right)+B_y\left(\frac{\Delta x}{2}\right)=0$$

Then,

$$-\dot{\tau}_{yx}+(\dot{\tau}_{yy}-\tau_{yy})\frac{1}{2}\left(\frac{\Delta x}{\Delta y}\right)+(\tau_{xx}-\dot{\tau}_{xx})\frac{1}{2}\left(\frac{\Delta y}{\Delta x}\right)+\dot{\tau}_{xy}-B_x\left(\frac{\Delta y}{2}\right)+B_y\left(\frac{\Delta x}{2}\right)=0$$

Now, what should you do next? We are trying to find a relationship at a point. So, as usual take limit $\Delta x \to 0$, $\Delta y \to 0$, it becomes a point. Now, as I repeated several times, it becomes very evident here, you should keep the aspect ratio same. You should not say that $\Delta x \to 0$, $\Delta y \to 0$, it becomes indeterminate. We shrink the plate keeping the expect aspect ratio same, so this $\frac{\Delta x}{\Delta y}$ is a finite value till the final point to string to a point. So, it becomes very clear this example, in this particular case. We are done told this several times for control volume, surface, etcetera. Here there is a ratio of the dimensions of the length and width. You are sinking this plate to a point and should always keep the aspect ratio, so $\frac{\Delta x}{\Delta y}$ is a finite value till even though both of them tends to 0. And as usual, when I write the moment balance they are all average values, now they become point values.

So, now as we proved earlier,

$$\tau_{xx} = \dot{\tau}_{xx}$$

$$\dot{\tau}_{yy} = \tau_{yy}$$

So, now, because $\Delta x \to 0$, $\Delta y \to 0$, the body terms also will not contribute. So, what is left out are only the shear stress terms. So, now, all the terms vanishes resulting in

$$\dot{\tau}_{yx} = \dot{\tau}_{xy}$$

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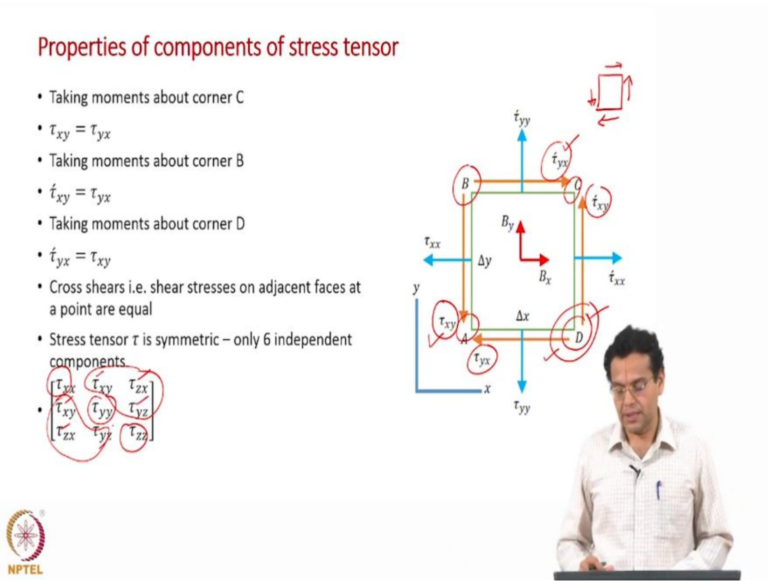

Now, we have taken this moment, moment about A and proved that $\dot{\tau}_{yx} = \dot{\tau}_{xy}$. So, it is intuition, by intuition we can say that to prove others we should correspondingly take moments about other axis. So, let us take moment about corner C. We take moment about corner C, so obviously, we will prove that

$$\tau_{xy} = \tau_{yx}$$

Now, taking moments about corner B, if you take about corner B then we will prove that

$$\dot{\tau}_{xy} = \tau_{yx}$$

Finally, taking moments about corner D. Corner D, then we will prove that

$$\dot{\tau}_{yx} = \tau_{xy}$$

Eventually, what happens? Though there are 4 terms shown there all of them are same. We have proved shown that though I have drawn to begin with the different variables all of them are the same values; all of them are τ_{xy} that is all. What is the conclusion? Cross, shears, the shear stresses acting on adjacent faces at a point are equal. You have a point, earlier it was same face opposite side. Now, we have two adjacent faces.

So, cross shears acting on adjacent faces are equal that is a conclusion. What is the implication of that? Stress tensor is symmetric, only 6 independent components, that is some relief for us, otherwise we are deal with 9 stress tensor components. Thanks to this equality, thanks to the moment balance that only 6 independent components.

$$\tau = \left[\tau_{xx}\ \tau_{xy}\ \tau_{xz}\ \tau_{yx}\ \tau_{yy}\ \tau_{yz}\ \tau_{zx}\ \tau_{zy}\ \tau_{zz}\ \right]$$

We proved that,

$$\tau_{xy} = \tau_{yx};\ \tau_{zx} = \tau_{xz};\ \tau_{yz} = \tau_{zy}$$

So,

$$\tau = \left[\tau_{xx}\ \tau_{xy}\ \tau_{xz}\ \tau_{xy}\ \tau_{yy}\ \tau_{yz}\ \tau_{xz}\ \tau_{yz}\ \tau_{zz}\ \right]$$

So, among the 6 off diagonal elements we need to consider only 3 of them, look at the way in this stress tensor has written. Earlier we have written as all the 9 to be independent. Here these lower diagonal elements below the diagonal are written as same as those above the diagonal. So, in this stress tensor representation only 6 independent components are shown, 3 along the diagonal and only 3 off diagonal elements; not 6 off diagonal elements. So, only 6 components have to be dealt with. Same thing can be extended to other planes as well.

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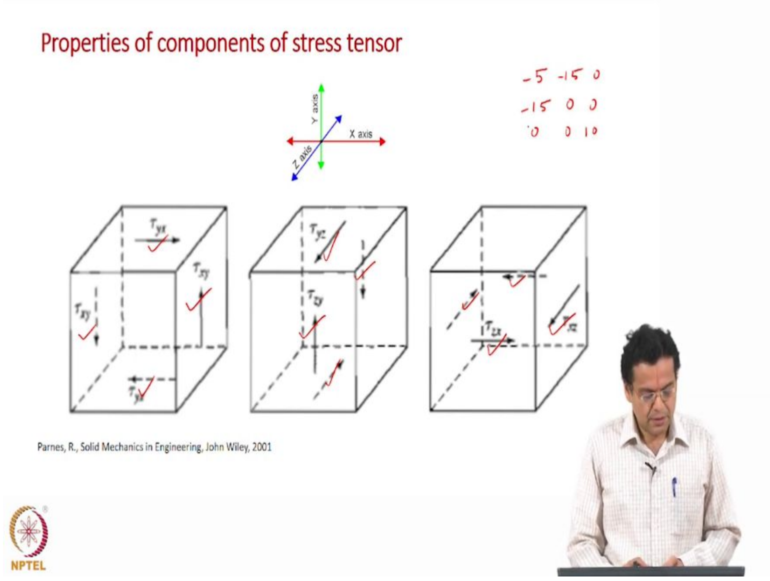

What you have discussed is these 4. Similarly of course, representation from Parnes, that $\tau_{yz} = \tau_{zy}$ that is what we have seen and similarly these two on the opposite faces; $\tau_{zx} = \tau_{xz}$ similarly opposite faces. Even the example which you have consider the numerical example to illustrate stress element, we took only a symmetric stress tensor. Though of course, I did not really mention that time, now we can recall saying that it says what a symmetric stress tensor.

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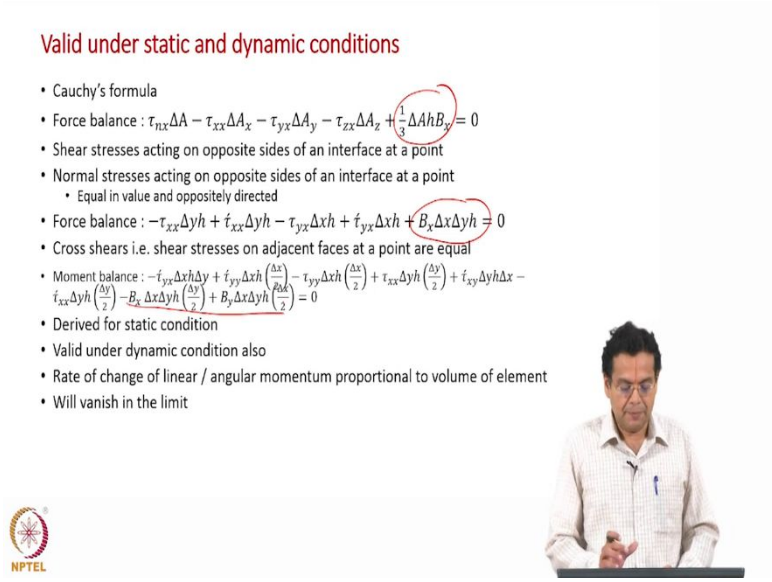

Now, what we have derived is under static condition because an equilibrium condition, but whatever results we have seen are valid under dynamic conditions also. And that is what we are going to just mention here, not really prove. Let us summarize what we have done, all the results whatever I discussed so far in this part.

- Remember we started with Cauchy's formula. We wrote a force balance.

$$\tau_{nx}\Delta A - \tau_{xx}\Delta A_x - \tau_{yx}\Delta A_y - \tau_{zx}\Delta A_z + \tfrac{1}{3}\Delta AhB_x = 0$$

- The force balance is shown here. On the right hand side was 0, all the time we took the right hand side to be 0 and,
- Next, we proved that shear stresses acting on opposite sides of interface at a point and normal stresses acting on opposite sides of an interface at a point are equal in value opposite direction. For that we use the force balance again, of course, geometry is no different for the case of Cauchy's formula it was tetrahedron, now it is a two dimensional plate. Here again, right hand side was 0.

$$-\tau_{xx}\Delta yh + \dot{\tau}_{xx}\Delta yh - \tau_{yx}\Delta xh + \dot{\tau}_{yx}\Delta xh + B_x\Delta x\Delta yh = 0$$

- Now, for cross shears we proved that shear stress on adjacent plane adjacent faces at a point are equal, we wrote a moment balance. Again in the moment balance we took the right hand side to be 0, all these equations the force balance at the right hand side 0, the moment balance the right hand side 0 are all under static conditions.

$$- \dot{\tau}_{yx}\Delta x h \Delta y + \dot{\tau}_{yy}\Delta x h \left(\frac{\Delta x}{2}\right) - \tau_{yy}\Delta x h \left(\frac{\Delta x}{2}\right) + \tau_{xx}\Delta y h \left(\frac{\Delta y}{2}\right) + \dot{\tau}_{xy}\Delta y h \Delta x - \dot{\tau}_{xx}\Delta y h \left(\frac{\Delta y}{2}\right) - B_x \Delta x \Delta y h \left(\frac{\Delta y}{2}\right) + B_y \Delta x$$

If you are writing the balance under dynamic conditions what will you have in the right hand side, you will have a rate of change of term, rate of change of momentum term. And then you will have rate of change of linear momentum for force balance. You will have rate of change of angular momentum for the momentum balance. But those terms are all proportional to volume, proportional to mass and,

- Remember, when we took the limit $\Delta x \to 0$, $\Delta y \to 0$ these terms wherever you had body force they are also proportion the volume. They never played a role in the final relationship. These terms never played final role in the final relationship. So, even if you include the right hand side, the rate of change of linear momentum rate of change of angular momentum those being proportional to the volume or mass, they will not play a role.
- In the surface area, you have some linear dimensions squared length into width or something, there you will have length width and some thickness also is there that is proportional to volume. When you shrink $\Delta x \to 0$, $\Delta y \to 0$, one linear dimension remains and when you shrink to 0 that will vanish. So, those will not contribute. So, that is why we are not, just to avoid complexity we are just mentioning you are not taking that into account the derivation, but intuitively reasoning out we can understand.
- So, derive for static condition valid and dynamic condition also. We have discussed that rate of change of linear angular momentum proportional to the volume of element will vanish in the limit $\Delta x \to 0$, $\Delta y \to 0$. So, we should not assume or come to the conclusion that these results are valid only under static condition. They are certainly valid under dynamic conditions also. That is the key thing I have to say.

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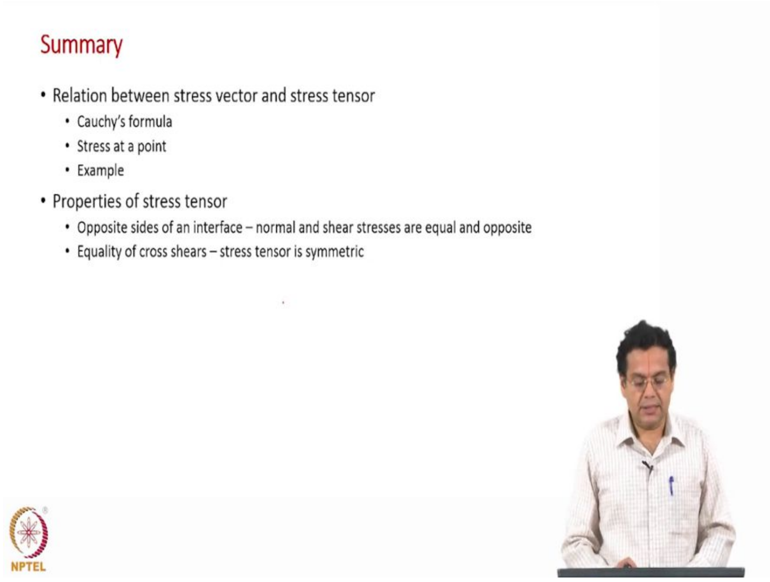

So, let us summarize we derived a relationship between the stress vector and stress tensor, namely the Cauchy's formula, simple force balance. So, now, we should you know what is that relationship, simply writing a force balance. And we also discussed what do you mean by stress at a point conceptually. It has all the information for finding out stress vector on any plane.

We looked at a numerical example represent that in terms of 3D representation in terms of vectors. Then we arrived the properties of stress tensor first is that opposite sides of interphase the shear stresses and the normal stresses are equal and opposite. And we also discuss about equality of cross shears, two adjacent planes, the shear stresses are equal. Important conclusion is that the stress tensor is symmetric, that is the most important to take away from the second property.

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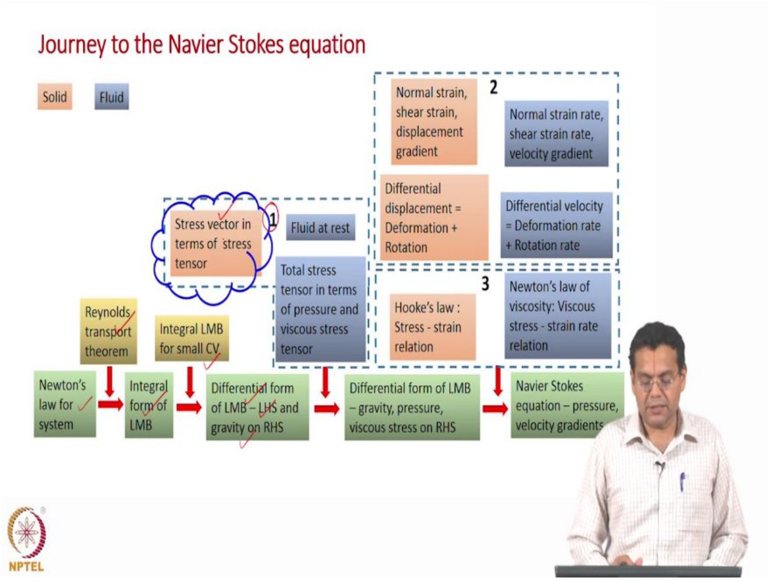

So, this brings us to the close of our first visit to solid mechanics. In terms of our journey to the Navier-Stokes equation we started with the Newton's second law of motion, applied the Reynolds transport theorem and derived the integral form of linear momentum balance. We applied that for a small control volume, derived the differential form of linear momentum balance, the left hand side and the gravity on the right hand side.

And we had surface forces in the right hand side understand surface forces, we took a diversion to solid mechanics. And now, this highlighted box is very clear to us, we discuss about stress vector, stress tensor, the relationship between the two and the properties of stress tensor.

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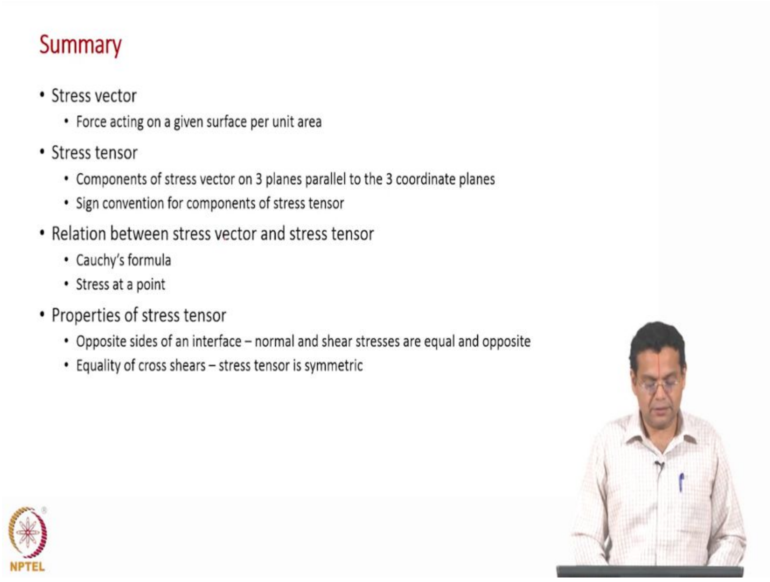

Summarize whatever I discussed during the first visit to solid mechanics. These are the topics which I discussed. We defined a stress vector, force acting on a given surface per unit area. We discuss about a stress tensor and a stress tensor that is 9 components and these are the components of stress vector acting on 3 planes parallel to the 3 coordinate planes. We also discuss about the sign convention for the components of stress tensor, and then we related the stress vector to the stress tensor by the Cauchy's formula and hence discussed the state of stress as stress at a point.

We derive the properties of stress tensor. What are the properties? On the opposite sides of an inter phase the normal and the shear stresses are equal, but opposite in direction and we also proved by using a moment balance that the cross shears are equal which means that the stress tensor is symmetric.