Continuum Mechanics And Transport Phenomena
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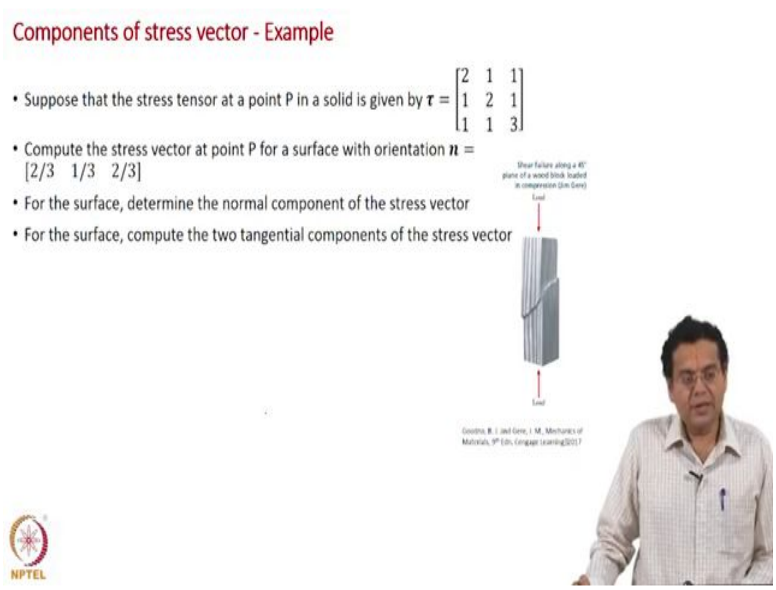
Lecture - 41
Components of Stress Vector: Example

Example: (Refer Slide Time: 00:14)



Now, we will look at an numerical example with an illustrate whatever we have discussed so far in terms of expressions. But before going to that I would say mathematical or theoretical example, what is shown right hand side of the figure, where whatever it discussed more practically applied. What is shown there is a bar, a vertical bar and subject to a compressive load, compressive because both are acting towards each other, it is a compressive load.

Now, based on the equations what we have derived we can find out that at 45 degrees you have maximum shear stress. Now, if the material stress which is can withstand is less then this maximum shear stress then It breaks along 45 degrees axes, 45 degrees plane. So, you can prove that based on this Cauchy's law that maximum shear stress occurs at an angle of 45 degrees. Because how can you do that? Our n can be any plane. So, find out the n or angle at which the maximum shear stress. And practically, you can observe that when you keep increasing the load, and if the maximum shear stress exceeds the allowable shear stress. Usually, materials have some limit for shearing direction or for normal direction. The normal values may be more, but shearing values may be less.

So, if your resulting shearing stress along 45 degrees exceeds then it starts breaking. That is the example shown here. So, if you look at solid mechanics book lot of such practical examples are given. Our objective of studying solid mechanics to take principles from here and applied to fluid mechanics, so we are not discussing in detail about such examples. What we will do is a more mathematical example, theoretical example with clearly illustrate what we had discussed so far. So, that is why this example is shown in this slide. There is no relationship between this diagram and this example apart from that both are applications.

Now, suppose that the stress tensor at a point in a solid is given by, let us say matrix.

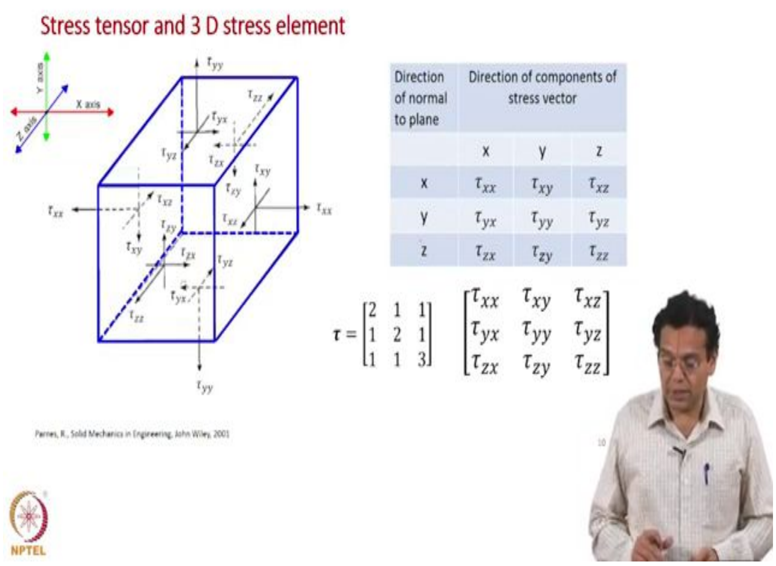
$$\tau = [2\ 1\ 1\ 1\ 2\ 1\ 1\ 1\ 3\ ]$$

We are given all the 9 components of stress tensor. Then we are asked to calculate the following,

- Compute the stress vector at point P. The stress tensor at a point P is given, we are asked to find out stress vector at point P for a surface with orientation $n = [2/3\ 1/3\ 2/3\ ]$, which means the normal vector to the plane is given to us.
- Now, for the surface determine the normal component of stress vector which in the normal stress.
- Next is compute the two tangential components meaning the shear stresses.

Remember this was our objective to calculate these values. As I told you the first step was done through Cauchy's formula, second step is more easily done through vector algebra that will be illustrate in this example. Second step is we have stress factor components along x, y, z, we have to have components along n, S_1, S_2, we will see that now.

Solution: (Refer Slide Time: 03:54)



What is given to us? Let us understand that in two ways. Remember we had this stress element with 9 independent components, we had 9 more which are same as the other 9 and earlier we had expression for τ just introduced variables. Now, we have numerical values for them in this example.

$$\tau = \left[\tau_{xx}\ \tau_{xy}\ \tau_{xz}\ \tau_{yx}\ \tau_{yy}\ \tau_{yz}\ \tau_{zx}\ \tau_{zy}\ \tau_{zz} \right] = [2\ 1\ 1\ 1\ 2\ 1\ 1\ 1\ 3\ ]$$

So, let us say each row represent components of stress vectors acting of x-plane, y-plane and z-plane.

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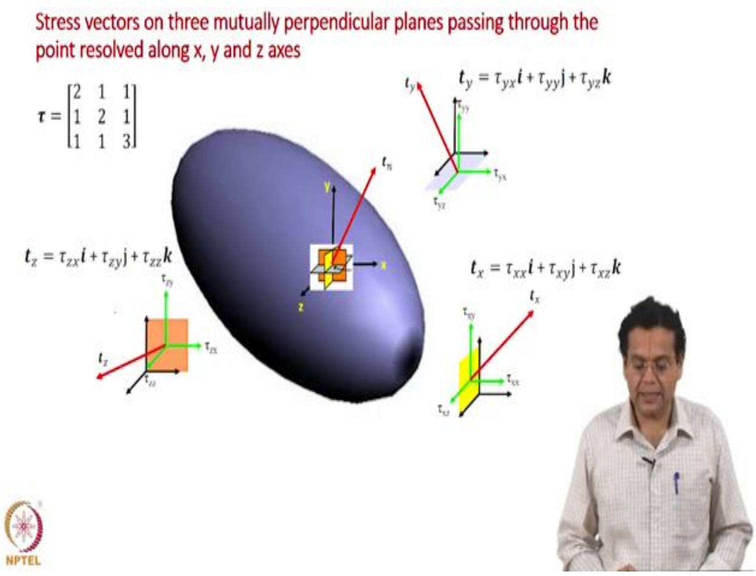

Looking at the other way, this is very good representation of what is given to us. We have three well defined planes and components of stress vector acting on x-plane for example, τ_{xx}, τ_{xy}, τ_{xz} are given as 2, 1, 1. And if you take the y-plane components of stress vector acting on the y-plane τ_{yx}, τ_{yy}, τ_{yz} are given as the second row, 1, 2, 1 and for the z-plane components of stress vector acting on z-plane τ_{zx}, τ_{zy}, τ_{zz} are given as the third row, 1, 1, 3. Just understand what is the value given to us, what is the matrix or stress tensor given to us.

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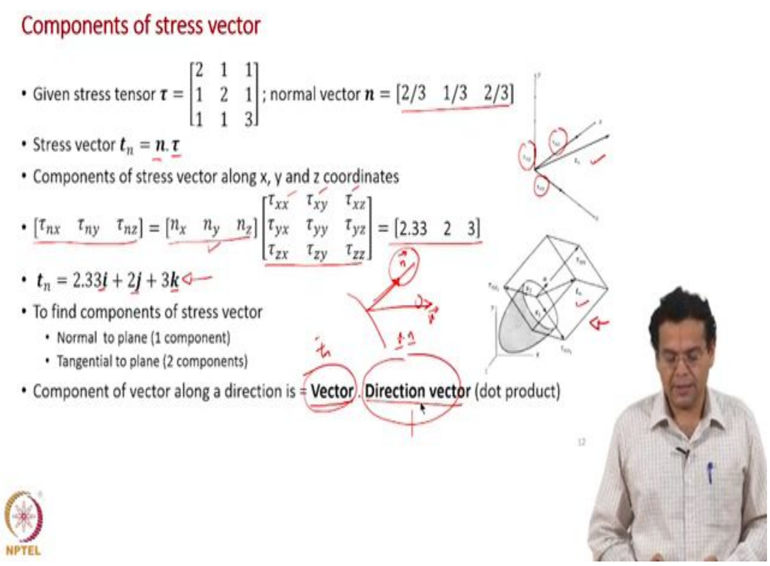

Now, we will find out components of stress vector in two ways, one along x, y, z axis which is shown here, then we will find along the normal and tangent to the plane. So, these are the values given to us, given stress tensor and the normal vector is given to us, it is a unit normal vector.

$$\tau = [2\ 1\ 1\ 1\ 2\ 1\ 1\ 1\ 3\ ]$$

And,

$$n = [2/3\ 1/3\ 2/3\ ]$$

Now, stress vector t_n just now seen the expression as

$$t_n = n.\tau$$

So, components of stress vector along x, y and z coordinates are the simple matrix multiplication of the n vector with the stress tensor.

$$\left[\tau_{nx}\ \tau_{ny}\ \tau_{nz}\ \right] = \left[n_x\ \ n_y\ n_z\ \right]\left[\tau_{xx}\ \tau_{xy}\ \tau_{xz}\ \tau_{yx}\ \tau_{yy}\ \tau_{yz}\ \tau_{zx}\ \tau_{zy}\ \tau_{zz}\ \right]$$

So, simple matrix multiplication of we have the values of n_x, n_y, n_z. We have values of all the 9 components here, simple matrix multiplication, you can use your calculator as well and get this three components of the stress vector, but they are along x, y, z axes.

$$\left[\tau_{nx}\ \tau_{ny}\ \tau_{nz}\ \right] = [2/3\ 1/3\ 2/3\ ]\ [2\ 1\ 1\ 1\ 2\ 1\ 1\ 1\ 3\ ] = [2.33\ 2\ 3\ ]$$

$$t_n = 2.33i + 2j + 3k$$

Now, this is a first step, where we related the component of stress tensor to components of stress vector along x, y, z axis. What is our objective? We have to find out along normal and tangential directions. So, next step is to find components of stress vector. Normal to plane, that is one component which is a normal stress and tangential to the plane, two components of them are there, they are the shear stresses.

Now, we have got a vector, we want component along a particular direction. Simple, what we should do just take projection of the vector along that direction. This we have already done. If you recall back when we discussed about Reynolds transport theorem, integral balances etcetera, remember we had a let us say an out flow surface, and then we had normal to the surface, and then we had velocity vector along some other direction. What do we do? We took $v.n$, which means that we are taking a component of v vector along the normal. So, to find projection of any vector along some other direction, it is enough if you just take a dot product of the vector with the direction vector that is what we have done here. We wanted when we say $v.n$, it is component of v along the n direction. So, we were interesting in getting what is the velocity along this normal direction, so we took up a projection of this.

Similarly, here we have one stress vector, we have the components of that, but we want components of the stress vector acting among some three well directions. So, we have to take a dot product of this vector with the direction vector which means that now, if you find out the three direction vectors this is nothing, but the stress vector. If you have vectors for all the three direction just take dot product that is what we are going to do now. This step is simple taking the dot product. What we are going to discuss now is what are the three direction vectors among n, S_1, S_2.

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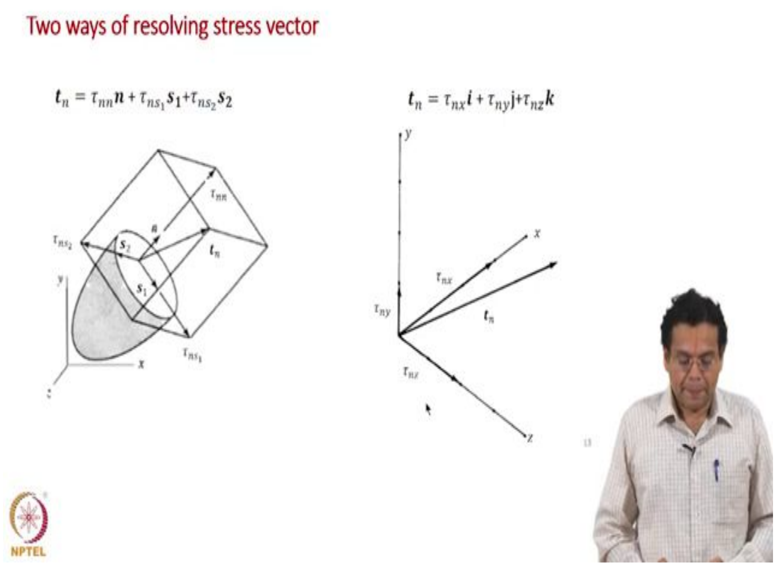

So, just to put it very clearly; we have evaluated τ_{nx}, τ_{ny}, τ_{nz}. We want to evaluate along the normal which is τ_{nn} and then other component is along S_1 and S_2, τ_{nS_1} another component is τ_{nS_2}. Right now what we are doing is, we are going to find out what is n vector, what is S_1 vector, what is S_2 vector. If you know that simple dot product of this stress vector with the direction vectors will give you the components along normal and the tangential directions.

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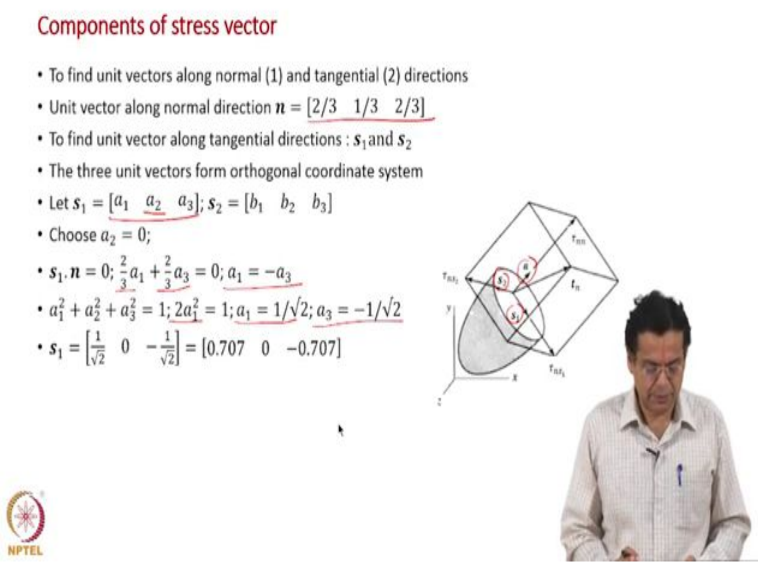

Among this n is already known, n is given to us. So, what is objective? To find unit vectors along normal and tangential directions that is objective, once you do that, simple dot product.

$$n = [2/3 \ 1/3 \ 2/3 \ ]$$

Now, unit vector along normal is given to us because that is a plane we are given, so n is given to us. Now, we have to find out the tangential directions.

Now, recall back that, when we introduce this coordinate system we introduce that coordinate system we introduce as a orthogonal coordinate system, n is normal to the plane, S_1, S_2 on the surface of the plane. All these three directions form a orthogonal coordinate system. And we will use this to find out S_1 and S_2, to find unit vectors along tangential directions that is the objective right now, because n is known to us.

Now, the three unit vectors from orthogonal coordinate system; what does it mean? The dot product of these vectors in pair should be 0,

$$n.S_1 = S_1.S_2 = S_2.n = 0$$

Let us call

$$S_1 = [a_1 \ a_2 \ a_3 \ ]$$

Those are the components which have to identify and then

$$S_2 = [b_1\ b_2\ b_3\ ]$$

And, those are the components which are the components to be identified.

Now, if you see there will be one degree of freedom which means that out of 6 components one can be arbitrarily or should be specified and we take one of the components to be 0. Here we have taken that

$$a_2 = 0$$

Now, will find out S_1, as we have discussed now, they form orthogonal coordinate system, so,

$$S_1.n = 0$$

$$\tfrac{2}{3}a_1 + \tfrac{2}{3}a_3 = 0$$

$$a_1 =- a_3$$

$S_1.n$ should be equal to 0 because they are perpendicular to each other.

Now, remember S_1 is a unit vector, which means that

$$a_1^2 + a_2^2 + a_3^2 = 1$$

So,

$$a_2 = 0; \quad a_1 =- a_3 \rightarrow a_1^2 = a_3^2$$

So,

$$2a_1^2 = 1 \rightarrow\ a_1 = \tfrac{1}{\sqrt{2}};\ \ a_3 =- \tfrac{1}{\sqrt{2}}$$

Which means that we have found out what is the S_1 vector,

$$S_1 = \left[\tfrac{1}{\sqrt{2}}\ 0\ -\tfrac{1}{\sqrt{2}}\ \right] = [0.707\ 0\ \ -0.707\ ]$$

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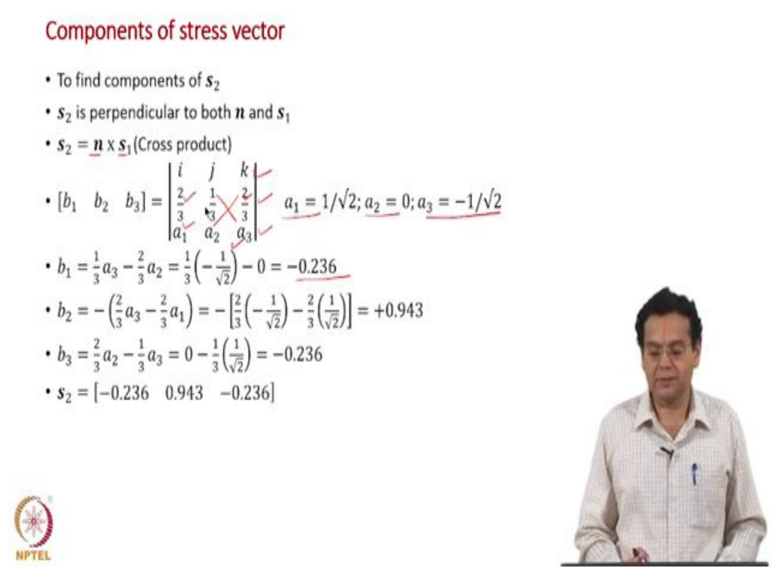

Now, we have to find out the S_2 vector; to find components of S_2. Now, once again will use the fact that all three form a orthogonal coordinate system, and S_2 is perpendicular to both n and S_1. So, how do you find out S_2? Simply, as the cross product of n and S_1. We know that result of a cross product is perpendicular to both the vectors. So, S_2 is perpendicular to both n and S_1, so we will use this fact to find out S_2.

$$S_2 = n\ x\ S_1$$

$$\left[b_1\ b_2\ b_3\ \right] = i\,j\,k\ \tfrac{2}{3}\ \tfrac{1}{3}\ \tfrac{2}{3}\ a_1\ a_2\ a_3$$

You have found out already $a_1,\ a_2,\ a_3$. Of course, a_2 was assumed. So, now, let us find out component wise,

The ith component of S_2 vector

$$b_1 = \tfrac{1}{3}a_3 - \tfrac{2}{3}a_2 = \tfrac{1}{3}\left(-\tfrac{1}{\sqrt{2}}\right) - 0 =- 0.236$$

Similarly, for b_2 of course,

$$b_2 - \left(\tfrac{2}{3}a_3 - \tfrac{2}{3}a_1\right) =- \left[\tfrac{2}{3}\left(-\tfrac{1}{\sqrt{2}}\right) - \tfrac{2}{3}\left(\tfrac{1}{\sqrt{2}}\right)\right] =+ 0.943$$

Then similarly the kth component, component on the z-axis is

$$b_3 = \tfrac{2}{3}a_2 - \tfrac{1}{3}a_3 = 0 - \tfrac{1}{3}\left(\tfrac{1}{\sqrt{2}}\right) =- 0.236$$

Of course, many of us would have used about cross product, used it in moments etcetera. It is another example where we are finding one direction is perpendicular to two other direction using cross product, and I would say application of cross product, yeah. So, now, you are determined S_2 vectors.

$$S_2 = [-0.236\ 0.943\ \ -0.236\ ]$$

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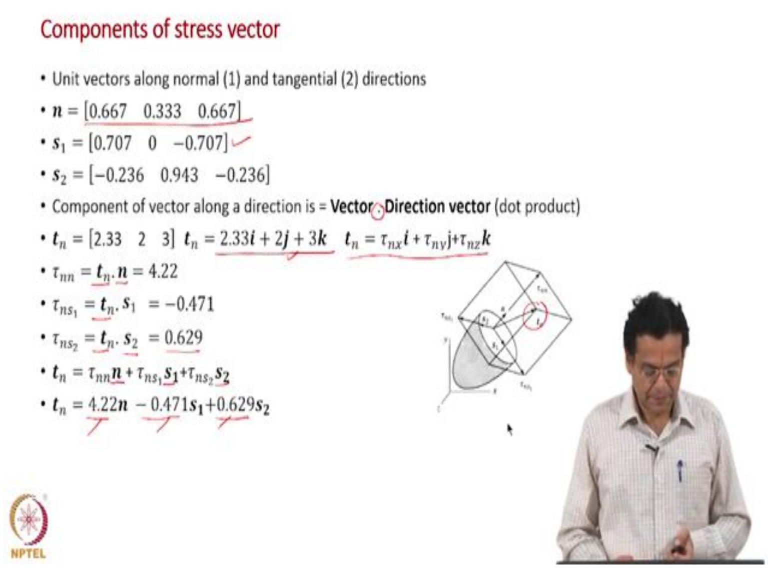

Let us summaries all the results, whatever we have got. Last one more step is left out. We have found out all the unit vectors along normal and tangential directions. Let us, list them

$$n = [2/3\ 1/3\ 2/3\ ]$$

$$S_1 = [0.707\ 0\ \ -0.707\ ]$$

$$S_2 = [-0.236\ 0.943\ \ -0.236\ ]$$

Then as we are discussed earlier simple dot product of stress vector with these direction vectors will give the component along n, S_1, S_2, and that is what you are going to do now.

Component of vector along a direction, just repeating what we are discussed earlier, it is vector dotted with direction vector, want to emphasis that the dot product. So, the stress vector is in terms of x, y, z component is

$$t_n = [233\ 2\ 3\ ]$$

$$t_n = 2.33i + 2j + 3k$$

And,

$$t_n = \tau_{nx}i + \tau_{ny}j + \tau_{nz}k$$

So, τ_{nn} is the component of stress vector along normal to the plane or the normal stress acting on the plane.

$$\tau_{nn} = t_n.n = 4.22$$

$$\tau_{nS_1} = t_n.S_1 =- 0.471$$

And,

$$\tau_{nS_2} = t_n.S_2 = 0.629$$

So, now, you are able to express the stress vector in terms of components along normal and tangential directions. Look at the way we expressed a t_n here in terms of i, j, k, they are the unit vectors. Here the unit vectors are n, and then S_1, and then S_2. The components are whatever we have found out right now 4.22, minus 0.471 and then plus 0.629. So, that was what we required.

$$t_n = 4.22n - 0.471\, S_1 + 0.629\, S_2$$

We wanted the components of stress vector along the normal to the plane and along the direction, tangential directions. And this is the normal stress and these two are the shear stresses. Only these are the normal stresses and shear stress acting on the plane that was our objective, we have done a complete cycle I would say. When we began with solid mechanics this is the diagram we saw. Then we went to x, y, z components, then we went to the 9 components. Then, now we are complying in the reverse directions, 9 components, x, y, z components and then components along n, S_1, S_2. In terms of discussion we came in the forward direction, in terms of application you going in the reverse direction that should be kept in mind. In fact, now only will be have a clear idea why do we start, where we start, where we end, how to connect etcetera.

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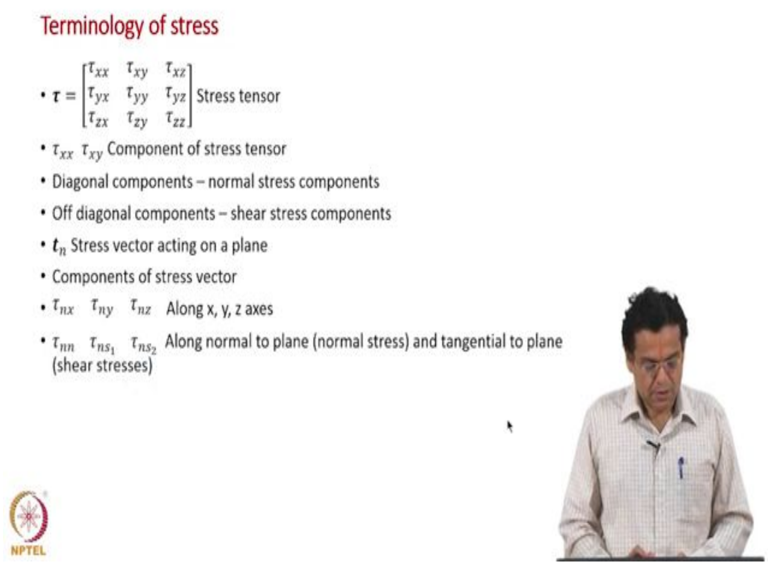

Now, there are so many terms on stress. So, I think it is time to summaries them, so that have a clear understanding and you really mean what you mention. So, far causally would have talked about stress, that could be in average, that could be in a vector, that could be in tensor.

$$\tau = \left[\tau_{xx}\ \tau_{xy}\ \tau_{xz}\ \tau_{yx}\ \tau_{yy}\ \tau_{yz}\ \tau_{zx}\ \tau_{zy}\ \tau_{zz}\ \right]$$

So, these stress with the 9 components this stress tensor. Remember all are at a point stress tensor, ok; it has 9 components, you call them as component of stress tensor. And the diagonal components are normal stresses, τ_{xx}, τ_{yy}, τ_{zz} because they are acting normal to the respective planes x-plane, y-plane, z-plane. Off diagonal elements they are all the shear stress components acting on the respective planes x-plane, y-plane, z-plane.

And now, t_n is the stress vector acting on any plane. Moment you see n, it is of course, represents any plane. And now, this stress vector has components, components of stress vector. Those components can be two ways, τ_{nx}, τ_{ny}, τ_{nz}, along x, y, z axis or it could be τ_{nn}, τ_{nS_1}, τ_{nS_2} which is are we interested in as the normal and shear stress acting on any plane. So, that kinds of summarizes what you are discussed in terms of terminology.

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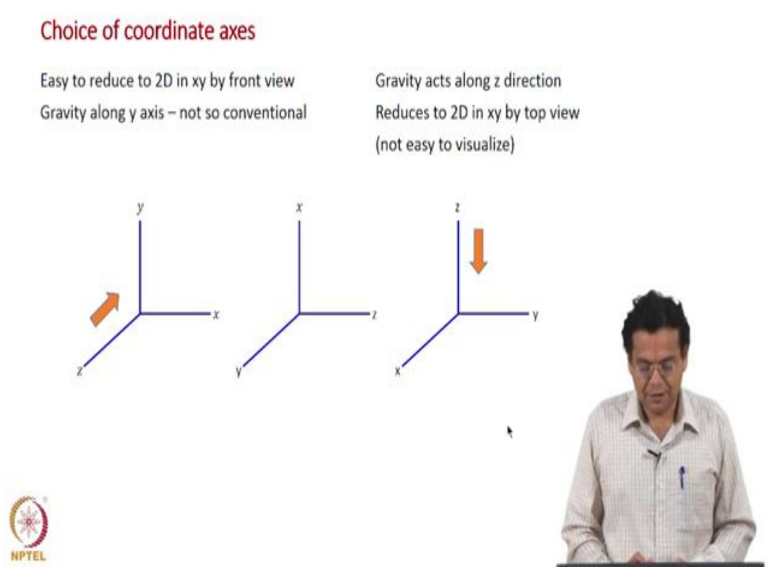

Remember, when we discuss ocean on ocean applying the continued equation to estimation of down the velocity in ocean, we discuss this coordinate axes. So, the first one is something very familiar to us, and if you switch this x, y, z twice then you get the third coordinate axis, no, and then we discussed what the advantages etcetera. You may be wondering why this coordinate axis now.

Reason is very simple. I am going to show all the results using three-dimensional graphs using MATLAB and MATLAB uses the third coordinate axis default. With lot of difficulty and some effort it can be changed, but just because to be easy and then, so that you can also try to reproduce in the results, I am just presenting the way in which MATLAB represents and that coordinate axis what is shown here. That is why just recollect showing these three coordinate axes. So, what we will see is whatever you have seen in terms of numbers and vectors, and set of elements matrices, etcetera, we will see pictorially in terms of 3D representation.

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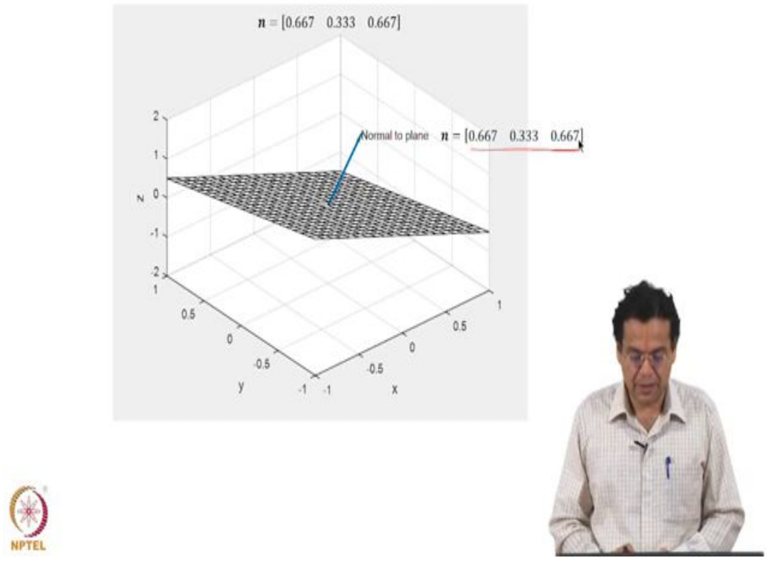

First all our attention on the plane and the normal to the plane, so that is the first graph. What you are seeing here is x, y, z coordinate axis and the plane is shown and of course, the plane is shown so large always imagine, it is a very small plane, it is a point. And we are given the components of the normal to the plane and so the plane and the normal to the plane are shown here. Our objective is to find out normal stress and shear stress on this plane. That will be the final a diagram, final 3D graph. In between, let us see what happens.

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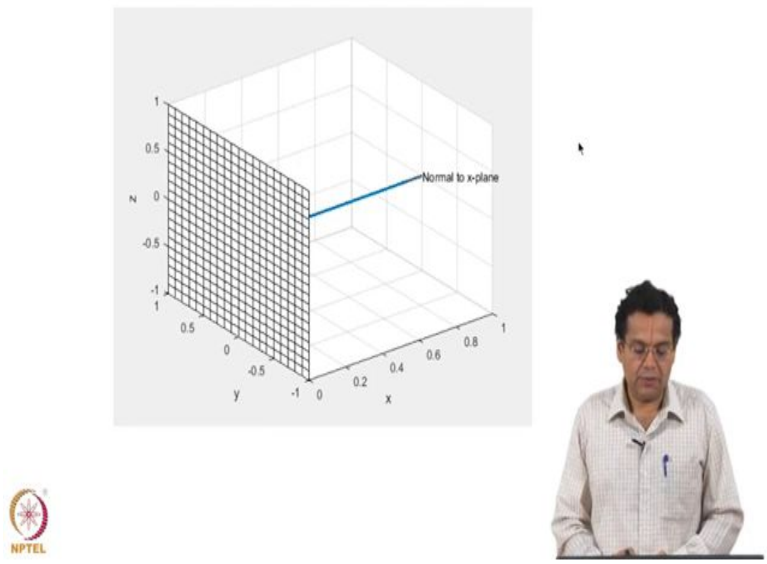

This is the plane whose normal is x axis. So, that that plane is shown. And remember the x axis here, so normal to that plane is also shown here.

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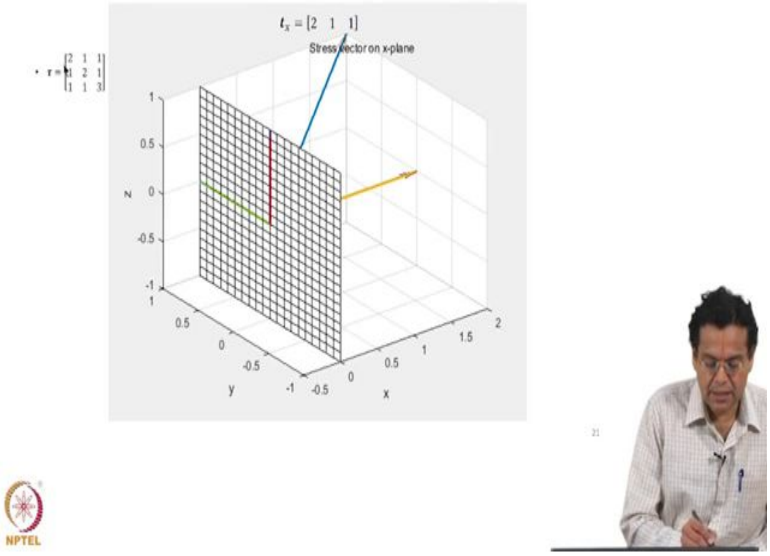

Now, remember when the stress tensor is given the first row represents the components of stress vector acting on x-plane. So, what is shown here is the stress vector acting on x-plane and those components are shown. The x component is 2, the y component is 1, z component is 1. Now, you are seeing the, in fact, we are also seeing the x-plane; x-plane is shown, stress vector in x-plane is shown, the components are also shown. Of course, this components are along x, y, z axis.

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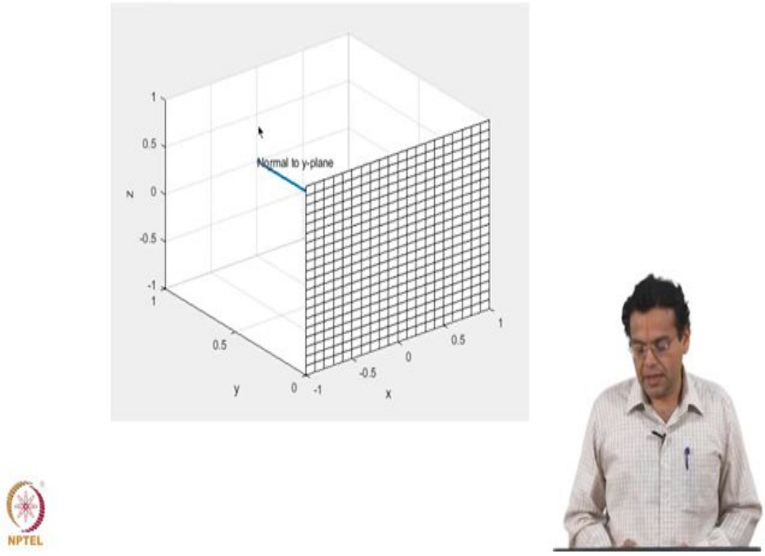

Now, analogously is an the y-plane and the normal to the y-plane is shown. Of course, we should understand that this is the y axis, so that is why this is the y-plane, and normal to the y-plane in facing something away from us.

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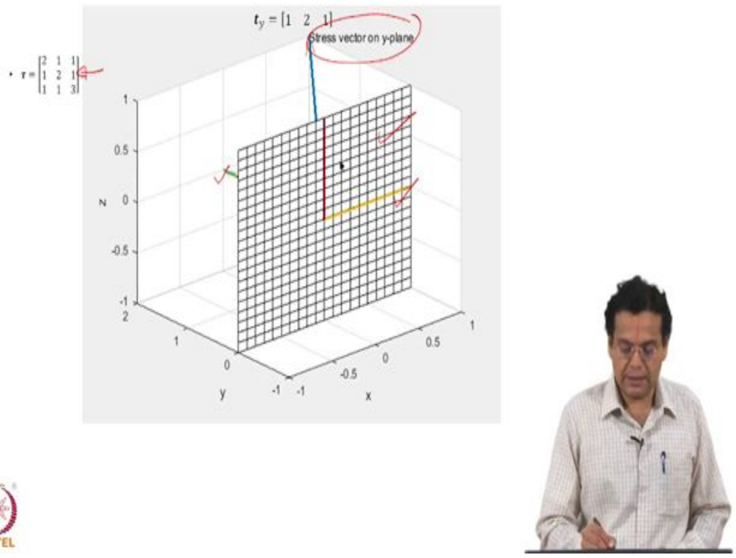

Now, similarly the second row represents the component stress, components of stress vector acting on the y-plane, so these are y-plane. And the stress vector acting on y-plane is shown, and the components are shown this is the x component this is the y component and this is the z component.

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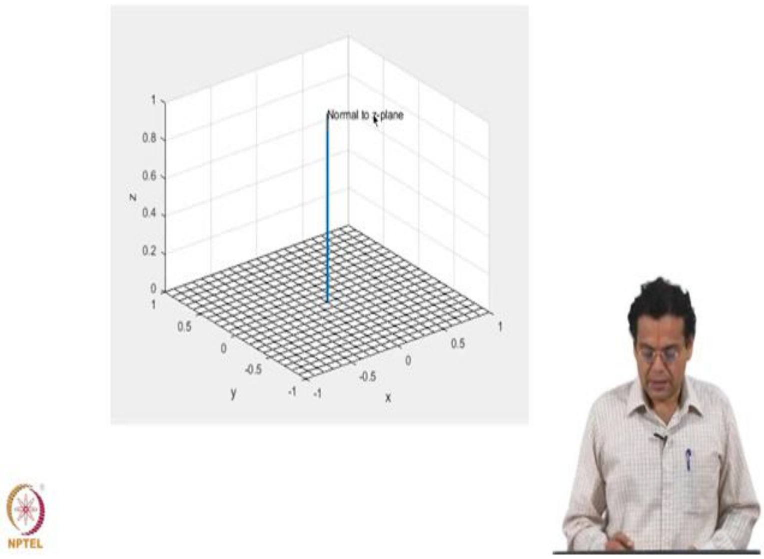

Now, similarly z-plane of course very easy to see because something in the floor, and the normal to the z-plane is also shown.

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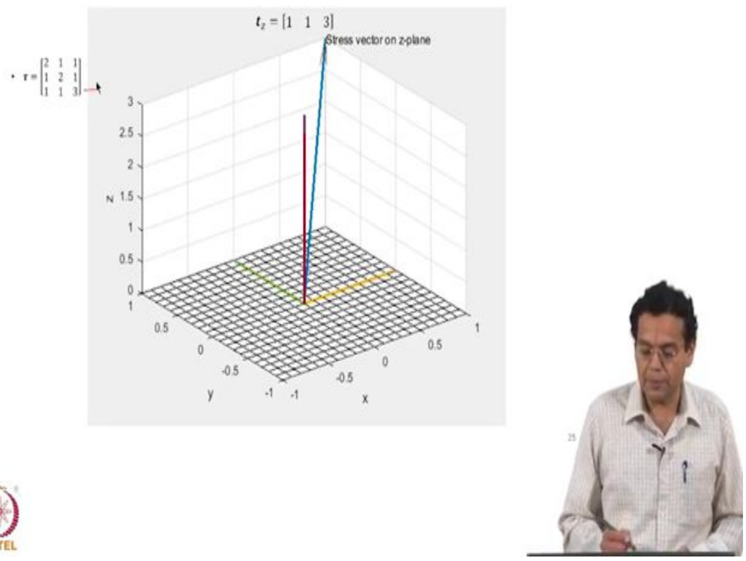

And the last row represent the components of the stress vector on z-plane. And the components are along the x axis 1, y axis 1 and z axis 3. Now, we have seen the representation information given to us, that is all we have done.

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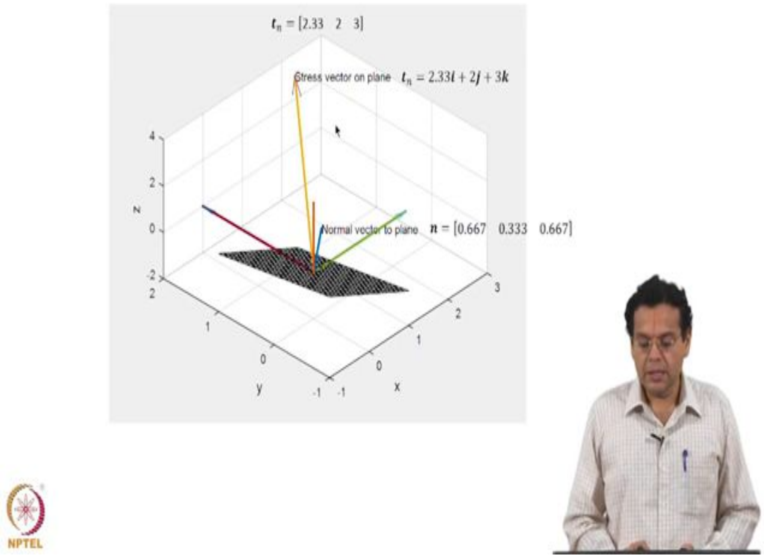

Now, what did you do? We applied the Cauchy's formula and found out the components of stress vector along x, y, z axis that is what is shown here. The plane is also shown. And then the normal vector to the plane is shown here, and the stress vector acting on the plane is also shown.

Now, what else is shown? The components of stress vector along x, y, z axis is shown and the component are 2.33, 2, and 3. Of course, you do not see as 2.33 because the angular view. So, this is representation of our intermediate result. That is a result which you obtain after applying Cauchy's formula; gives a components of stress vector, but along x, y, z axis. Now, the final result.

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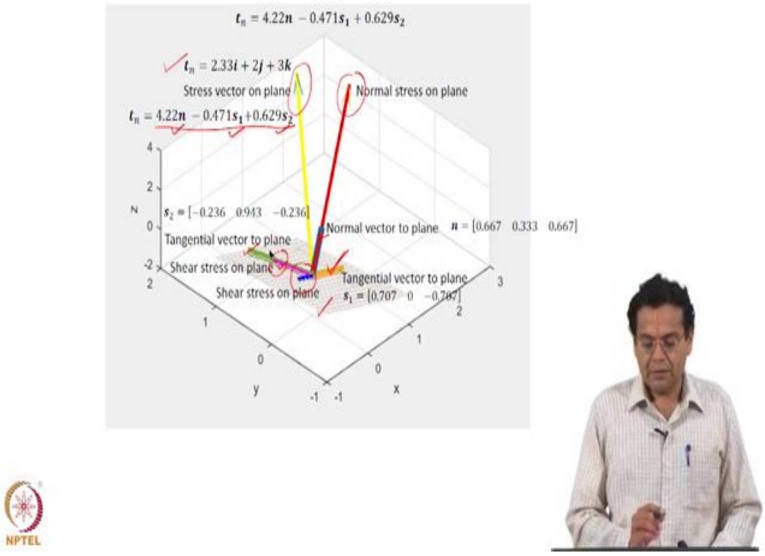

Now, let me explain what is shown here, the plane certainly and then first let me explain the axis. Axis important is for this figure are the normal and the tangential directions. So, the normal to the plane is shown. The first tangential direction S_1 is shown, second tangential direction S_2 is also shown.

Now, the stress vector acting on the plane is shown by this yellow arrow mark. What I shown here is a representation which used in the previous slide in terms of x, y, z axis. More importantly what you see here is a representation of the along in terms of components, along our chosen coordinate axis which is attached to the plane that is why you see here n, S_1, S_2.

Now, let us look at the components. 4.22 acts along the normal, then minus 0.471, but it is minus 0.471 that is why you have along the negative direction to S_1 is along this direction because we have minus 0.471, it acts opposite to that. Finally, you have S_2, tangential vector the values 0.629 and that is what you see in the magenta colour.

So, we have shown the plane, and then the three coordinate directions, and then we also shown the stress vector, and also shown the normal stress component and the stress components. Of course, the arrows are not sharp. I made them very thick, so that becomes easy to visualize.