Continuum Mechanics And Transport Phenomena
Prof. T. Renganathan
Department of Chemical Engineering
Indian Institute of Technology, Madras

Lecture – 40
Cauchy's formula

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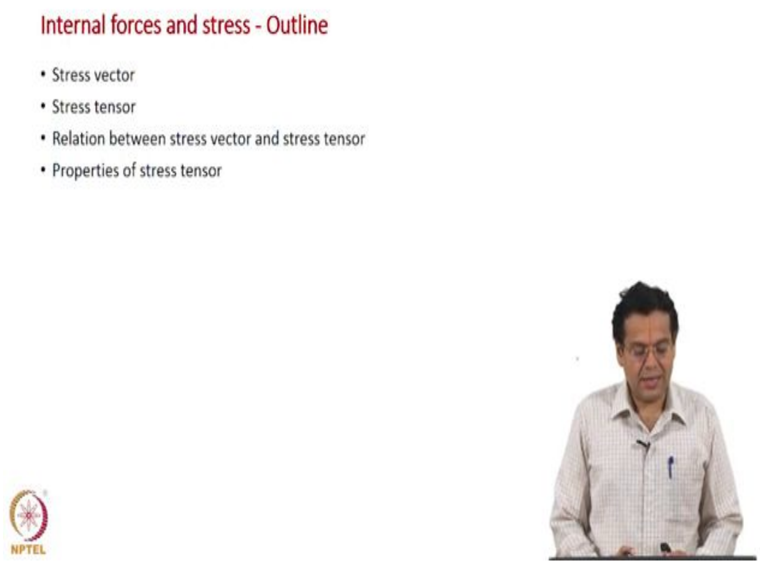

We are in solid mechanics. We have take a diversion to solid mechanics to understand the surface forces on the right hand side of the linear momentum balance. And in this first visit to solid mechanics we are discussing internal forces and stress and in this we have discussed about the stress vector and the stress tensor.

Now, we are discussed the other two topics namely the relationship between stress vector and stress tensor what is the relationship between them and what are the properties of stress tensor, of course, you will have an example in between, numerical example in between.

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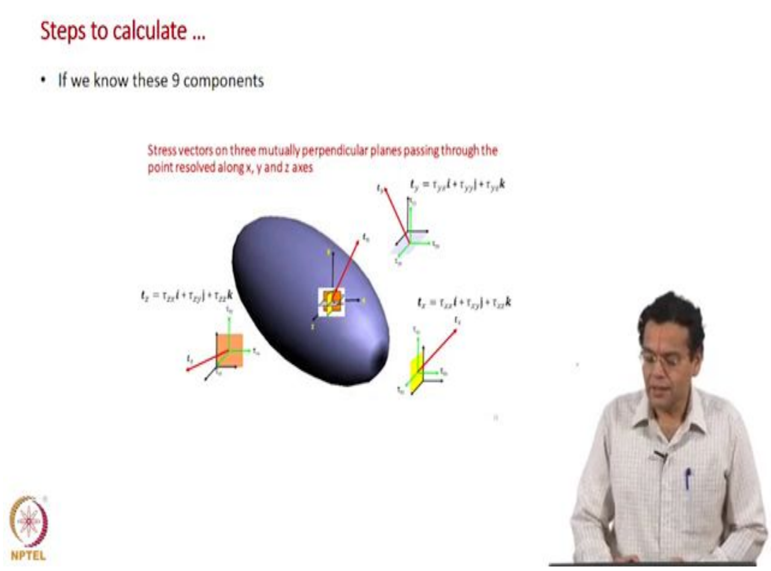

Just to set the stage for today's discussion, we will recall this particular slide which you have seen earlier, where I summarized about stress vector stress tensor in the reverse direction that will set the stage for today. We said if you know the 9 components, these are the components of stress vectors acting on three planes parallel to the coordinate axis.

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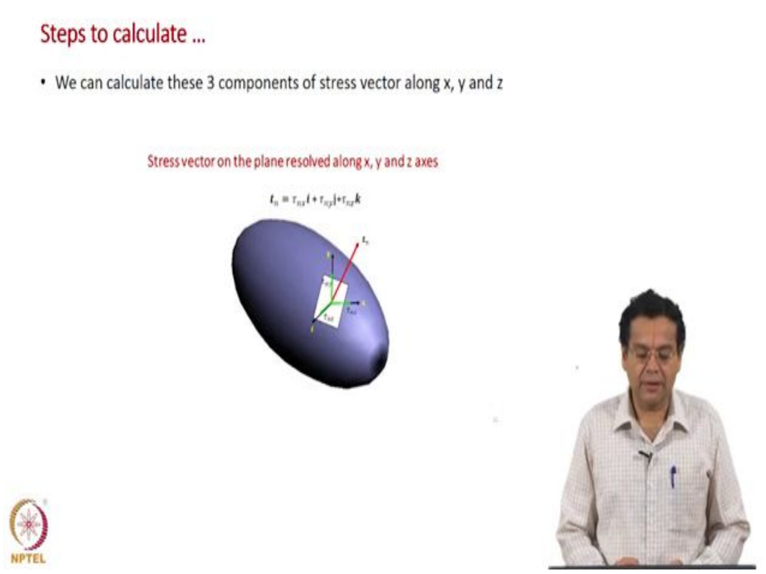

So, if we know this then we can calculate the three components of the stress vector. Those components are along x, y, z axis of the stress vector. So, this is what we are going to do now, how to relate the nine components to the three components that is the first step.

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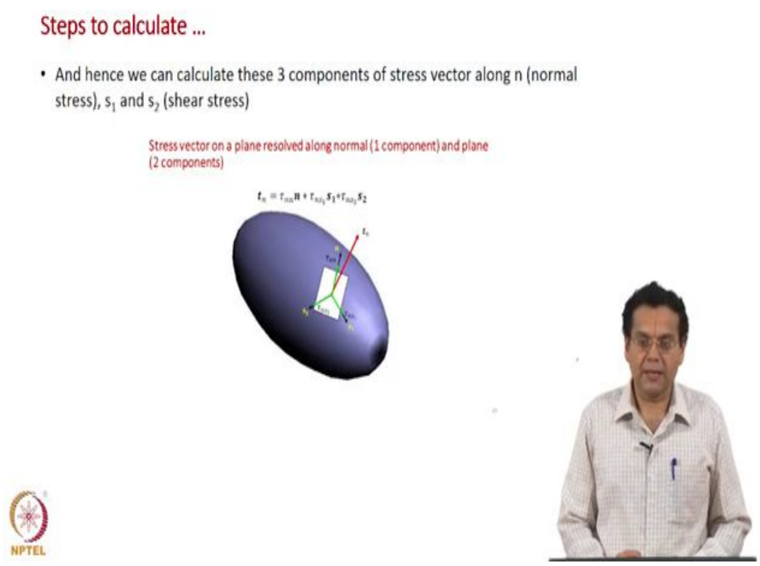

Second step is much more simpler because if you know these three components just use simple vector algebra to find out the components normal to the plane and tangent to the plane. This way much more simpler; the second step is much simpler, first step involves some concepts.

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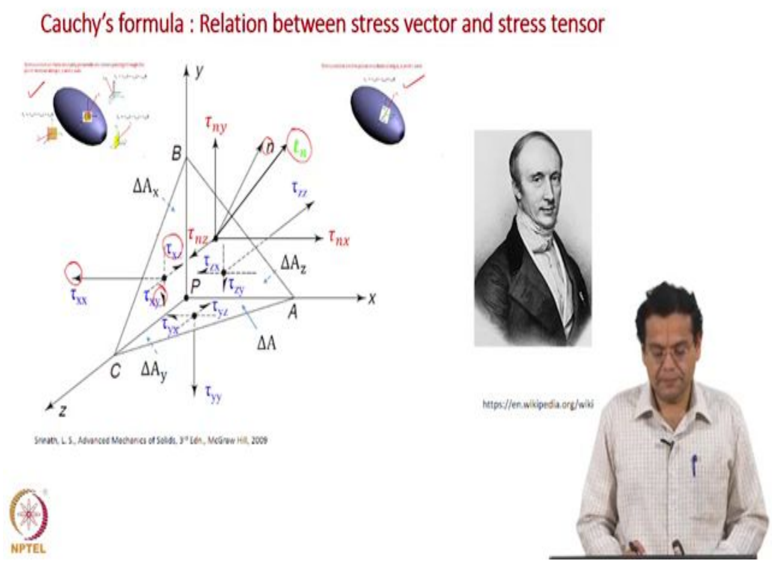

So, we will proceed the first step now where we are going to relate the 9 components to the 3 components. The expression what you are going to derive is called the Cauchy's formula named after the who derived it namely Cauchy whose photograph is shown here. We are going to relate stress vector and stress tensor or to put short nine components with three components. And, to have a quick idea what you see here in the left hand side is the same slide minimized there to show that we are given those nine components and we have to find these three components which are the components respective along x, y, z axis.

Now, we said that we have an arbitrary plane and then we have normal to the plane, at the point we are considering three planes parallel to the coordinate axes. So, now, I should consider the geometry with includes all these four planes such a geometry is a tetrahedron. As you see here A I will mark minimally on the slide. So, already lot of things are, there lot of text is there. So, if you look at ABC is arbitrary plane whose normal is n vector here and the three planes parallel to coordinate planes are BPC whose normal is x-axis. And then we have the plane CPA whose normal is y-axis, and then we have the plane BPA whose normal is the z-axis. Of course, all the normal are towards the negative axis not the positive axis.

So, we choose tetrahedron for this reason that we have to relate the components of stress vector acting on this plane we call as arbitrary plane; arbitrary because n can have any value. And to the components of stress vector acting on those three well defined planes, when I say well defined planes please understand that they are parallel to coordinate planes. Now, what else is shown here the stress vector acting on the arbitrary plane is shown here t_n and then the components of the stress vector along x, y, z axis are also show here in red τ_{nx}, τ_{ny}, τ_{nz}, this what you have to find out.

Now, the given information to us are the components of stress vector acting on those three planes. Let us consider the plane whose normal is x-axis that is the plane CPB. Now, it is a plane whose normal is towards the negative x-axis and based on our sign convention all the forces are to be directed towards the negative axis. So that why you see the τ_{xx} directed towards the negative x-axis, τ_{xy} towards negative y-axis, τ_{xz} towards negative z-axis. We always show in terms of a positive sense.

Now, similarly we have the plane APC, the components of stress vector acting on that plane are shown, let us start with τ_{yx}, τ_{yx} once again all arrows are shown in the towards the negative axis τ_{yx}, τ_{yy}, τ_{yz}. And, then if you take the plane whose normal is z-axis, what is that plane? APB and once again all the components are shown towards the negative axis τ_{zx}, τ_{zy} and then τ_{zz}. Whatever are shown in blue the nine components which are given to us we will have to find out the components shown in red which are the components of stress vector on the arbitrary plane.

What else is shown here are the areas ΔA is the area of the arbitrary plane ΔA_x , ΔA_y , ΔA_z which are the planes whose normal are x-axis, y-axis, z-axis. Now, how should we imagine this tetrahedron? The way in we should imagine as tetrahedron is you have a solid object in that you have taken a small tetrahedron element. So, your tetrahedron is somewhere inside the solid object that is how we should interpret this.

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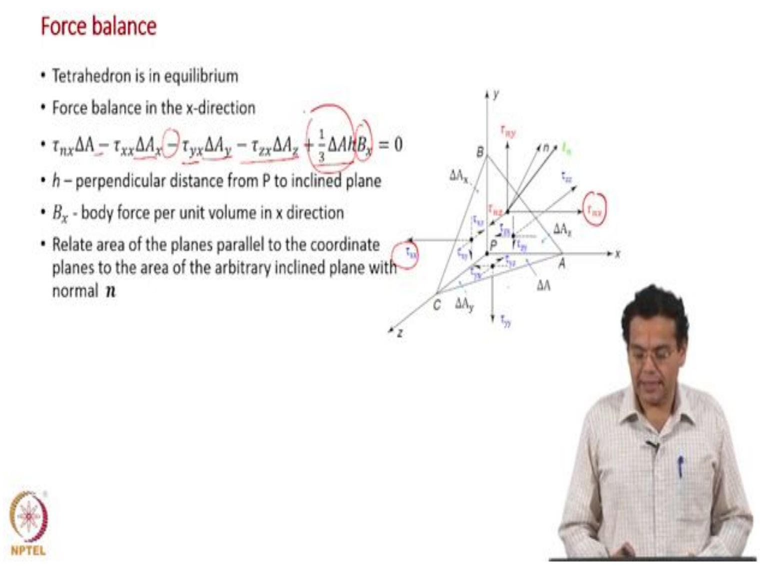

Now, we are considering the object under equilibrium as we have discussed earlier any part of the body is also under equilibrium. So, this tetrahedron also should be under equilibrium. So, we do a force balance and we do a force balance first along the x direction, then analogously we can write for other directions. Now, when you write a force balance you multiply the stress values with the area on which it acts. So, we are writing a force balance x direction. So, we should account for all the components of stress vector acting along the x direction.

$$\tau_{nx}\Delta A - \tau_{xx}\Delta A_x - \tau_{yx}\Delta A_y - \tau_{zx}\Delta A_z + \tfrac{1}{3}\Delta A h B_x = 0$$

So, let us take the component of the stress vector acting on the arbitrary plane τ_{nx} and that is along positive axis because when we dissolve vector along the coordinate axis they are along the positive axis. So, τ_{nx} along the positive axis so, it is positive and the area over which it acts is ΔA. Now, next coming to the x face τ_{xx} is the component acting along the x-direction and it is along the negative x-axis. So, we have an negative sign here and multiplied by the area over which it acts which is ΔA_x So, we expect force balance of multiplying the stress by the corresponding area.

Now, for the y face I should consider the component which is acting along x-direction which is τ_{yx} and then of course, it is along negative x-axis multiplied by the area over which acts it is ΔA_y, similarly for the z direction $-\tau_{zx}\Delta A_z$. Now, this tetrahedron will also have a body force, we will denote that body force along the x direction; along the x-axis positive x-axis as B_x and that you need a body forces body force per unit volume. And, we know that the volume of tetrahedron is $\frac{1}{3}\Delta A h$, where h is the perpendicular distance from P to the surface of the arbitrary plane. So, this is the volume of the tetrahedron multiplied by the body force per unit volume; h is the perpendicular distance from P to the incline plane, B_x body force per unit volume in x direction.

Now, if look at this expression you have the area of the arbitrary plane and then you have the area of the planes parallel to the coordinate planes. So, now, we will have to find the relationship between these two, so that you can cancel out some common factors that is what we will do next. We are going to relate that ΔA_x, ΔA_y, ΔA_z which are the areas of the planes parallel to the coordinate planes to the area of the arbitrary plane which is ΔA.

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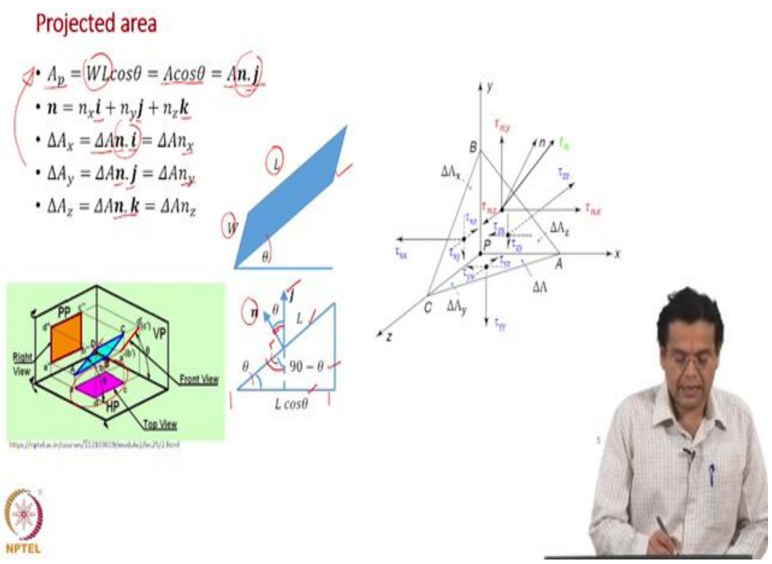

Let us do that. For that if you look at the area of let us say CPB, ΔA_x we can easily understand that that area is nothing, but the projection of ΔA on a plane whose normal is x-axis. You have ΔA which is the area of the arbitrary plane if you project on a plane on the let us say z y plane you get the area ΔA_x or putting in the other way you project on a plane whose normal is x-axis, now we will have to relate this. To relate, I have a plane which is shown and then you are going to look at the area of this projected plane on this plane. This arbitrary plane and this plane for our case is the y plane because it is normal is y-axis.

Now, if look at the projection then this width remain same, but the L becomes L cos(θ) where θ is angle between this plane and the horizontal plane. That is what is shown that the actual area is L x W, the length and the multiplied by width if θ is the angle which it makes with the horizontal plane, then the length of the projection is L cos(θ) x W. So, let us write down that.

$$A_p = WL\, cos\theta = Acos\theta = A\, n.j$$

A_p is the projected area, and area of the plane is A, where theta is angle between the plane and in this case the horizontal plane. Now, how do we interpret this theta in the more general way that is what is shown here. If you look at this triangle which is the front view of this, this one is the arbitrary plane and this is a horizontal plane and θ is the angle between them.

So, the angle which the plane makes the horizontal plane is equal to the angle between normal to the plane and the normal to the plane on which you are projecting; j vector is the normal to the plane on which you are protecting, n vector is the normal to your plane.

So, both these angles are same; angle between the plane and the horizontal plane in this case and the angle between the normal to the plane and the normal to the plane on which you are projecting. So, which means that this $cos\theta$ can be written as n.j that is what I have done here, n.j can be represented as $cos\theta$ remember both are unit vectors. Now, we can generalize this; now, of course, other way of representing is shown here from reference.

So, this is the plane and what you see here is the projection of the plane; probably you would have drawn this in the engineering drawing course. So, remember n vector is a unit vector and it has components along the i, j direction which I call them as n_x, n_y, n_z.

$$n = n_x i + n_y j + n_z k$$

We have seen that ΔA_x represents projection of Δ A on the x plane which, means

$$\Delta A_x = \Delta A\ n.i = \Delta A n_x$$

Similarly, for the y direction now this becomes almost same as this,

$$\Delta A_y = \Delta A\ n.j = \Delta A n_y$$

And, similarly

$$\Delta A_z = \Delta A\ n.k = \Delta A n_z$$

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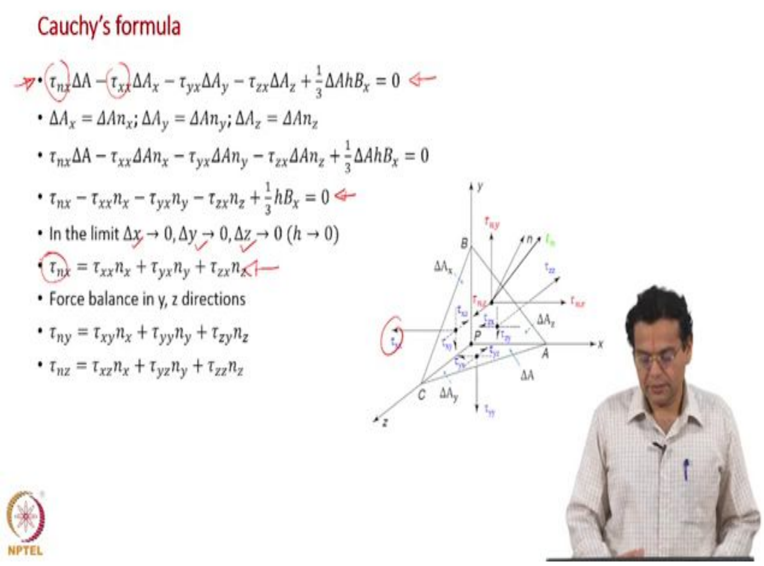

So, now let us substitute all this in the force balance equation.

$$\tau_{nx}\Delta A - \tau_{xx}\Delta A_x - \tau_{yx}\Delta A_y - \tau_{zx}\Delta A_z + \tfrac{1}{3}\Delta A h B_x = 0$$

We have,

$$\Delta A_x = \Delta A\ n.i = \Delta A n_x$$

$$\Delta A_y = \Delta A\ n.j = \Delta A n_y$$

$$\Delta A_z = \Delta A\ n.k = \Delta A n_z$$

We said that was our objective, we have Δ A and ΔA_x, ΔA_y, ΔA_z we said we will relate so that we can cancel out ΔA. Let us do that.

$$\tau_{nx}\Delta A - \tau_{xx}\Delta A\, n_x - \tau_{yx}\Delta A\, n_y - \tau_{zx}\Delta A\, n_z + \tfrac{1}{3}\Delta A h B_x = 0$$

ΔA can be cancelled out that was our object

$$\tau_{nx} - \tau_{xx}\, n_x - \tau_{yx}\, n_y - \tau_{zx}\, n_z + \tfrac{1}{3} h B_x = 0$$

Then, we take limit $\Delta x \to 0$, $\Delta y \to 0$, $\Delta z \to 0$ and $h \to 0$ some explanation here. First if you look at the first equation we had cancelled out area this is similar to what we had done earlier when you wrote a balance for a control volume we divided by the volume because we wrote it for a control volume. Now, we are writing over a surface so, dividing by the area of the surface;

Second, when we derived the conservation equation let us your mass left side of momentum balance. We wrote the balance of the control volume and then we shrink the control volume to a point we do the exactly the same thing here as well in the limit of $\Delta x \to 0$, $\Delta y \to 0$, $\Delta z \to 0$ which means that we are shrinking this tetrahedron to a point and as we have done earlier what did we say we shrink so that this aspect ratios maintained same.

We should shrink as a cube, we should not shrink as this length becoming 0 or this length very large we should shrink symmetrically, so that the shape is maintained. Similarly, here also we shrink the tetrahedron to your point maintaining the shape of tetrahedron. So, all the $\Delta x \to 0$, $\Delta y \to 0$, $\Delta z \to 0$ which means that h also goes to 0. What is the h? The distance between, the normal distance between P and the surface that also goes to 0 and why do we do that as always we want always a relationship it is valid at every point in the object.

Remember, we are going to substitute bag use it in the right hand side of the conservation equation, our conservation equations are at every point in the fluid there. Now, when we go back we substitute a relationship which is valid at every point. So, this relationship which you are going to write now is valued at every point inside the solid object, which means that when we wrote this expression they are just average values, exactly analogous to what it were control volume.

Remember, τ_{nx} is average over the arbitrary plane when I write τ_{xx} it is average over this area whose normal is x-axis, when we take limit they become point value. So, conceptually they are same as what we have done earlier; objective is to get a relationship which is valid at every point inside the solid object.

$$\tau_{nx} = \tau_{xx}\, n_x + \tau_{yx}\, n_y + \tau_{zx}\, n_z$$

Now, we can do a similar force balance along the y direction and then z direction and then get

$$\tau_{ny} = \tau_{xy}\, n_x + \tau_{yy}\, n_y + \tau_{zy}\, n_z$$

And,

$$\tau_{nz} = \tau_{xz}\, n_x + \tau_{yz}\, n_y + \tau_{zz}\, n_z$$

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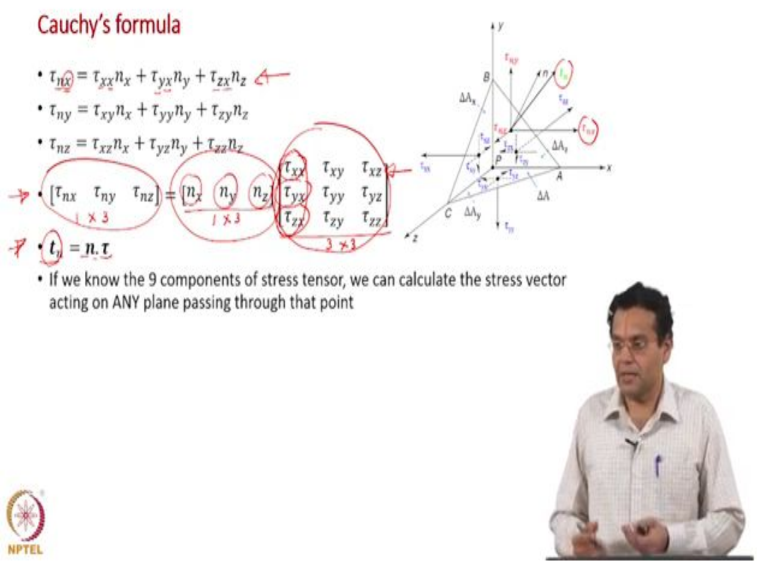

So, now let us summarize all of them the three relationships which we have derived we have achieved our objective of relating the components of the stress vector to the components of stress tensor and similarly.

$$\tau_{nx} = \tau_{xx}\, n_x + \tau_{yx}\, n_y + \tau_{zx}\, n_z$$

$$\tau_{ny} = \tau_{xy}\, n_x + \tau_{yy}\, n_y + \tau_{zy}\, n_z$$

$$\tau_{nz} = \tau_{xz}\, n_x + \tau_{yz}\, n_y + \tau_{zz}\, n_z$$

Now, if you observe this, if you look at the second subscript we know that second subscript tells the direction of the force. So, that is why we have x as a second subscript and here you have τ_{xx}, τ_{yx}, τ_{zx}. In fact, that is how we wrote the balances as we took all the components acting along the x direction.

In the left hand side what we have is n because the component for arbitrary stress vector and then that relates to on the right hand side you have x component of the three different stress vectors. That is why you have x-axis, y-axis, z-axis; left hand side is only one stress vector acting on the arbitrary plane, right hand side you have the x components of three different stress vectors acting on the x, y, z plane that is why the first subscript is x, y, z. Of course, moment you look at the second subscript that tells you the direction in which you made the force balance.

Now, what we will do is we have three equations, well let us put in a more simple way and more formal way or elegant way.

$$\left[\tau_{nx}\ \tau_{ny}\ \tau_{nz}\right] = \left[n_x\ \ n_y\ n_z\right]\left[\tau_{xx}\ \tau_{xy}\ \tau_{xz}\ \tau_{yx}\ \tau_{yy}\ \tau_{yz}\ \tau_{zx}\ \tau_{zy}\ \tau_{zz}\right]$$

Express the left hand side as 1 x 3 matrix, and then right hand side can be expressed as a simple matrix multiplication of this n vector with the stress tensor.

$$t_n = n.\tau$$

So, this three equations can be simply represented as a matrix multiplication of matrix 1x3 matrix made of the components of the normal vector and another 3x3 matrix whose components are the components of stress tensor. This is why to begin with I said for our scope, the stress tensor is almost a matrix. All the matrix multiplication rules are applicable here as well.

So, how do we represent this? The stress vector the earlier expression, this equation represent in terms of components, this equation represents in terms of a vector. Left hand side you have a stress vector acting on plane n, right hand side you have n vector dotted with the stress tensor. You are not going to discuss about dot product formally I should say as dot product of n vector with stress tensor within our scope for understanding you can understand that is just as a matrix multiplication of n vector with stress tensor.

So, this is the question we asked and we have answered that. If you know the nine components of stress tensor that is our right hand side, we can calculate the stress vector acting on any plane; when I say any plane our n is given to us; passing through that point. Remember, all are at a point, so, you should imagine a small point you have a small plane and you are replacing that with 3 planes of the same point. So, that should be your imagination 3 planes are there and you have arbitrary plane all very small at a point,.

So, if you know the 9 component of stress tensor, we can calculate stress vector, $\tau_{nx}\ \tau_{ny}\ \tau_{nz}$ on any plane because any plan is described by the components of the normal vector to it and so, this is a relationships which we are seeking.

Also, I like to mention if you look at the stress tensor if you look at one row what does it represent? Components of stress vector acting on x plane; if you look at one column what does it represent? It represents x component of three different stress vectors.

So, the way in which you have arranged nicely when you multiply what happens you multiply $n_x\tau_{xx}$ plus $n_y\tau_{yx}$ plus $n_z\tau_{zx}$ and that is exactly what you have here. So, the way in which arranged it when you multiply you get all the components or the contribution of all the x components of three different stress vectors to the x component of the stress vector acting on arbitrary plane.

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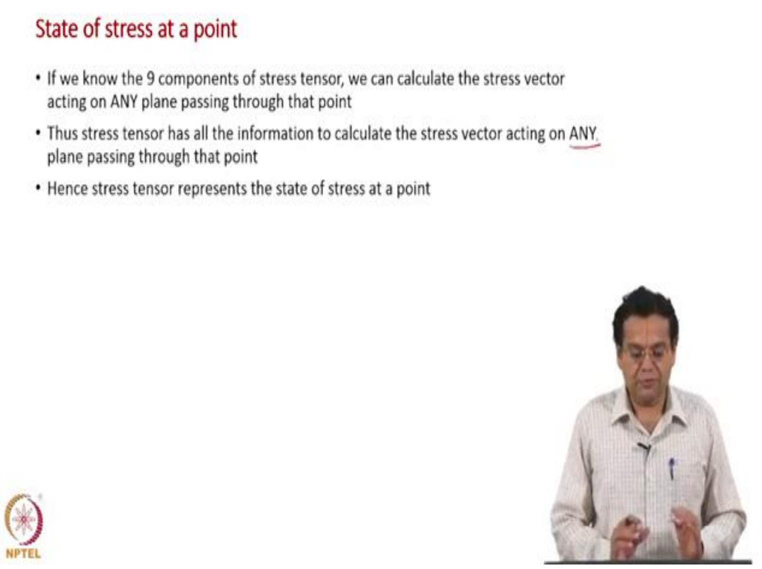

Now, at terminology state of stress at a point; what do we mean by state of stress at a point?

- The first sentence is same as last sentence, if we know the 9 component of stress tensor we can calculate the stress vector acting on any plane passing through that point. Which means that at a point I can have infinite number of planes and for any plane I can find out the stress vector acting on it which means that whatever information is required is contained in your stress tensor that is advantage. That is why, we came from a arbitrary plane arbitrary axis to arbitrary plane and x, y, z coordinate axis; that means, millions of possibilities are there infinite number of possibilities are there. By going to a stress tensor we are limited to or we can get those values, but what needs to be given is only those 9 components, moment you give this 9 components I can find out 3 components for any plane. So, we are reducing infinite number of some information to a just 9 components and then of course, the n vector.

- Thus stress tensor has all the information to calculate the stress vector acting on any plane passing through that point, can have an infinite number of planes you are just at a point. Let us say you are at a point, you can have infinite number of planes at a same point and it is enough that if you have the stress tensor at that point you can get stress vector acting on any of these planes, and
- Hence stress tensor represents the state of stress at a point.

So, of course, little bit not as easy as vector to understand if you look at a stress tensor you should imaging that tells about state of stress at a point. Why does it tell about state of stress, because it has all the information to calculate stress vectors on any plane passing through that point. So, by this way we answered the first part of the question; second part is of course, more of vector algebra.