Continuum Mechanics And Transport Phenomena
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Lecture – 35
Differential linear momentum balance: Transient, convection and body force terms

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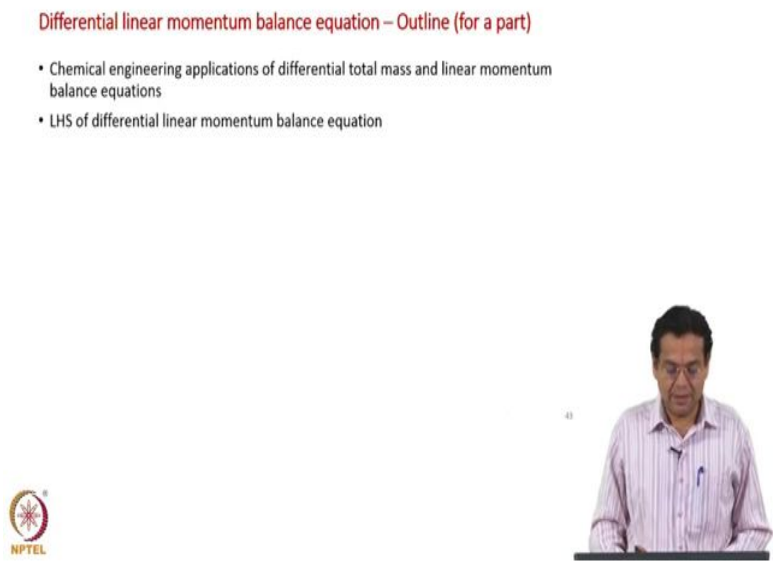

We are in the first part of Differential linear momentum balance equation, we have discussed few chemical engineering applications of differential total mass and linear momentum balance equations. Now, we will derive the left hand side of the differential linear momentum balance equation, we will understand why we do only the left hand side as we go along.

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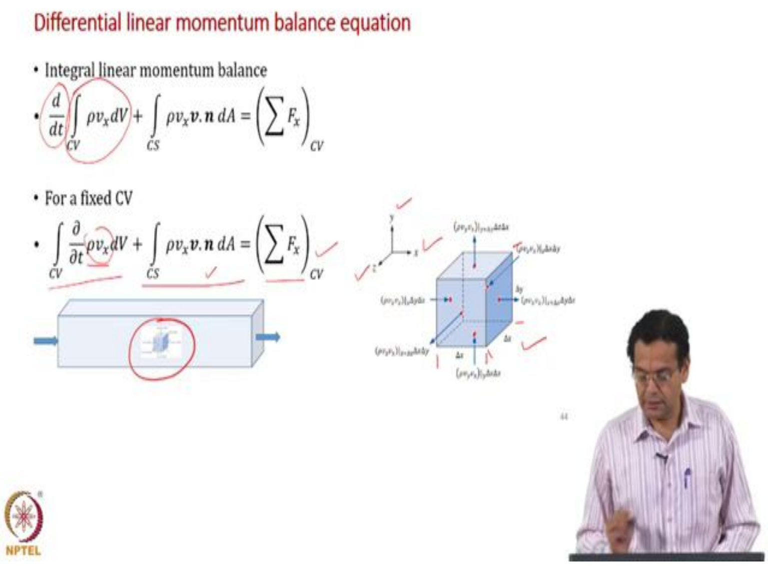

So, as we have derived the differential mass balance for the differential momentum balance, we started with the integral balance equation. So, we start with the integral linear momentum balance equation and because of vectorial equation we write only the component of that x component of the integral linear momentum balance equation. So, we will also drive throughout only the x component of differential momentum balance equation.

Now, we will consider a small control volume as shown (in above slide) and this small control volume is inside a pipe of rectangular cross section, as explained earlier we take rectangular cross section so, that we can work in Cartesian coordinate system.

$$\frac{d}{dt}\int_{CV} \rho v_x dV + \int_{CS} \rho v_x v.n\, dA = \left(\sum F_x\right)_{CV}$$

Now, we consider a fixed control volume. So, we can bring in the time derivative inside the integral sign

$$\int_{CV} \frac{\partial}{\partial t}\rho v_x dV + \int_{CS} \rho v_x v.n\, dA = \left(\sum F_x\right)_{CV}$$

And, when we bring inside ρv_x can be function of both space and time. So, becomes a partial derivative when you integrate all the special variations which are taken into account. So, only time variation remind. In this case because it is inside the integral ρ *and* v_x can be functions of time hence a partial derivative of course, these terms are remaining same. So, this tells about the rate of change of x momentum with in the control volume, net rate at which x momentum leaves through the control surface and some of forces acting on the control volume.

Now, the control volume is shown here and the coordinate axes are same x, y and z and its cuboidal control volume of length Δx, and then Δy and then Δz.

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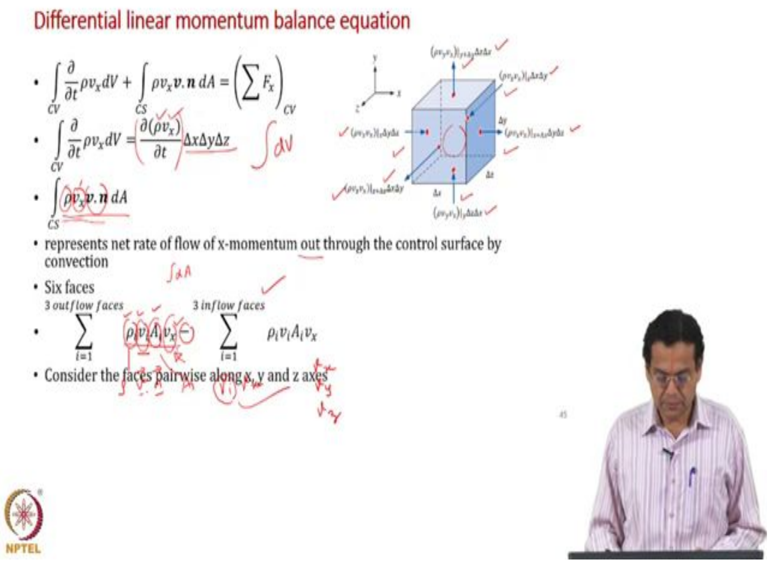

Now, let us simplify the different terms in the integral balance equation; when I say simplify, we are going to apply this integral balance equation for this small control volume. And, the way you should imagine is that this control volume is in inside our equipment; a small region inside our equipment and then as certain for mass balance will shrink to a point. Now, let us take the rate of change of x momentum term, we will take an average density and x velocity in this region, in this small control volume and because it is average it is just a constant.

$$\int_{CV} \frac{\partial}{\partial t}\rho v_x dV = \frac{\partial(\rho v_x)}{\partial t}\Delta x\Delta y\Delta z$$

Right know these represent average values when we take limit and make it as a point, they become point values that you already seen under mass balance. So, it is very important to understand the derivation for the differential mass balance. So, that they can easily extend to differential momentum balance later on any in fact, any balance as well. Now, this term represents the rate of accumulation momentum in the control volume.

$$\int_{CS} \rho v_x v.n\, dA$$

Now, let us move on to the net rate at which x momentum leaves the control surface which is the convective momentum. So, this represents net rate of flow of x momentum out, (out is important because n is outward normal) through the control surface by convection.

Now, there are six faces to the for this control volume cuboidal control volume. If you express this convective momentum for the six faces this is how we represent as we are saying here (in the above slide).

Now, what are the six faces? The left hand side face and the right hand side face and then we have the bottom face and the top face and then we have the rear face and the front face. Now you have in flow through the left face, you have inflow through the bottom face, you have in flow through the rear face and then other three faces there is outflow which is the right face, the top face and the front face. Now,

$$\int_{CS} \rho v_x v.n\, dA = \sum_{i=1}^{3\ outflow\ faces} \rho_i v_i A_i v_x - \sum_{i=1}^{3\ inflow\ faces} \rho_i v_i A_i v_x$$

In writing, we are made two assumptions, it is not an assumption, it is a valid for this case. The first assumption is that the velocity is perpendicular to the face that is why this $v.n$ has been just written as v_i and then we have also taken the $\rho v_x v.n$ to be constant so, I have left with integral dA that is why I have A_i here.

So, we have two velocities one is v_i and then v_x. This v_i depends on the face for which we are writing and that v_i could be v_x; if you are considering the faces along x direction, it could be v_y if you are considering faces along y direction and v_z face along z direction. So, $\rho_i v_i A_i$ account for the mass flow in a particular direction entering and leaving in this case of course, leaving and this v_x accounts for the direction of momentum.

Hence, this v_x is the velocity in the x direction until $\rho_i v_i A_i$ we have mass flow in a particular direction, but moment you multiply with v_x you get x momentum flow or x momentum in a particular direction. So, we have similarly we write for inflow faces and we take care of the negative sign. So, now let us consider the faces pair wise.

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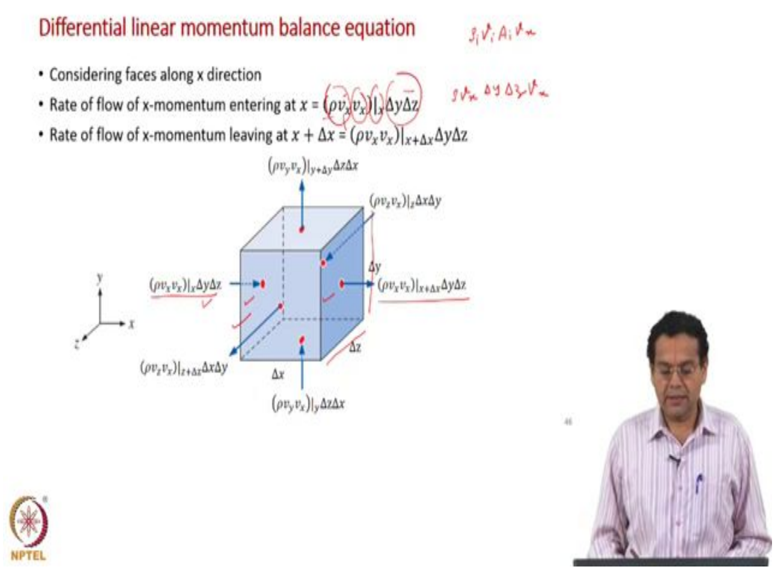

Now, considering the faces along the x direction I am going to consider the left hand side face and the right hand side face.

Rate of flow of x momentum entering at x is = $(\rho v_x v_x)|_x \Delta y \Delta z$

Let us understand this, many terms are familiar here based on mass balance. What are the terms that are familiar? $(\rho v_x)|_x \Delta y \Delta z$ is very familiar to us. What is the meaning of that? As we have seen earlier two ways of interpreting; this is the velocity multiply by area gives volumetric flow rate, multiply by density gives the mass flow rate.

Other way of interpreting v_x is the volumetric flux multiply by density gives mass flux, multiply area gives you mass flow rate. So, that is the significance of that term. We are interested in the x momentum so, multiply with v_x.

Now, the density and velocity can change special location, that is why and we are evaluating the rate of flow of x momentum entering at this face and this face is at x and that is why we evaluated x. If we understand this clearly all other terms become very simple. Now, let us write it for the x momentum living at x + Δx same expression, but evaluated at x + Δx.

Rate of flow of x momentum leaving at x + Δx is = $(\rho v_x v_x)|_{x+\Delta x} \Delta y \Delta z$

So, the inflow rate of x momentum entering shown and the rate of x momentum leaving is shown here.

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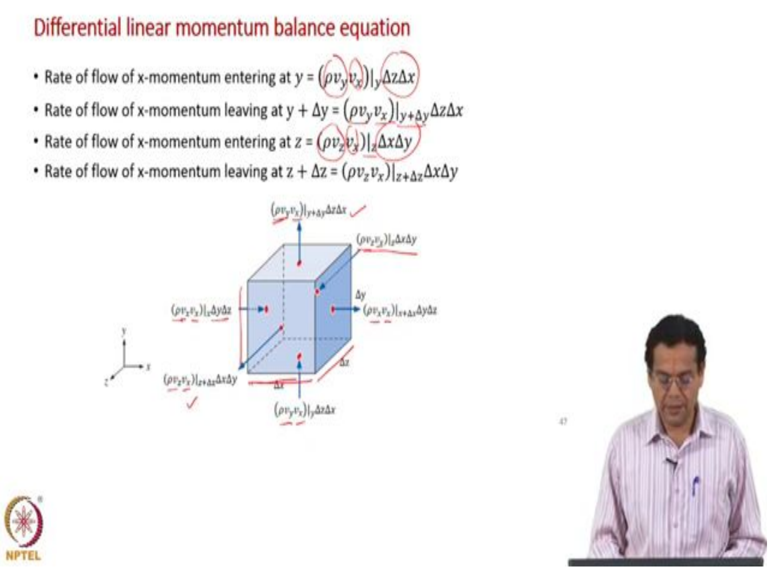

But in this case I like to repeat for y and z directions to understand that better.

Rate of flow of x momentum entering at y is = $(\rho v_y v_x)|_y \; \Delta x \Delta z$

Remember once again it is rate of flow of x momentum, but now because of mass flow entering and leaving along y direction, So, please pay attention that always the second velocity represents the velocity component along the direction in which we are writing the momentum balance.

Now, let us write the rate of flow of x momentum living at y + Δ y

Rate of flow of x momentum leaving at y + Δy is = $(\rho v_y v_x)|_{y+\Delta y} \; \Delta x \Delta z$

Now, let us do that for the z direction as well.

Rate of flow of x momentum entering at z is = $(\rho v_z v_x)|_z \; \Delta x \Delta y$

Similarly, rate of flow of x momentum leaving at z + Δz = $(\rho v_z v_x)|_{z+\Delta z} \; \Delta x \Delta y$

So, now if you look at this, let us focus on the flows shown in this diagram. If you look at the second velocity component is always v_x. So, if we look at the diagram itself we should be able to tell that, we are writing the linear momentum balance along the x direction. Look at the first component v_x we have ρv_x of course, if it along take with $\Delta y \Delta z$ that tells about mass flow and the x direction entering and leaving similarly mass flow on the y direction entering and leaving, mass flow on the z direction entering and leaving. So, the way in which you write should go along the physical interpretation.

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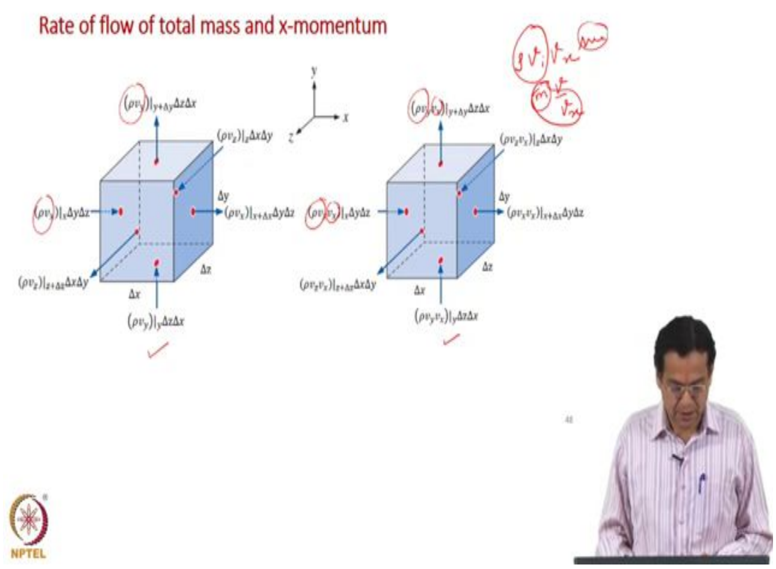

Now, if you look at this it has a close analogy with what you have written for differential mass balance. Now if you look almost same only difference is the presence of v_x in all the terms ok. This is a left hand side is the control volume shown with all the inflow and outflow mass flows, right hand side is a diagram with all the inflow and outflow x momentum what is the difference? For example, one have ρv_x , and another have $\rho v_x v_x$.

So, that shows that if you multiply with the v_x you get the x component of velocity. Suppose if your writing y component etcetera you will multiply by v_y all the terms. So, for this momentum term is not something new to you, mass into velocity something very well known to you, but only difference is you would have so, for written in a physics class etcetera for a solid object. So, mass was known mass of the solid object into v vector. So, there was no discussion about the mass at all, mass is given and v could be any direction could be along x direction, y direction, and z direction.

But now this mass represents mass of a fluid and hence that mass flow can happen in x direction y direction z direction that is why we are giving importance to mass also the direction and the component of velocity. This was just written as of course, multiply by area etcetera area all these put together as it must just m and here we had v vector that v vector we have taken in one component as v_x.

So, you would written as m v x now further discussion would have been required for just for just a solid object. Because it is a fluid flow which can take place in three directions we also discuss the direction of the mass flow as well.

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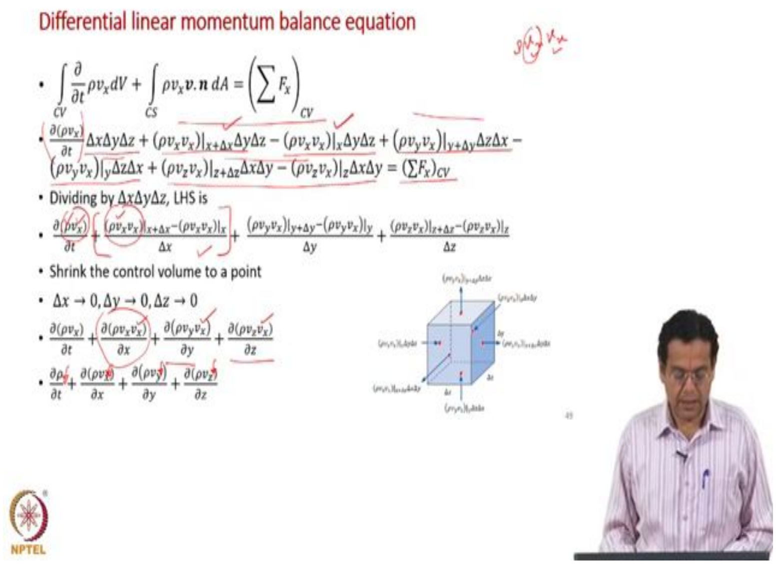

Now, let us put them all together in the integral linear minimum balance equation both for the rate time rate of change term and the net convection term.

$$\int_{CV} \frac{\partial}{\partial t}\rho v_x dV + \int_{CS} \rho v_x v.n\, dA = \left(\sum F_x\right)_{CV}$$

$$\frac{\partial(\rho v_x)}{\partial t}\Delta x\Delta y\Delta z + (\rho v_x v_x)|_{x+\Delta x}\,\Delta y\Delta z - (\rho v_x v_x)|_x\,\Delta y\Delta z + (\rho v_y v_x)|_{y+\Delta y}\,\Delta x\Delta z - (\rho v_y v_x)|_y\,\Delta x\Delta z + (\rho v_z v_x)|_{z+\Delta z}\,\Delta x\Delta y$$

The mass flow associate with different directions carries with them the x momentum. So, first two represent mass flow leaving and entering along x direction, next two for y direction and next two terms for z direction and right hand side we have some of forces acting on the control volume ok. Now, as we have done for the differential mass balance will divide by $\Delta x\Delta y\Delta z$ the left hand side alone.

$$\frac{\partial(\rho v_x)}{\partial t} + \frac{(\rho v_x v_x)|_{x+\Delta x} - (\rho v_x v_x)|_x}{\Delta x} + \frac{(\rho v_y v_x)|_{y+\Delta y} - (\rho v_y v_x)|_y}{\Delta y} + \frac{(\rho v_z v_x)|_{z+\Delta z} - (\rho v_z v_x)|_z}{\Delta z}$$

So, now significance in the earlier x terms where they were representing for example, if you take these two terms, they represent net rate of flow of momentum leaving along x direction. Now this term represents same significance per unit volume per unit volume ok. So, if you significance of that is that these two terms put together is that net rate of flow of x momentum leaving through the control surface per unit volume.

Now, we will shrink the control volume, we have discuss this when we derived the differential form of mass balance, we will shrink the control volume to a point objective is that we need a equation which as evaluated every point on the flow domain so, that when you solve see remember we are going to solve this and finally, get the velocity profile, I need velocity profile on the entire section and every region. So, I need equations valid at every point in the domain and hence shrink the control volume to a point, also we said that the geometry should be maintained same should remain as a cuboid.

So, length along every direction also goes to 0 and what happens? Now all these where average values within the region they become point values; and the average values and the face they also become point values and hence you get a differential equation there now all the variables present point values. And of course, the derivative as limit $\Delta x \to 0$, $\Delta y \to 0$, $\Delta z \to 0$ becomes a partial derivatives

$$\frac{\partial(\rho v_x)}{\partial t} + \frac{\partial(\rho v_x v_x)}{\partial x} + \frac{\partial(\rho v_y v_x)}{\partial y} + \frac{\partial(\rho v_z v_x)}{\partial z}$$

As I have told you earlier you can understand the term in one direction and analogously extend into other directions, but with a caution to take care of the correct velocity component when we extend other direction make sure for example, if we understand the first two terms it has $\rho v_x v_x$. When extend other direction if we understand which velocity is to be replaced with v_y. It is this velocity the left hand side velocity which is to be replaced with v_y. May be wondering y so, particular about this direction it has a implication letter on as well after we discuss surface forces will understand that right know will stick on to that also becomes easy look at the differential equation left hand side, all the second velocity components are v_x tells you that it is a momentum balance along the x direction.

Now, let us compared with the continuity equation or the differential mass balance and see what are the difference as we expect based on the derivation etcetera. Based on the derivation all these terms are multiply by v_x of course, not outside that can also vary. So, you multiply here with a v_x. So in fact, if you vary from clear with about the differential mass balance, you can write the x component of linear momentum balance very easily. Just multiplying everywhere with v_x and we have force including the accumulation term as well.

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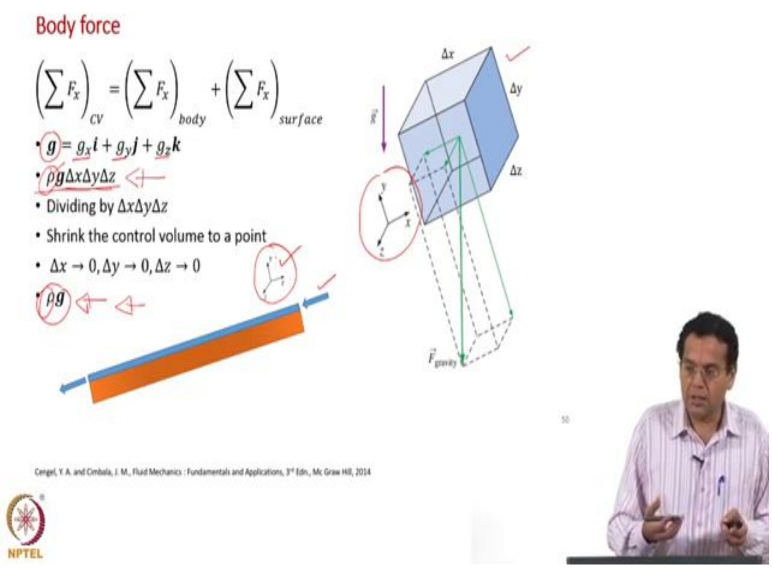

Now, we also look at predominantly we looked at the left hand side we also look at the right hand side forces; we had two forces body force and surface force, we are going to look at the body force we are going to look at the body force only now we will see the reason shortly.

$$\left(\sum F_x\right)_{CV} = \left(\sum F_x\right)_{body} + \left(\sum F_x\right)_{surface}$$

The control volume has two components spilt in the body and surface forces. Now the body force is gravitational force as I told you that is the only body force which we are going to consider.

Now, so, for in the earlier examples, we had a well defined control volume and so, gravity was always acting along one access it could be z axis, y axis etcetera. But I can have a geometry aligned any particular angle for example, one example shown here (in referred slide) what is a control volume aligned along general coordinate axes shown here need not be aligned along our usually x, y, z axis.

Now, what is application? For example, we have inclined plate which is shown here and then let us say water flows along that. In this case will be convenient for me to take a coordinate axis which is aligned along the flow direction I will take x, y, z which mean that this is not your gravity always acts along the vertically downwards. But now we are coordinate axis is not aligned along a usual you are usually x and y z axis which means that you can have components of gravity along your chosen x, y, z axis.

You know gravity acts vertically downwards if we are coordinate axis also as one axis along that vertical direction, that will be only one component of gravity let us say g_y and g_z. But suppose we are coordinate axis is not is own general, which is not aligned for the vertical axis not aligned along the gravity then you can have components of gravity along your chosen x, y, z axis which are not your regular x, y, z axis. That is why in general we represent g as

$$g = g_x i + g_y j + g_z k$$

So, remember the body force is a volumetric force or expresses per unit mass or unit volume, g is the gravitational force per unit mass.

- $\rho g \Delta x \Delta y \Delta z$

So, we multiply by the mass of this control volume what is that mass? It is density into the volume of control volume that there is a mass was there, at this point it is an average value ρ is average of course, g just a constant. As we are done for the left hand side we also divide by the volume Δx, Δy, Δz. Shrink the control volume to a point and now what you get is

- ρg

What is the difference between this term and this term? Earlier this is the average of rho in this control volume when you divide and shrink to a point. So now, this represents this represents the body force because of the gravitational field gravitational force per unit volume ones again remember this is gravitational force per unit volume. First expression is gravitational force or body force here when you have divided represents the body force or gravitational force per unit volume. Of course, once we pool all the terms we look at the significance of the each term there, but as we go along also should be familiar that all the terms are per unit volume bases.

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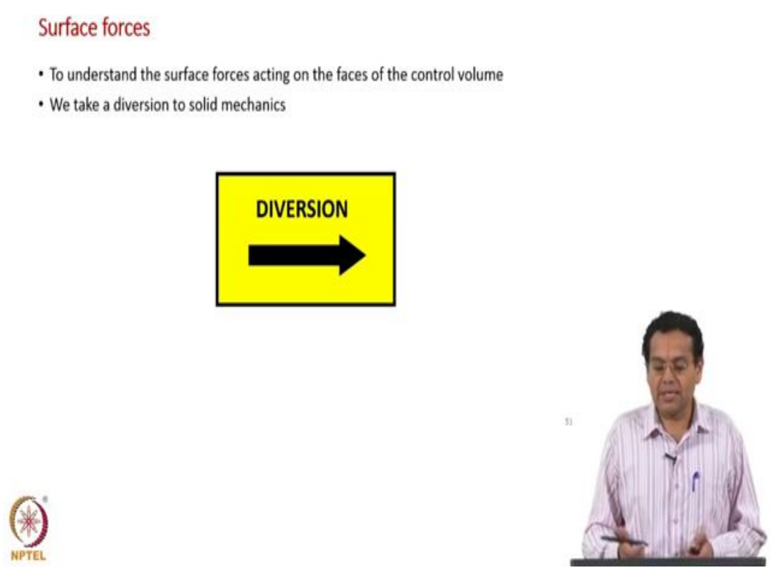

Surface forces are all the more involved and hence we pay special emphasize that, to understand the surface forces acting on the faces of control volume we take a diversion to solid mechanics. Why we should take a diversion? First of all you are in highway travelling at least a 100 kilometer per hour and you look at diversion board, you get highly irritated that you take a diversion, but if on the diversion let us we are going to meet fluid mechanics and if on the diversion meet solid mechanics and the solid mechanics tells that I know fluid mechanics very well, good friend of mine, we share lot of commonalities, then we are happy exactly same situation now.

Meaning of you may be wondering why take a diversion solid mechanics may be prospect also. But based on feedback from students, that question has been there but, as we go back and forth solid and fluid mechanics, there very much satisfied about the progress of take a diversion. And other reason for going to solid mechanics is that easier to understand about surface forces when you discuss through solid mechanics.

And of course, other advantages that we can see the parallel between solid mechanics and fluid mechanics, we say that is one of the object of the course as well that we discuss the commonalities between solid and fluid mechanics of course, justifying the term continuum mechanics. So, one is look at the generalities between them, second is easier to understand the surface forces when we discuss through solid mechanics.

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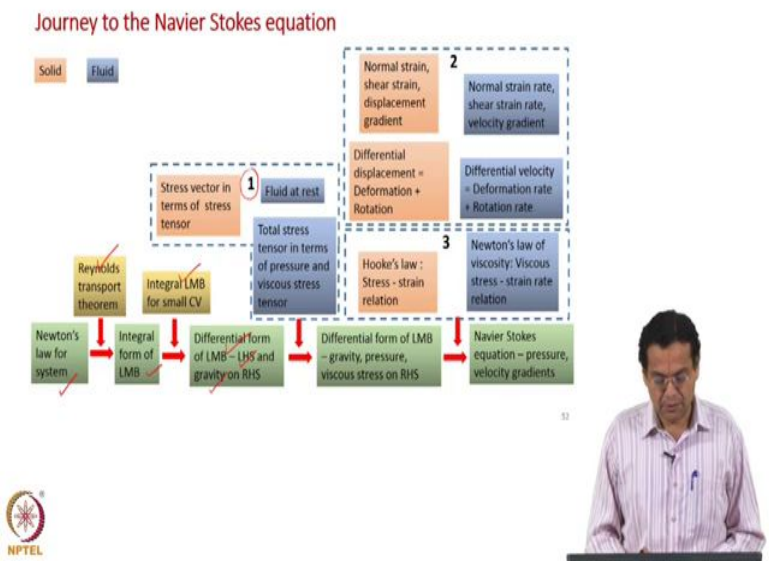

Very colourful slide after several lectures as I told you we will derive the differential form of the linear momentum balance, name as the Navier-Stokes equation, the author who, derived that Navier and Stokes. And, this slide shows that various steps involved in the derivation which I fondly call as the journey to the Navier-Stokes equation.

We are somewhere in the beginning, I will not discuss the entire slide now; you may not understand that, let me go to a point where we are now. Some of the steps should be familiar to you. We started the Newton's second law of motion for the system, we applied the Reynolds transport theorem and then derived the integral form of linear momentum balance; LMB it as well Linear Momentum Balance.

Now, we use the integral minimum balance for a small control value that is what we have discussed in the last few slides and then derived the differential form of linear momentum balance, not the complete form. What we are derived is the left hand side which was almost similar to our mass balance that is way we could derive that, and then we consider only the body force on the right hand side which is the gravity on the right hand side that is where we stand. Just to understand the surface forces we take a diversion, this 1 represents the first time we visit a solid and come back to fluid mechanics.

So, first time we take a diversion to solid mechanics, you will understand what are the boxes here, what do they represent later. So, just to show that we take a diversion to solid mechanics to understand about surface forces and then rest I will explain later. This 2 and 3 represents second time third time when we go and come back to solid and fluid mechanics in schedule between solid and fluid mechanics ok.

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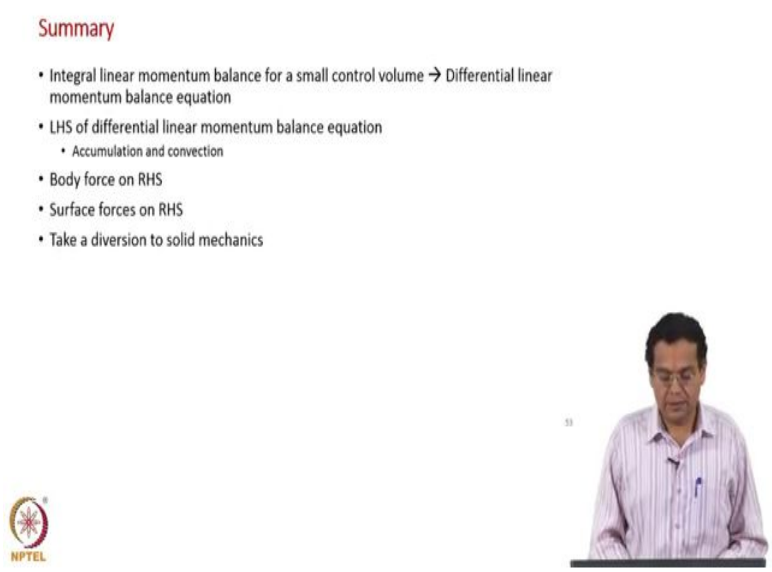

That is a summary of what we have done so far under differential linear momentum balance. We started with the integral momentum balance, applied for a small control volume, derived the differential form of linear momentum balance equation. We have derived the left hand side which has the rate of change of transfer to time term and then the convective momentum term. And, we have on the right hand side we have only at the body force, we have to look at the surface force on the right hand side to understand that we have take a diversion to solid mechanics, that is where we stand.