Continuum Mechanics And Transport Phenomena
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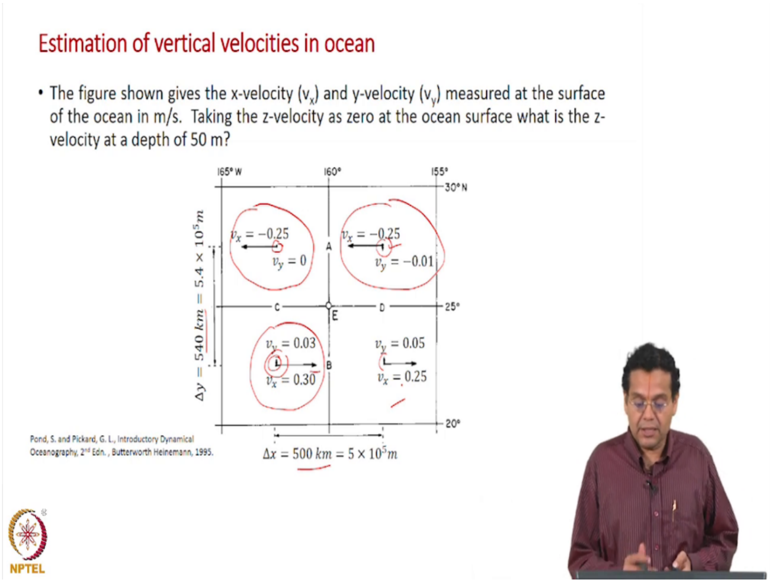
Lecture - 28
Differential Total Mass Balance: Examples - Part 2

Example: (Refer Slide Time: 00:15)



Once again a nice interesting example I would say, where we link measurements and equation of continuity; though not a core chemical engineering example, an example from ocean engineering I would say, but anyway, fluids are common to everybody. So, we will still discuss this, first what is the measurement we take. So, you have to imagine can go to a seashore and imagine yourself standing on a seashore or ocean shore, and on the surface, you are measuring the x and y velocity components. So, these velocity components are measured, suppose that it is easier to measure these velocity components compare with the downward velocity. So, what we do we measure x and y velocity and use the continuity equation to estimate the downward velocity that is why the title says estimation of vertical velocities in the ocean, and those velocities are shown here.

So, let us read the problem the figure shown gives the x velocity and y velocity measured at the surface of the ocean in m/s. Taking the z velocity as 0 at the ocean surface what is z velocity at a depth of 50 meters. Now, let us look at this figure shown here (above slide image); first, what is shown are four measurement points.

Something like taking a region of ocean and divide into four zones but of course, we are in the ocean; so, look at the distances their order of kilometers, 500 kilometers not even order of kilometers order of hundreds of kilometers. So, the x distance between the measuring point is 500 kilometers, the y distance is 540 kilometers. Now, let us take one point where the x velocity is in a negative direction so, -0.25 and y velocity component = 0. So, just the arrows are shown proportional to the velocity values.

Let us take the second region, in this region once again velocity is in the negative direction same value -0.25, you see a small component of velocity in the y-direction it is -0.01. Similarly, let us take the third zone; now the x velocity is in the positive direction 0.3 because the ocean can have any x velocity, y velocity can vary from region to region; y velocity is also positive a small value 0.03. Similarly, in the last fourth zone the x velocity, y velocity are 0.25 and 0.05 that is how we understand the measurements shown.

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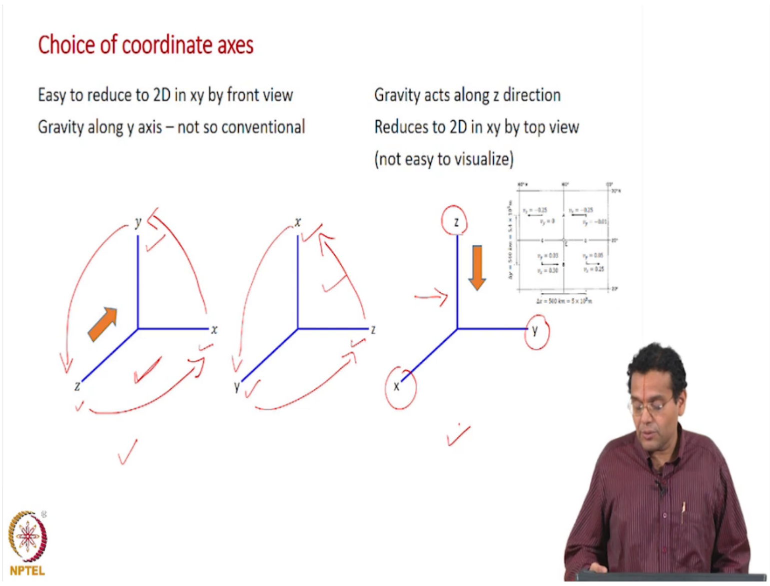


Now, before we proceed with solving, we need to discuss about the choice of co-ordinate axis. Different books use different co-ordinate axis and that is relevant to this particular example; you will understand why I discuss now. Now, this choice of co-ordinate axis remembers all of them are right hand coordinate system. Now, the first coordinate system (above slide image) is an axis that has been chosen so far and that is what we will also continue to choose almost over the entire course.

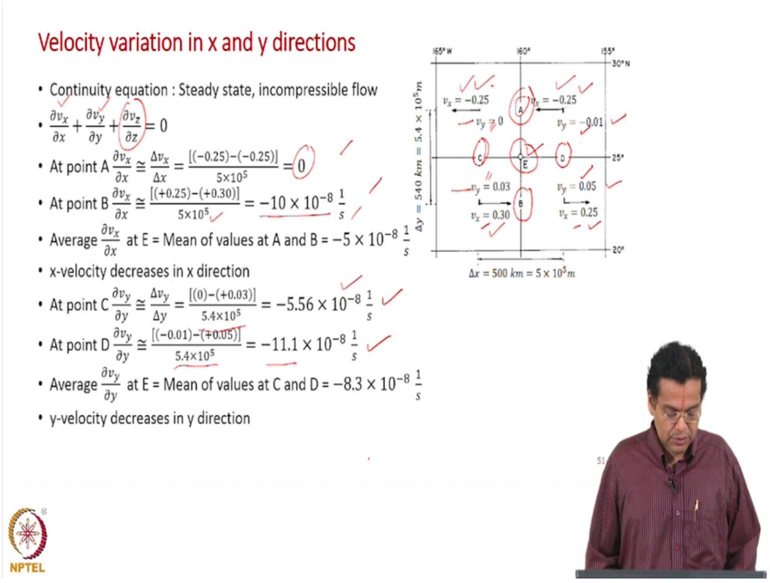
Now, in this coordinate system, the horizontal axis is x, the vertical is y and of course, the axis towards us is a z-axis. Now, what I have shown in the second figure is that have changed the axis. I moved x, y, and z anti-clockwise. So now, the horizontal axis becomes z, the vertical is x and the axis towards us is y. Now, I do one more rotation in the same way so, I move x, y, and z anti-clockwise result in this axis. So, the horizontal axis is y, the vertical is z, the axis towards us is x.

Now, our attention is the first figure and third figure, not the usual coordinate axis which are used in books are the first and the third; so, we will discuss not the second one. If you look at the first choice coordinate axis what is the advantage, if you look at the front view you can easily reduce it to two-dimensional case; you just have x-axis and y-axis. What is a slight disadvantage? Gravity acts along the y-axis, of course, negative y-axis which is not so conventional, the moment you say some position you always say z; let us say someday term is there z = 0.5. So, traditionally and always the direction along which gravity acts is z so, that is why it is slightly unconventional.

So, let us come to the third diagram, where gravity acts along z-direction so, along the conventional z-direction. But what is the disadvantage? Suppose, if you want to reduce to 2D you will have to look down, of course, maybe little turn then you will have x-axis and y-axis, that way slightly inconvenient. But, this is the axis which is used in this problem; I say on the surface of the ocean I have x, y, and then z-axis of course, into the ocean will be negative. So, the third choice is what is applicable for this particular problem with z-axis along the vertical direction.

So, these are the different choice of co-ordinate system, we will usually come across the first or the third, second is rarely used. Let us proceed with this co-ordinate axis keeping this co-ordinate axis, the third one keeping in mind that is what is of course used here.

Solution: (Refer Slide Time: 06:05)



Now, we are going to use the continuity equation, once again we will assume it is a steady state and assume incompressible flow in the ocean and, remember it is a three dimensional case because we have all three components. We are going to apply this continuity equation at the point E.

$$\frac{\partial(v_x)}{\partial x} + \frac{\partial(v_y)}{\partial y} + \frac{\partial(v_z)}{\partial z} = 0$$

I need values for this derivative at E then I will find out what is v_z that is the whole idea; I want to evaluate the first two derivatives at E. To evaluate the $\frac{\partial(v_x)}{\partial x}$, I will evaluate at A. How do I evaluate? to evaluate the derivative I need the x velocity at two points which are separated along the x-axis, similarly, I will use the two v_x values at point B and find out what is the derivative at B, take an average and I will get the value at E, similarly, we will do for the other directions.

So, at point A

$$\frac{\partial(v_x)}{\partial x} \cong \frac{\Delta v_x}{\Delta x} = \frac{[(-0.25)-(-0.25)]}{5\ x\ 10^5} = 0$$

Now let us consider point B

$$\frac{\partial(v_x)}{\partial x} \cong \frac{\Delta v_x}{\Delta x} = \frac{[(+0.25)-(+0.30)]}{5 \, x \, 10^5} = - \, 10 \, x \, 10^{-8} \frac{1}{s}$$

Now, to find out $\frac{\partial(v_x)}{\partial x}$ at E I take the average of values found out A and B.

$$\frac{\partial(v_x)}{\partial x} \, at \, E = Mean \, of \, values \, at \, A \, and \, B = \; - \, 5 \, x \, 10^{-8} \frac{1}{s}$$

We do all this because it can be highly fluctuating, we need to arrive at some average value that is why we make measurements divide into four regions, make measurements, find make four measurements, find from the two find two derivatives, find average all these are done at average out. Let us exactly do the same thing for the y-direction.

At point C,

$$\frac{\partial(v_y)}{\partial y} \cong \frac{\Delta v_y}{\Delta y} = \frac{[(0)-(+0.03)]}{5.4 \, x \, 10^5} = - \, 5.56 \, x \, 10^{-8} \frac{1}{s}$$

Now, at point D

$$\frac{\partial(v_y)}{\partial y} \cong \frac{\Delta v_y}{\Delta y} = \frac{[(-0.01)-(+0.05)]}{5.4 \, x \, 10^5} = - \, 11.1 \, x \, 10^{-8} \frac{1}{s}$$

Let us take an average of these two values which will give us $\frac{\partial(v_y)}{\partial y}$ at E.

$$\frac{\partial(v_y)}{\partial y} \, at \, E = Mean \, of \, values \, at \, C \, and \, D = \; - \, 8.3 \, x \, 10^{-8} \frac{1}{s}$$

So, at the same point, you have got the first value for $\frac{\partial(v_x)}{\partial x}$ which is the change of v_x with respect to x; you have also obtain $\frac{\partial(v_y)}{\partial y}$ change of v_y by y-direction that is also negative. So, y velocity also decreases along the y-direction, that was our objective at the same point, evaluate these derivatives.

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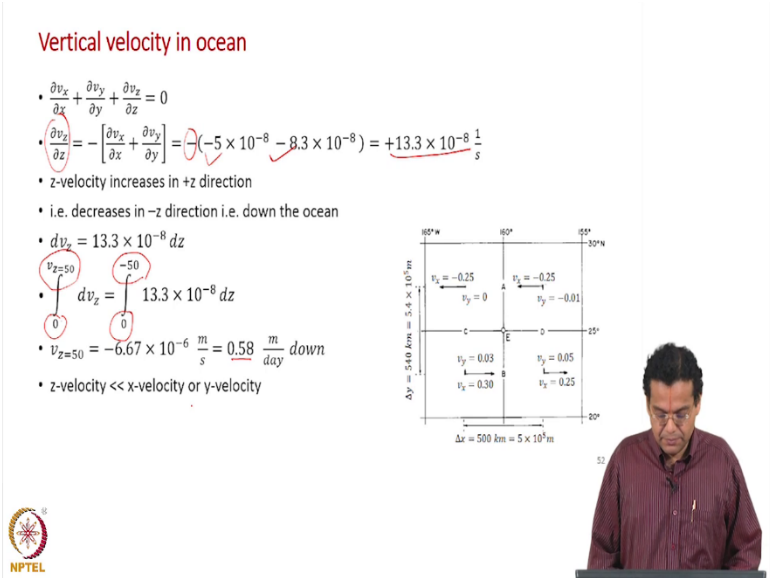


Now, let us substitute in the continuity equation to find out the $\frac{\partial(v_z)}{\partial z}$.

$$\frac{\partial(v_x)}{\partial x} + \frac{\partial(v_y)}{\partial y} + \frac{\partial(v_z)}{\partial z} = 0$$

$$\frac{\partial(v_z)}{\partial z} = -\left[\frac{\partial(v_x)}{\partial x} + \frac{\partial(v_y)}{\partial y}\right]$$

$$\frac{\partial(v_z)}{\partial z} = -\left[-5x10^{-8} - 8.3x10^{-8}\right] = 13.3x10^{-8}\frac{1}{s}$$

So, the z velocity increases along the positive z-axis which mean down the ocean it decreases, z velocity increases the positive z-direction in the ocean of course, in the negative z-direction or to go down in the ocean the z velocity decreases.

So, if we start with the ocean it decreases. So now, we are asked to find out what is a velocity at a particular depth, which means that we will have to integrate this. We will assume all these do not vary with spatial location, that is why it is a constant and then integrate, you integrate from the surface of the ocean down the ocean. So, the limits of velocities are we are said that the z component of velocity is 0 at the surface and we are asked to find out what is the z velocity at a distance of 50 meters. So, the limits for z are $z = 0$ at the surface of the ocean because we are going down the ocean and $z = 50$ of the depth of 50 meters.

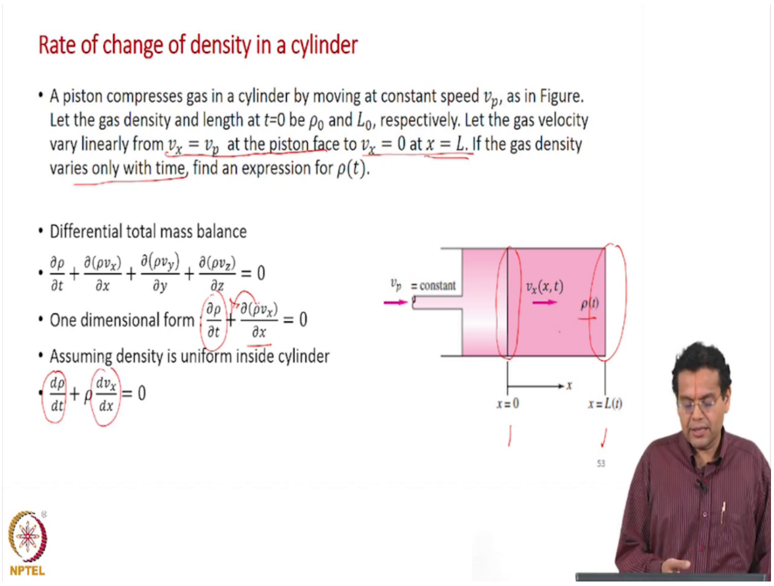
$$\int_{0}^{50} dv_z = \int_{0}^{50} 13.3x10^{-8} dz$$

$$v_{z=50} = -\ 6.67x10^{-6}\frac{m}{s} \ = 0.58\ \frac{m}{day} down$$

Through this integration and you get the velocity which is negative. So, to get a quick idea of what this velocity is 0.58 meters or you will be traveling 0.58 meters per day. So, in a day you will travel only point roughly is 0.6 meters which is very low velocity. Remember our x and y velocities are extremely large, they were order of some meters per second etcetera, but now our z velocity extremely low; maybe that is reasons it is difficult to measure the z velocity but measure the x and y velocity.

So, a very good example I would say linking measurements along x and y-axis, finding that using that to estimate the z velocity component using the continuity equation.

Example: (Refer Slide Time: 13:35)



The last example as usual including the transient term, let us discuss the geometry first. I have a cylinder with the piston a well known geometry, let us say you would have come across in thermodynamics and mechanical engineering class etcetera. This could be a cylinder in an engine or reciprocating pump etcetera, of course, we are considering a gas here and we have a piston that moves at a constant velocity v_p.

So, let us read the problem. A piston compresses gas in a cylinder by moving at a constant speed as shown in the figure. Now, let the gas density and length at the time some time t = 0 be ρ_0 and L_0. What does it mean? the density is a function of time at some time t = 0 it is ρ_0 and the length keeps changing with time at some time t = 0 it is L_0. Let the gas velocity vary linearly from $v_x = v_p$ at the piston phase to $v_x = 0$ at $x = L$. What is the meaning of that? Remember we have been telling always that at a surface the fluid clings to the surface which means that it gets its same velocity at that of the surface. So now, this velocity this surface moves at a velocity of v_p. So, the velocity at the fluid and that which is attached or which is clinging to that surfers also v_p.

Now, this surface of the cylinder stationary so, the velocity of the fluid in this region or at the surface is also 0, that is why it says let the gas velocity vary, of course, I mean the assumption that the variation is linear. What I explained is the condition that $v_x = v_p$ at the piston phase to $v_x = 0$ at $x = L$; they are nothing, but the velocity of the piston at the phase of the piston and, the velocity of the surface of the cylinder which is 0 at the other side that is why the conditions are $v_x = v_p$ and $v_x = 0$ and we are assuming it will vary linearly.

Remember, L is as a function of time we cannot say fix time that is why it is L. If the gas density varies only with time, what does it mean? In the cylinder, we are assuming density to be constant throughout the volume. So, that is why gas density varies only with time. Find an expression for $\rho(t)$, how does the density vary as a function of time; we are not accounting for spatial variation.

Solution:

Now, let us start the differential total mass balance, now remember density varies spatially. So, I cannot use a simplified form and we just have only one-dimensional case. So, I simplify it and write as

$$\frac{\partial(\rho)}{\partial t} + \frac{\partial(\rho v_x)}{\partial x} + \frac{\partial(\rho v_y)}{\partial y} + \frac{\partial(\rho v_z)}{\partial z} = 0$$

For one dimensional case,

$$\frac{\partial(\rho)}{\partial t} + \frac{\partial(\rho v_x)}{\partial x} = 0$$

Remember that, I have not taken density out of this spatial derivative. Now, if you assume density is uniform inside the cylinder, after this assumption I am not saying density is not a function of time, density is assumed to be uniform inside the cylinder. Now, I take density out of the second derivative.

$$\frac{\partial(\rho)}{\partial t} + \rho\frac{\partial(v_x)}{\partial x} = 0$$

So, density plays a role in this form of the equation, in all other earlier equations we never saw the density at all. So, in this particular case after we assume the density is uniform inside the cylinder, then I take out the derivative and, both of them become total derivatives because density is a function of time only so,

$$\frac{d\rho}{dt} + \rho\frac{dv_x}{dx} = 0$$

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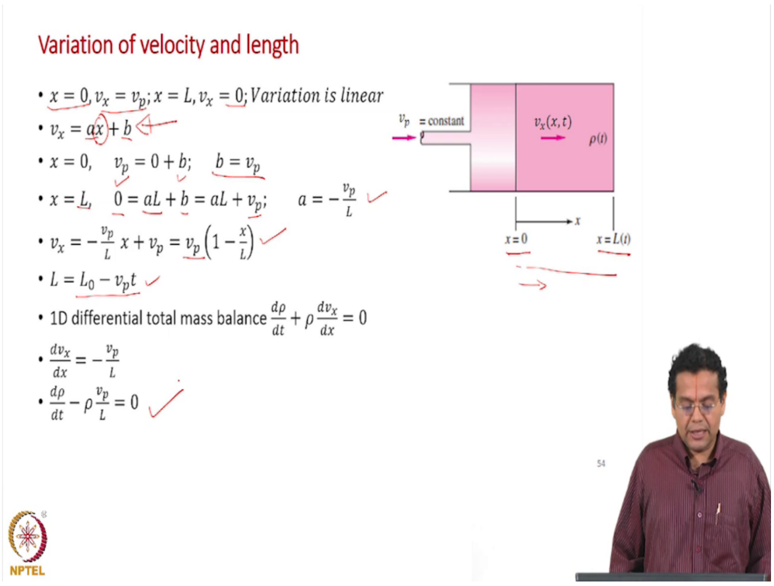

Now, we need to find an expression for the velocity, we are told that the x velocity varies linearly, and let us find out what is the expression, a simple expression. The given condition are

$$x = 0,\ v_x = v_p\ ; \quad x = L,\ v_x = 0;\ \ variation\ is\ linear$$

So, if you take $v_x = ax + b$ we have two conditions to find the two constants, that is what we are going to do. Let us assume $v_x = ax + b$ and find out the constant,

$$x = 0, \quad v_p = 0 + b \; ; \quad b = v_p$$

$$x = L, \quad 0 = aL + b = aL + v_p; \quad a = -\frac{v_p}{L}$$

So, let us substitute in the constants in the equation and then find out that the velocity variation with respect to distance follows this relationship

$$v_x = -\frac{v_p}{L}x + v_p = v_p\left(1 - \frac{x}{L}\right)$$

When $x = 0$ we have $v_x = v_p$, and $x = L$ we have $v_x = 0$ of course, it should obey those conditions. What is that we have found out? We have found out v_x as a function of x. Why do we do that because we want to evaluate the derivative of course,

$$L = L_0 - v_p t$$

So, the remaining length is $L_0 - v_p t$, time t = 0 it is L_0 overs time period t the piston moves a distance $v_p t$. So, the remaining length is $L_0 - v_p t$ that is what I have always been emphasizing, that L is a function of time and it decreases with time.

Now, the one dimension differential total mass balance as we have written is

$$\frac{d(\rho)}{dt} + \rho\frac{d(v_x)}{dx} = 0$$

So, let us substitute here, we have got an expression for v_x let us differentiate that and get

$$\frac{dv_x}{dx} = -\frac{v_p}{L}$$

So,

$$\frac{d(\rho)}{dt} - \rho\frac{v_p}{L} = 0$$

We are proceeding towards evaluating rho as a function of time.

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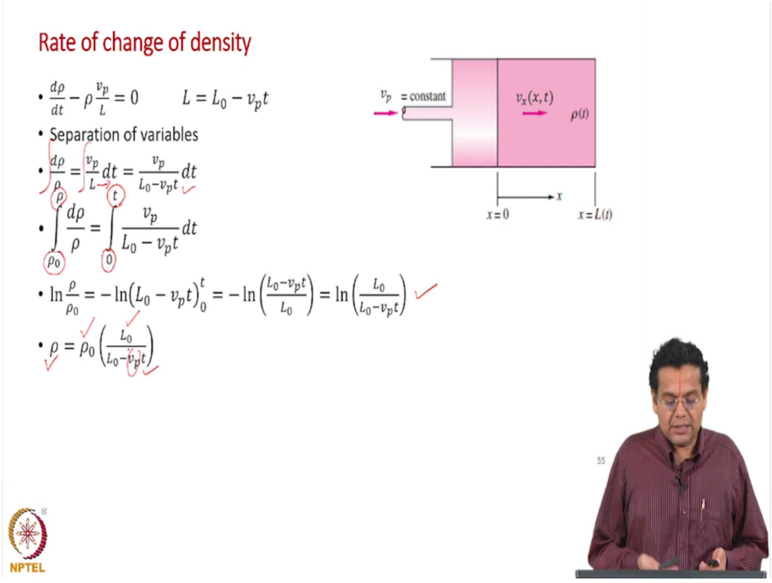

So, let us do that

$$\frac{d(\rho)}{dt} - \rho\frac{v_p}{L} = 0$$

$$L = L_0 - v_p t$$

So, let us use separation of variables,

$$\frac{d(\rho)}{\rho} = \frac{v_p}{L}dt = \frac{v_p}{L_0 - v_p t}dt$$

We have substituted for L it is a function of time when I integrate here right hand side L is not a constant. So, before integrating we should substitute in terms of time. Now, let us integrate at time t = 0 density is ρ_0, at any time t the density is ρ. Those are the limits of the left hand side, right hand side, of course, limits are 0 and then time any time t.

$$\int_{\rho_0}^{\rho}\frac{d(\rho)}{\rho} = \int_0^t \frac{v_p}{L_0 - v_p t}dt$$

$$\ln ln\left(\frac{\rho}{\rho_0}\right) =- \ln ln\left(L_0 - v_p t\right)_0^t =- \ln ln\left(\frac{L_0 - v_p t}{L_0}\right) = \ln ln\left(\frac{L_0}{L_0 - v_p t}\right)$$

$$\rho = \rho_0\left(\frac{L_0}{L_0 - v_p t}\right)$$

So, density depends on time for given initial densities, initial length, and velocity of the piston those are the variables. So, if you give me the velocity of the piston and initial density, initial length; this expression gives us how does density varying as a function of time.

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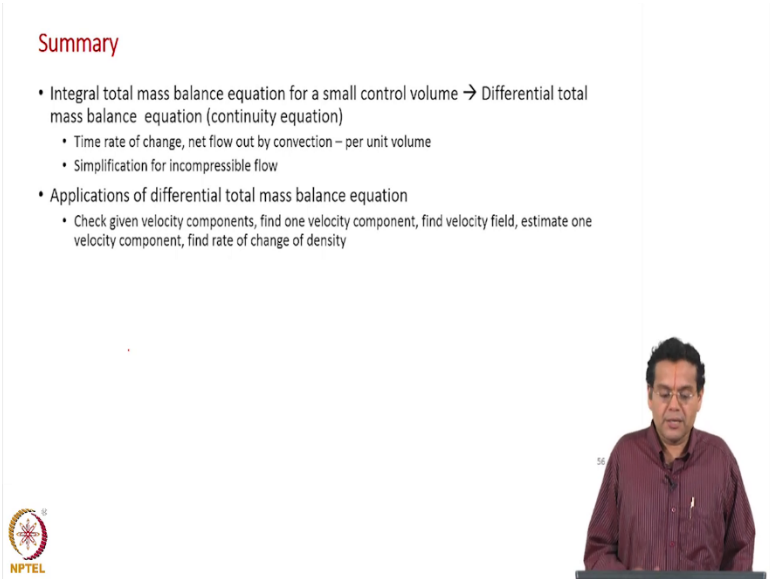

So, with that, we have concluded all the applications. So, let us summarize this part on the differential total mass balance equation.

- We started with the integral total mass balance equation for a small control volume and obtained the differential total mass balance equation by shrinking it to a point, also called the continuity equation and,
- It has two terms: one of the time rate of change of mass, another term of net flow out of mass by convection, but the emphasis on per unit volume that is why it is density. And, we simplified for of course, compressible or incompressible within the scope of this course, and most of the courses in your engineering curriculum you will be restricted to incompressible flow. So, that form is more important which we will come across repeatedly later.
- We looked at applications of differential total mass balance equations. We checked given velocity components, we found out one velocity component given the other, we have also used the integral and differential balance to find velocity field and we estimated one velocity component based on experimental measurements on two other

components and finally an example on the rate of change of density with respect to time.