# Continuum Mechanics And Transport Phenomena

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## Lecture - 27
## Differential total mass balance: Examples Part 1

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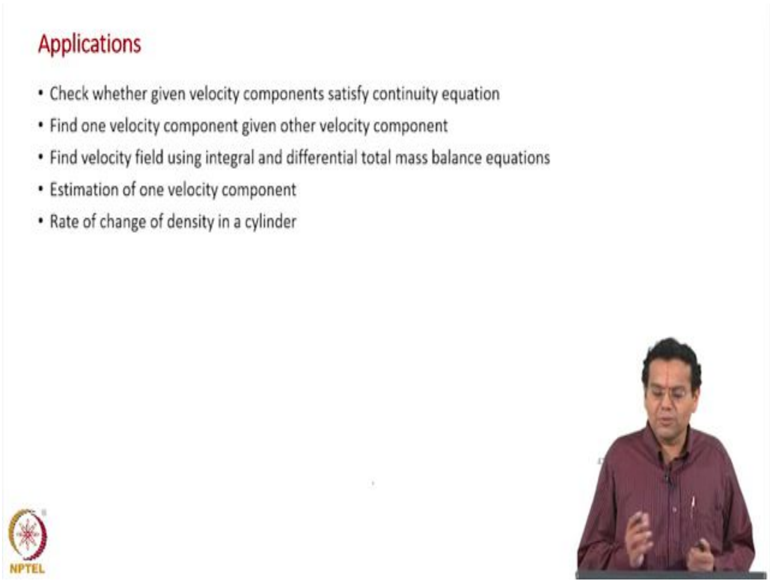



So, let us look at applications of differential total mass balance equation or the continuity equation.

- The first application is that suppose if you have arrived at some velocity field meaning I found out the expressions for $v_x$, $v_y$, $v_z$ you can check whether the given velocity components or found out components satisfy the continuity equation.

- The second, if you know one velocity component you can find the other velocity component as we have seen that continuity equation puts a restriction on how one velocity component depends on the other etcetera. So, if we specify one let us say 2D you cannot independently specify another it is constrained by the mass balance.

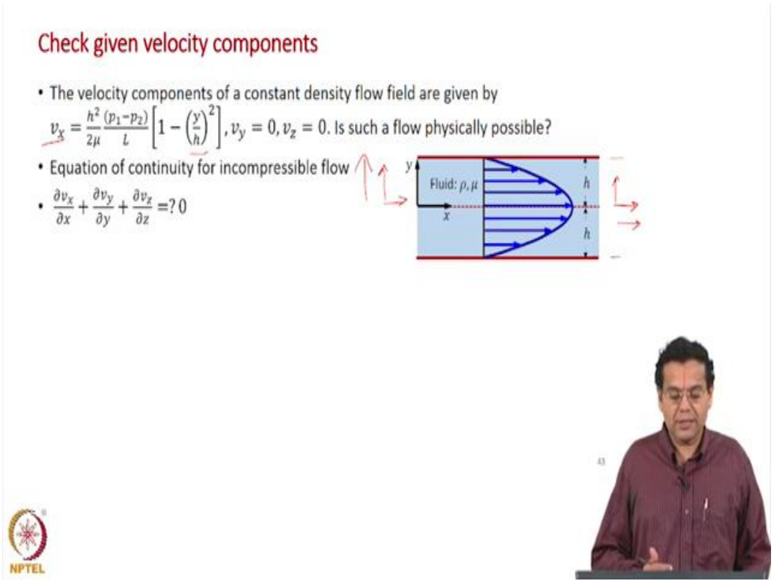
- The third example is that under certain conditions you can use the integral and differential mass balance equation to arrive at the velocity field that is a good example there and

- The fourth one is once again interesting we make some measurements on velocity components along let us say two directions, the example of measurements of velocity in an ocean surface we make measurements along x and y direction and find out what is the vertical component of velocity using a mass balance or the continuity equation. That is why it says estimation of velocity component and

- Final example including the transient term that we find out what is the rate of change of density in a cylinder that cylinder could be in an engine or in a pump etcetera.

The scope of the whole course is that we will be using almost always only the incompressible form of the continuity equation. Even throughout the engineering chemical engineering course any engineering course, we will be almost be restricted only to the incompressible form mechanical engineers do use the compressible form as well.

**Example:** (Refer Slide Time: 02:12)




Look at the first application is that, the geometry is well known to us its flow between the two parallel plates. The velocity components of a constant density may implies that it's an incompressible flow, constant density flow field are given by. We are given the x component of velocity and that depends on the y-direction.

$$v_x = \frac{h^2}{2\mu}\left(\frac{p_1-p_2}{L}\right)\left[1-\left(\frac{y}{h}\right)^2\right], \quad v_y = 0, \quad v_z = 0$$

The configuration geometry is shown, the horizontal axis x, and the vertical axis is y. So, the velocity in the x-direction varies along the y-direction we also called as a lateral direction. So, flow is in one direction perpendicular to the variation is there. Flow takes place in the x-direction and variation is in the y-direction and there is no variation along the direction of flow, $v_x$ depends only on y of course, there is no velocity component in the vertical direction, there is no flow in the y-direction, of course, there is no flow in the z-direction as well. Now, the question is in such a flow physically possible?

**Solution:**

As I told you let us say we have found out this expression and we need to check whether it satisfies the continuity equation. Let us write down the equation of continuity for incompressible flow.

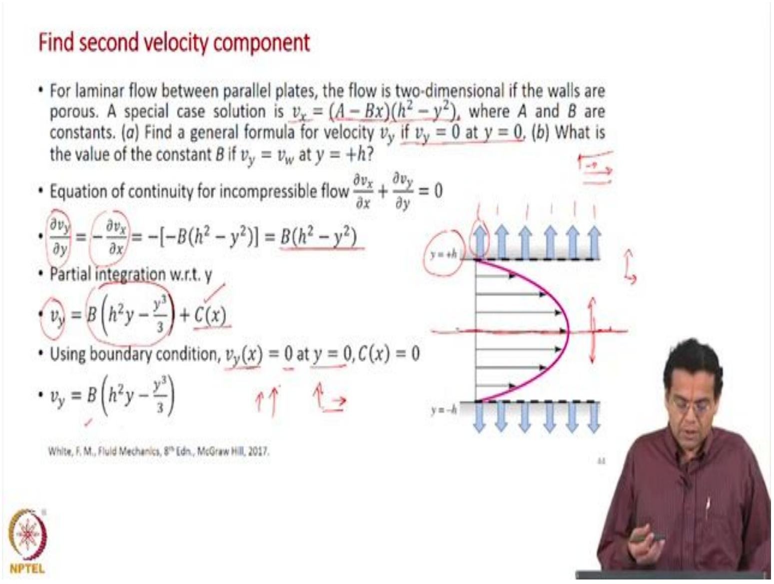
$$\frac{\partial(v_x)}{\partial x} + \frac{\partial(v_y)}{\partial y} + \frac{\partial(v_z)}{\partial z} = 0$$

So, there is no density term rate of change of density with time is also not there, there is no density in the net mass flow term also. You need to check whether this is indeed 0. So,

$$v_x = f(y) \rightarrow \frac{\partial(v_x)}{\partial x} = 0, \quad v_y = 0, \qquad v_z = 0$$

The first derivative is 0, not $v_x$, in second and third terms the velocity itself is 0. So, which means that the equation of continuity is satisfied. So, we can conclude that the constant density of flow describes the velocity components is possible. A flow is possible only if it satisfies the basic conservation of mass. At every point in the flow field remembers its a point form of a conservation equation at every point in the flow field mass balance is satisfied that is the meaning of this.

**Example:** (Refer Slide Time: 04:38)




Let us take the second example first let me explain the geometry of two parallel plates as we have always seen, but now the difference is that the plates are porous. So, there is flow in the y-direction as well this results in a two dimensional velocity field what does it mean that is variation along with the x-direction variation along the y-direction. Now when there were two nonporous plates then the x velocity varied in the y-direction only. Now because there is outflow through this porous wall x velocity varies along the flow direction also. So, that is why it becomes two dimensional and we can expect that because there is outflow the velocity will decrease along the direction of flow just to repeat. If the walls were not porous, then the x velocity varies along the y-direction only. But now because the walls are porous and water leaves through that now along the flow direction the velocity decreases.

Let us now read the problem, for laminar flow between parallel plates the flow is two dimensional if the walls are porous if the walls were not porous it was one dimensional. A special case solution meaning under certain conditions you can derive the expression for x component of velocity.

$$v_x = (A - Bx)(h^2 - y^2)$$

Look at the expression it has 2 terms, $(h^2 - y^2)$ this term tells you the variation in the y-direction. Now we have seen that physically the velocity should decrease along the x-direction that is denoted by $(A - Bx)$ it has a minus Bx term which means that the velocity decreases along the x-direction where A and B are constants now what is that we are asked to find out?

a) Find the general formula for velocity $v_y$, I told you one use of major use of continuity equation is that, we are given one velocity component we will have to find out the other velocity component in this case $v_y$. We are also given one condition which says that if $v_y = 0$ at $y = 0$ what is that line y = 0? The axis of the central line is y = 0 So, along the axis, there is no velocity that is why vy is 0 something like a dividing line flow goes towards the +h and then goes towards −h. So, flow division takes place and so, along this line, there is no vertical component of velocity, that is why this says $v_y = 0$ at $y = 0$.

b) Now, the second part is what is the value of the constant B, if $v_y = v_w$ at $y =+ h$. $y =+ h$ denotes the top plate and at that plate, you are given the velocity. Let us say the flow through the porous wall is at a certain value given by $v_w$ if that is the case how do you find out B.

**Solution:**

So, let us proceed with this start the equation of continuity as usual for incompressible flow,

$$\frac{\partial(v_x)}{\partial x} + \frac{\partial(v_y)}{\partial y} = 0$$

Now, because it is two dimensional case so, I am not considering the third term. Now, we want to find out the $v_y$ the y component velocity. So, let us say rearrange for y component of velocity,

$$\frac{\partial(v_y)}{\partial y} =- \frac{\partial(v_x)}{\partial x} =- \left[-B\left(h^2 - y^2\right)\right] = B(h^2 - y^2)$$

So, if we differentiate this what you get. So, if simplify you get $B(h^2 - y^2)$. Now we have a partial differential for $v_y$ we have to get $v_y$ which means that we will have to integrate it. Now it's not usual integration where we are given $\frac{dv_y}{dy}$, but now we have $\frac{\partial(v_y)}{\partial y}$ it's a partial differentiation hence the integration is also partial let us see what is the implication of that. Now, the partial integration w.r.t y

$$\int \frac{\partial(v_y)}{\partial y} dy = \int B(h^2 - y^2)\, dy$$

$$v_y = B\left(h^2 y - \frac{y^3}{3}\right) + c(x)$$

Now, because it is partial differentiation and we are doing a partial integration, the constant can be a function of x also. To be more generic we are writing it as C as a function of x that that is what differentiates on a usual integration from this partial integration where the constant can be a function of x. Now just like usually evaluate the constant based on a boundary condition in this case also we are given a condition. So, let us use that. So, using the boundary condition, boundary condition meaning that, at a particular location you are given the value for the dependent variable

$$v_y(x) = 0 \text{ at } y = 0$$

$$0 = B\left(h^2 0 - \frac{0^3}{3}\right) + c(x) \rightarrow C(x) = 0$$

So, that is how we evaluate the constant similar to usual integration use the boundary condition and evaluate C = 0. So, now, your expression for the y velocity component becomes

$$v_y = B\left(h^2 y - \frac{y^3}{3}\right)$$

This tells you the variation of $v_y$ as a function of y, $v_x$ the x velocity component varied along y direction and x direction also. The vertical component of velocity $v_y$ varies along only y direction look at the dependency and we have a constant B which can be evaluated let us proceed to evaluate that.

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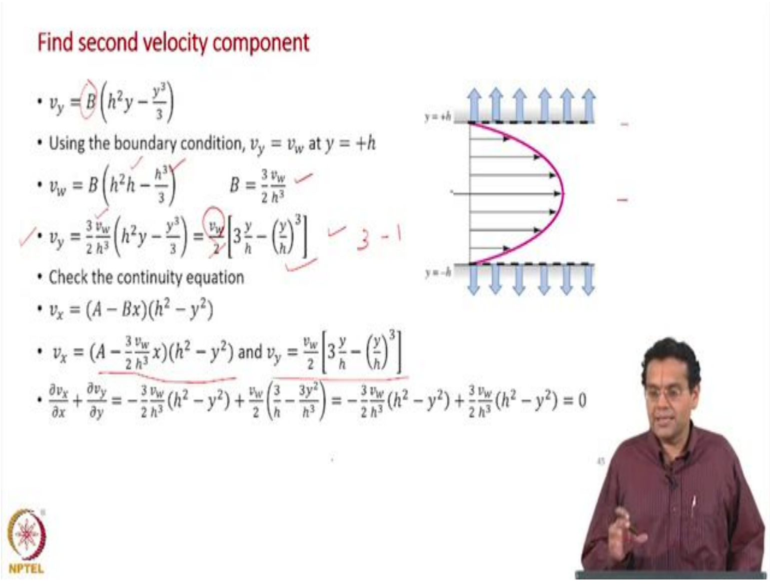



So, that is the expression for $v_y$ ,

$$v_y = B\left(h^2 y - \frac{y^3}{3}\right)$$

Now, we are given another boundary condition. Using the boundary condition

$$v_y = v_w \text{ at } y = +h$$

So, let us substitute that

$$v_w = B\left(h^2(+h) - \frac{(+h)^3}{3}\right)$$

$$B = \frac{3}{2}\frac{v_w}{h^3}$$

Let us substitute the B in expression for completing the expression for $v_y$ .

$$v_y = \frac{3}{2}\frac{v_w}{h^3}\left(h^2 y - \frac{y^3}{3}\right)$$

If you look at the question it says a general expression for $v_y$ because B is a constant value, but once you use the second boundary condition you get a specific value for B and hence becomes velocity expression for the specific problem. Of course, you can simplify and put in this nice form

$$v_y = \frac{v_w}{2}\left[3\frac{y}{h} - \left(\frac{y}{h}\right)^3\right]$$

Now, this velocity profile is satisfied by both the boundary conditions. Of course, when you substitute y = 0 this becomes 0 ( $v_y = 0$ ) and when you substitute y = h it results in $v_w$ So, it satisfy; obviously, those two conditions. Now the problem is solved, but we can check whether the expressions are correct by substituting in the continuity equation just as an add on we can do that. So, let us do that

$$v_x = (A - Bx)(h^2 - y^2)$$

So, we will substitute the expression for B which I have found out now

$$v_x = (A - \frac{3}{2}\frac{v_w}{h^3}\, x)(h^2 - y^2)$$

and then, of course, we can also take the expression for $v_y$ which we derived here.

$$v_y = \frac{v_w}{2}\left[3\frac{y}{h} - \left(\frac{y}{h}\right)^3\right]$$

Let us substitute in the continuity equation

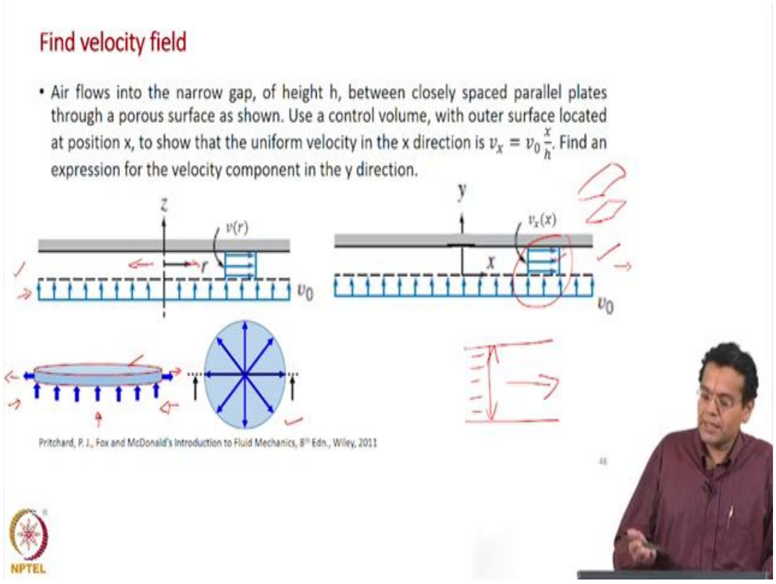
$$\frac{\partial(v_x)}{\partial x} + \frac{\partial(v_y)}{\partial y} = 0$$

Now, differentiate $v_x$ and $v_y$ partial with respect to x and y respectively then

$$\frac{\partial(v_x)}{\partial x} + \frac{\partial(v_y)}{\partial y} =- \frac{3}{2}\frac{v_w}{h^3}\left(h^2 - y^2\right) + \frac{v_w}{2}\left[\frac{3}{h} - \frac{3y^2}{h^3}\right] =- \frac{3}{2}\frac{v_w}{h^3}\left(h^2 - y^2\right) - \frac{3}{2}\frac{v_w}{h^3}\left(h^2 - y^2\right) = 0$$

So, that is how you check using the continuity equation, here it is very obvious, we use the continuity equation to get the velocity profile, but just to for illustration it also checking back whether the continuity equation is satisfied. If it were just two parallel plates, it would be interesting just because we made it porous we got a two dimensional field and so, it became a little more interesting. That is the selection of the problem in a book. The books when they write they choose give problems either they are simple, at the same time put for the point for example, in this case, simple modification resulted in a two dimensional flow field.

**Example:** (Refer Slide Time: 15:46)




A good example which combines the integral mass balance and the differential mass balance. Let me first explain the geometry which is shown here (above slide image) we have two plates and a thin gap is exist between them and the bottom plate is porous just like the earlier case, but the top plate is not porous that is a solid surface. Now, this case air enters through the porous plate from the bottom and leaves around the pheri – pheri. So, that is the front view the top view shown here. So, the flow takes place radially flow. So, we have a plate, air enters and then flows radially out and that is what is shown as a top view here and the front view is, of course, shown the air entering here, and then flow flowing if you look at the front view it flows along the radial direction, now analysis of this will require the cylindrical coordinate system. Once again we want to restrict to the Cartesian coordinate system what do I do? I take just two plates not a circular geometry anything just to plates of width w and then the bottom plate is porous and analyze the same situation in terms of Cartesian coordinates and that is what is shown on the right hand side.

So, in this case, these are plates and the top plate is not porous, the bottom plate is porous and the gap is very thin and then we are seeing how flow takes place in x-direction what is the velocity profile etcetera.

Let us now read the problem, air flows into the narrow gap of height h between closely spaced parallel plates to the porous surface as shown. The problem for this figure is shown on the right hand side I have also taken shown on the left hand side if we have to understand the configuration the flow of geometry etcetera. Use a control volume with an outer surface located at position x, to show that the uniform velocity in the x-direction is

$$v_x = v_0 \frac{x}{h}$$

Such a profile is shown here when I say uniform velocity we have seen this earlier also for the application of integral mass balance. Suppose if you have a flow between two parallel plates when I say a uniform velocity the velocity does not vary in the lateral direction putting in the other way the velocity does not vary in the direction perpendicular to the flow.

So, such a profile is shown and we are asked to find out this uniform velocity in the x-direction and in this case, it happens to be a function of x, we will have to prove that we will have to arrive at this expression. Once we get one velocity component as we have seen in the previous example, we can get the other velocity component also. So, find expression for the velocity compound of the y-direction.

**Solution:**

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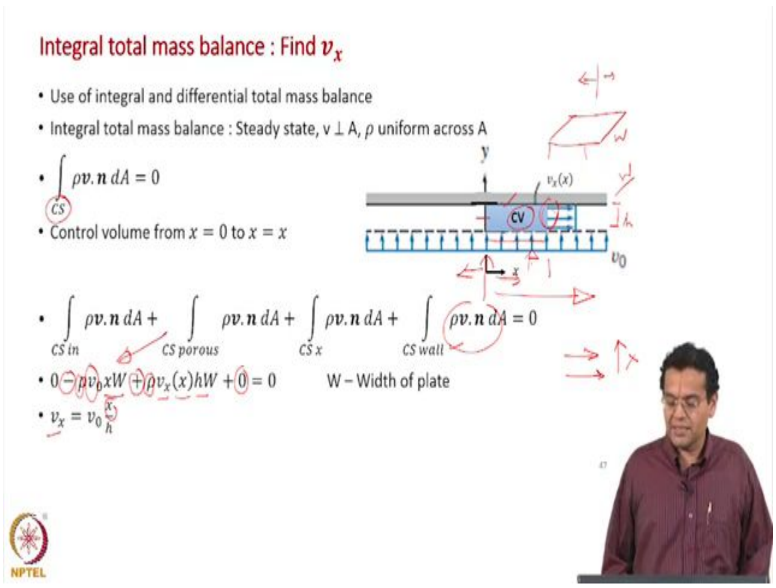


First, we will use the integral balance to get $v_x$ then second we use the differential balance. So, that way this example nicely combines both from the balance equation. Now, the use of integral and differential total mass balance is the object of this example. So, let us write the integral total mass balance equation at a steady state and the velocity is perpendicular to area and the density is uniform across the cross sectional area.

$$\int_{CS} \rho\, v.n\, dA = 0$$

So, this is an integral form of the mass balance equation, there is no transient term and I am writing only the net mass flow term alone. Now, the control volume is shown here, the question says that the control volume should extent from x = 0 to a distance x = x that is why this control volume is shown here. Now, this control volume has four surfaces let us split this control surface for those four surfaces.

$$\int_{CS\ in} \rho\, v.n\, dA + \int_{CS\ porous} \rho\, v.n\, dA + \int_{CS\ x} \rho\, v.n\, dA + \int_{CS\ wall} \rho\, v.n\, dA = 0$$

The left side is the in surface and then the bottom porous surface and then the right side surface at x and then the top wall. Now as we have seen in the previous case a flow takes place and divides, similarly here also flow is along this direction and divides this side and this side. So, along that line, there is no velocity the velocity is 0.

$$0 - \rho v_0\, xW + \rho\, v_x(x)\, hW + 0 = 0$$

So, the first term does not contribute once again it's x = 0 where flow division takes place, and hence there is no velocity at that particular surface. Now coming to the bottom surface it is an inflow surface because flow enters through the porous wall which is shown by a negative sign we have seen $v.n$ is negative for inflow and then the magnitude of velocity is, $v_0$ now, what is the area? Now we have considered a plate whose width is W and this distance the length of the control volume along that direction is x. So, the area = $xW$.

Now coming to the surface where there is a flow out of surface at the x its outflow that is why it is positive, density is written as such what is the magnitude of the velocity $v_x$? But now that $v_x$ depends on x that that is shown explicitly here what is area now? This is the h is the height between the two plates and we have width W hence the area = $hW$. Of course, at the wall there is no penetration through the wall hence the last term also does not contribute once again that is 0.

So, now, let us simplify this our objective is to define what is $v_x$. So,

$$-\rho v_0\, xW + \rho\, v_x(x)\, hW = 0$$

$$v_0\, xW = v_x(x)\, hW$$

$$v_x = v_0 \frac{x}{h}$$

So, the velocity increases along this direction why does it increase? There is flow from this surface. So, whatever flows from the wall contributes to velocity along the x-direction hence velocity increases along the x-direction given by this expression.

So, this is the case where the velocity varies along the flow direction just like in the last case and it does not vary in the perpendicular direction. v x is not a function of y, but does not vary in the y-direction, but varies only in the x-direction, increasing linearly with x. So, we use the integral total mass balance equation under certain conditions like this and find out what is the x component of velocity and for which is the same as given in the question. So, we have kind of verified.

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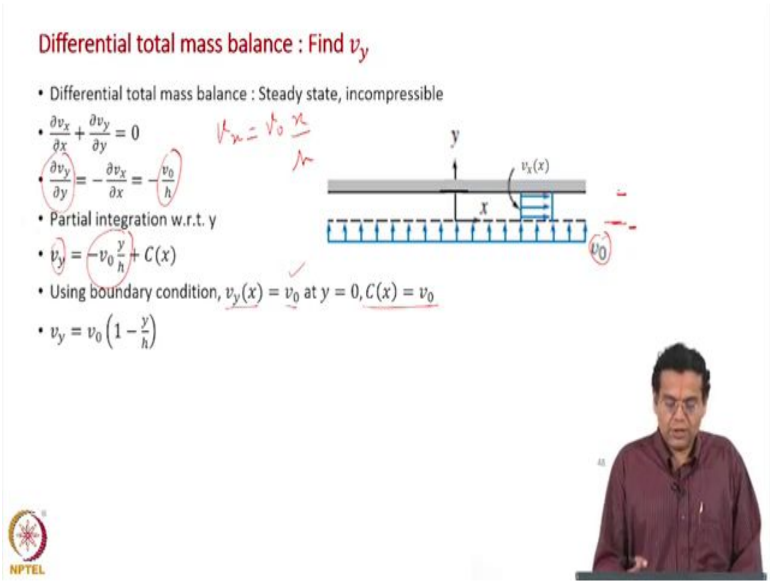


Now, this part is the same as what we have done earlier we know one velocity component use the continuity equation and then find out the other velocity components. So, this is not something new to this example, but the combination of integral differential balance is something new to this example.

So, we will start with the differential total mass balance the continuity equation for steady state incompressible flow and then write for the two dimensional case only the first two terms are considered.

$$\frac{\partial(v_x)}{\partial x} + \frac{\partial(v_y)}{\partial y} = 0$$

This is exactly what we have done in the previous example no difference at all, but for a different velocity field, and then we have seen the velocity profile

$$v_x = v_0 \frac{x}{h} \quad \rightarrow \quad \frac{\partial(v_x)}{\partial x} = \frac{v_0}{h}$$

Now, with partial integration with respect to y, as we have done earlier.

$$\frac{\partial(v_y)}{\partial y} = -\frac{\partial(v_x)}{\partial x}$$

$$\frac{\partial(v_y)}{\partial y} = -\frac{v_0}{h}$$

$$v_y = -v_0 \frac{y}{h} + C(x)$$

Now, using the boundary condition we evaluate the constant C so, the boundary condition at this location at this surface is

$$v_y(x) = v_0 \; at \; y = 0, \quad \rightarrow \quad C(x) = v_0$$

So, C the constant which can be a function of x, here it's just a constant $v_0$. So, let us substitute back we get

$$v_y = -v_0 \frac{y}{h} + v_0$$

$$v_y = v_0 \left(1 - \frac{y}{h}\right)$$

It is a simple expression for the vertical component of velocity. When y = 0, $v_y(x) = v_0$ and when y = h, $v_0 =$ 0 because there is no flow through the top surface. So, it satisfies the conditions. So, what is that we have done? Here is I used the integral mass balance equation and got one velocity component use the differential mass balance got the other velocity component.